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Existentially closed ordered difference fields and rings

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We describe classes of existentially closed ordered difference fields and rings. We show an Ax-Kochen type result for a class of valued ordered difference fields.

1 Existentially closed real-closed difference fields

In the first part of this paper we will consider on one hand totally ordered *difference fields* (a difference field is a field with a distinguished automorphism σ) and on the other hand preordered difference fields.

By a well-known theorem of A. Tarski, the theory RCF of real-closed fields is the model-companion of the theory of totally ordered fields and a direct consequence of results of H. Kikyo and S. Shelah is that the theory of real-closed totally ordered difference fields, RCF_σ, does not have a model-companion (see [17]).

Note that in a difference field (K, σ) one has automatically a pair of fields, namely $(K, \mathrm{Fix}(\sigma))$, where $\mathrm{Fix}(\sigma)$ denotes the subfield of elements of K fixed by σ, and if K is real-closed, then so is $\mathrm{Fix}(\sigma)$. W. Baur showed that the theory of all pairs of real-closed fields (K, L) with a predicate for a subfield is undecidable ([1]). However, he also showed that, adding to the language of ordered rings a new function symbol for a valuation v, the theory of the pairs (K, L) such that v is convex, the residue field of L is dense in the residue field of K and each finite-dimensional L-vector space of K has a basis $a_1, \dots, a_n$ satisfying $v(\sum_i b_i \cdot a_i) = \min_i\{v(b_i \cdot a_i)\}$ for all $b_i \in L$, becomes decidable ([1]).

First, we describe a class of existentially closed totally ordered difference fields (even though it is not an elementary class). We also consider the case of a proper preordering, using former results of A. Prestel and L. van den Dries (see Section 1.3).

Then we consider valued totally-ordered fields and we assume on one hand that σ is strictly increasing on the set of elements of strictly positive valuation and on the other hand that in the pair $(K, \mathrm{Fix}(\sigma))$ the residue field of K and the residue field of $\mathrm{Fix}(\sigma)$ coincide (and so we are trivially in the Baur setting).

We proceed as for the case of valued difference fields with an ω-increasing automorphism treated by E. Hrushovski ([6]) and we show an Ax-Kochen-Ersov type result.

In the second part, we consider commutative von Neumann regular lattice-ordered rings with a distinguished automorphism σ which fixes the set of the maximal ℓ-ideals and we use transfer results due to S. Burris and H. Werner ([4]) in certain Boolean products in order to describe the class of existentially closed such lattice-ordered rings.

In [15], we showed certain undecidability results for Bezout difference rings. One of the consequences was that any commutative lattice-ordered ring with a distinguished automorphism σ with an infinite orbit on the set of its maximal ℓ-ideals has an undecidable theory, whenever the subring fixed by σ is an infinite field ([15, Corollary 8.1]). On the positive side, we also showed that the theory of von Neumann regular commutative f-rings with a pseudo-inverse and a distinguished automorphism was a Robinson theory and so we obtained the existence of a universal domain for its subclass of existentially closed models.

Let us motivate our study of difference totally ordered fields by the following two well-known examples.

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By a classical result due to A.I. Malcev, H. Hahn and B.H. Neumann, any totally ordered field K embeds in a power series field of the form $k((G))$, where k is a totally ordered archimedean field (and so a subfield of $\mathbb{R}$) and G is a totally ordered abelian group whose underlying set is the set of archimedean classes of elements of K. An automorphism of K induces an automorphism of G. More generally, we consider power series fields of the form $F((G))$, where G is any totally ordered abelian group and F any totally ordered field. The elements of $F((G))$ are formal sums of the form $s := \sum_{g\in G} c_g \cdot x^g$, where $c_g \in F$ and $\mathrm{supp}(s) := \{g \in G : c_g \neq 0\}$ is a well-ordered subset of G. There is a natural valuation v on $F((G))$ which sends $s \neq 0$ to $g_s := \min(\mathrm{supp}(s)) \in G$, the ordering on $F((G))$ is defined by $s > 0$ if $c_{g_s} > 0$ (see [13, Chapter 8, Section 5]). Assume now that (K,τ) is an ordered difference field with automorphism τ and that ϱ is an automorphism of the totally ordered abelian group G, then we can define the following automorphism σ of $K((G))$: $\sigma(s) := \sum_{g\in G} \tau(c_g)\cdot x^{\varrho(g)}$. Let $d \in \mathbb{N} - \{0\}$, let ϱ_d be the automorphism of G sending g to $d\cdot g$. Then define σ_d on $K((G))$ which sends $\sum_{g\in G} c_g \cdot x^g$ to $\sum_{g\in G} c_g \cdot x^{d\cdot g}$. Consider the ultraproduct $\prod_U (K((G)),\sigma_d)$, where U is a non principal ultrafilter on ω. Let $\sigma := [\sigma_d]_U$ and $\varrho := [\varrho_d]_U$, then ϱ is an automorphism on $G^{\omega}{}_U$ which is ω-increasing (see Notation 2.1).

Our second example is the field $\mathbb{R}((t))^{\mathrm{LE}}$ of real exponential-logarithmic series (see [11]) constructed as follows: One starts with the field $R_0 := \mathbb{R}((x^{-1}))$ of Laurent series ordered by $x > \mathbb{R}$ of elements $f(x)$ of the form $r_n \cdot x^n + \cdots + r_1 \cdot x + r_0 + r_{-1}\cdot x^{-1} + r_{-2}\cdot x^{-2} + \cdots$, consisting of an infinite part $f_1 := r_n \cdot x^n + \cdots + r_1 \cdot x$, a standard part r_0 and an infinitesimal part $f_{-1} := r_{-1}\cdot x^{-1} + r_{-2}\cdot x^{-2} + \cdots$. The field $K = R_0$ can be decomposed as a direct sum of an additive subgroup $K_\infty = K - \mathcal{O}_K$ consisting of its elements of valuation > 1, and a multiplicative (convex) subgroup consisting of its elements of valuation ≤ 1. One defines the exponentiation operation E on finite elements $r_0 + f_{-1}$ as follows:

$$E(r_0 + f_{-1}) := e^{r_0}\cdot \textstyle\sum_{m=0}^{\infty} \dfrac{1}{m!}\cdot f_{-1}^m,$$

where e is the usual exponentiation operation on $\mathbb{R}$. Then, taking a strictly increasing homomorphism E_1 from the additive group of K into the multiplicative subgroup of its strictly positive elements, one defines $E(f(x)) := E_1(f_1)\cdot E(r_0 + f_{-1})$. Then one considers the field $R_1 := R_0((E_1(K_\infty)))$ and iterate this construction in ω steps, obtaining the field $\mathbb{R}((x^{-1}))^{\mathrm{E}}$ and then one closes off by the logarithmic function, obtaining $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ as a countable union of exponential fields. This last construction uses the substitution map $\Phi : R^E \longrightarrow R^{\mathrm{E}}$ defined (informally) by $\Phi(f(x)) := f(E(x))$, and so is the identity on $\mathbb{R}$. This is used to define a logarithm operation for the elements in its image (see [11, Section 2.6]). Then one can verify that Φ is an automorphism of $\mathbb{R}((t))^{\mathrm{LE}}$ and that it is ω-increasing [11].

1.1 Preliminaries

Let $\mathcal{K} := (K, +, -, \cdot, <, \sigma, 0, 1)$ be a totally ordered difference field and let K^+ denote the *strictly* positive elements of K. Let $\mathcal{L} := \{+, -, \cdot, 0, 1\}$ (respectively $\mathcal{L}_< := \{+, -, \cdot, <, 0, 1\}$) be the language of rings (respectively ordered rings) and $\mathcal{L}_\sigma$ (respectively $\mathcal{L}_{<,\sigma}$) be its expansion by two unary function symbols σ, σ^{-1}. Let L be a difference field and let A be a subset of L, we will denote by $\langle A\rangle_\sigma$ the $\mathcal{L}_\sigma$-substructure of L generated by A; we will denote by $\mathrm{acl}_\sigma(A)$ the model-theoretic algebraic closure of A in L.

In the following, we will also consider the reduct of $\mathcal{K}$ to its difference field structure. To ease the notation, we will distinguish the two cases by denoting $\mathcal{K}$ by $(K, <, \sigma)$ and its reduct as a difference field by (K, σ). Let K^{ac} be the algebraic closure of K and K^{rc} its real closure.

A field is *formally real* if -1 is not a sum of squares; it can be endowed with a total order if and only if it is formally real.

Recall that RCF denotes the theory of real-closed fields $(F, +, \cdot, <, 0, 1)$; it can be axiomatized by the scheme of axioms expressing that F is a totally ordered commutative field where every monic polynomial with coefficients in F of odd degree has a root and every positive element of F is a square. A. Tarski showed that RCF is a complete theory and that it admits quantifier elimination in the language $\mathcal{L}_<$.

Let us quickly review some basic facts on definable subsets. One has a cell decomposition result for models of RCF. Namely any non-empty definable subset A of F^n is a finite union of disjoint $(i_1, \ldots, i_n)$-cells, where $i_1, \ldots, i_n$ is a sequence of zeroes and ones (see [10, 2.11]). Moreover, if A is defined over a finitely generated subfield F_0, then the cells occurring in the above decomposition are also F_0-definable. The dimension of a

$(i_1, \dots, i_n)$-cell is by definition $i_1 + \dots + i_n$ and the dimension of A is the maximum of the dimensions of the cells that it contains (see [10, Chapter 4, 1.1]).

Equivalently, one can define the dimension of A over F_0 as the maximum of the dimension of the tuples $\bar{a} \in A$ over F_0, where $\dim(\bar{a}/F_0)$ is the cardinality of any maximal algebraically independent subtuple of $\bar{a}$ ([21, Lemma 1.4 and Note p. 244]). We have the following equivalence: $\dim(A) = \ell \leq n$ if and only if some projection of A onto F^ℓ has interior in F^ℓ (see [21, Lemma 1.4]). Therefore, we can tell in a first-order way what the dimension of A is. The tuple $\bar{a}$ is called a *generic point of* A if its dimension is equal to the dimension of A ([21, Note (ii)]).

Let $\bar{f} := (f_1, \dots, f_n)$ be a generic point of A. Let τ be a permutation of the indices $1, \dots, n$. Then there exists an invertible matrix M over F_0 such that the tuple $\bar{t} = (t_1, \dots, t_r) = M \cdot (f_{\tau(1)}, \dots, f_{\tau(r)})$ is such that $t_1, \dots, t_r$ is a transcendence basis for $F_0(\bar{f})$ and the t_i's (as well as the f_i's) are integral over $F_0[t_1, \dots, t_r]$. So, there exist $n - r$ monic polynomials $q_i \in F_0[t_1, \dots, t_r][X]$ such that $q_i(f_i) = 0$ for $n - r + 1 \leq i \leq n$.

With $\bar{f}$ we will associate the ideal $\mathcal{I}(\bar{f}) = \langle q_1, \dots, q_{n-m} \rangle$ of $F_0[x_1, \dots, x_n]$. Note that

$$(*) \qquad \frac{\partial q_i}{\partial x_{m+i}}(\bar{f}) \neq 0, \quad \text{for } 1 \leq i \leq n - m.$$

We will call any such tuple $\bar{f}$ satisfying these conditions $(*)$ *non-singular*. Michaux and Rivière showed that in any neighbourhood of a non-singular point one can find a generic point of A (see [20, Proposition 1.6]).

Now, we will assume in addition that F is a difference field, i.e., a field with a distinguished automorphism σ.

We will denote by $\mathrm{Fix}_F(\sigma) := \{x \in F : \sigma(x) = x\}$ the subfield of F consisting of the elements fixed by σ. Note that since F is a model of RCF, then so is $\mathrm{Fix}(\sigma)$ (see Corollary 2.2 below).

If we forget the order, it is now well-known that the class of existentially closed models of the theory of difference fields is elementary and has a recursive axiomatization called ACFA (see for instance [5, 1.1]). Let ACFA_0 denotes the theory ACFA plus the scheme of axioms expressing the field has characteristic 0. Both theories ACFA and ACFA_0 are decidable (see for instance [5, 1.4, 1.6]).

Notation 1.1 Let K be a difference field and let $X = (X_1, \dots, X_m)$ be a finite tuple of indeterminates and let X^σ be the tuple $(X_1^\sigma, \dots, X_m^\sigma)$. Let $K[X]_\sigma$ be the σ-polynomial ring, i.e., the polynomial ring in infinitely many indeterminates $X, X^\sigma, \dots, X^{\sigma^n}, \dots$, $n \in \mathbb{N}$. Let $P \in K[X]_\sigma$ and suppose that, for some $1 \leq j \leq m$, $X_j^{\sigma^n}$ occurs non trivially in P, then the order of P in X_j is greater than or equal to n ([8, p. 65]); it is equal to n if n is the highest such natural number. The *effective order* of X_j in P is $n_1 - n_2$, where n_1 is the order of X_j in P and n_2 is the lowest natural number such that $X_j^{\sigma^{n_2}}$ occurs non trivially in P.

As usual we can write "$P(X) \in K[X]_\sigma$ of order n, as $P^*(X_1, \dots, X_m, X_1^\sigma, \dots, X_m^{\sigma^n})$ for some element $P^*(Y_1, \dots, Y_{m\cdot(n+1)}) \in K[Y_1, \dots, Y_{m\cdot(n+1)}]$, and we define $\dfrac{\partial}{\partial X_i^{\sigma^j}} P := (\dfrac{\partial}{\partial Y_{i\cdot(j+1)}} P^*)(X_1, \dots, X_m^{\sigma^n})$.

Let $(\tilde{K}, \tilde{\sigma}) \models \mathrm{ACFA}$ containing (K, σ) and let (F, σ) be a difference subfield containing (K, σ). We recall below certain facts about difference algebras which can be found either in [8] or [5]. We will use the term σ-ideal for an ideal which is closed under σ; it is *reflexive* if whenever $\sigma(a) \in I$, then $a \in I$; it is *perfect* if whenever a product of images of a by powers of σ belongs to I, then a belongs to I.

Let A be a subset of F^n and let $\Phi_F(A) \subset F[X]_\sigma$ (respectively $I_F(A) \subset F[X]$) be the set of difference polynomials (respectively ordinary polynomials) in n variables annulled by all elements of A. The ideal $I_F(A)$ is prime and $\Phi_F(A)$ is a σ-ideal which is reflexive and perfect. The perfect σ-ideals of $F[X]_\sigma$ satisfy the ascending chain condition ([8, Chapter 3]). Therefore, a perfect σ-ideal I is the perfect closure of a finite set S of σ-polynomials (see [8, Chapter 3]); we will use the notation $I = \{S\}$. If we want to stress in which difference polynomial ring we are taking the closure, we add a subscript as follows: $I_{\tilde{K}} = \{S\}_{\tilde{K}[X]_\sigma}$ is the perfect closure of S in $\tilde{K}[X]_\sigma$.

As usual, we will say that a subset $V \subset F^n$ is a *difference variety* (respectively an *irreducible variety*) if it is the set of zeros of some perfect reflexive σ-ideal of $F[X]_\sigma$ (respectively some prime ideal of $F[X]$). Recall that a variety V is absolutely irreducible if $I_{F^{\mathrm{ac}}}(V)$ is a prime ideal of $F^{\mathrm{ac}}[X]$. We will denote by $V^\sigma(F)$ the set of zeroes of $I_F^\sigma(V)$, where $I_F^\sigma(V)$ denotes the ideal of $F[X]$ obtained by applying σ to the coefficients of the elements of $I_F(V)$.

By the above there exists a finite set S_V such that $\Phi(V) = \{S_V\}$. We will denote the perfect closure of S_V in $\tilde{K}[X]_\sigma$ by $\Phi_{V,\tilde{K}} = \{S_V\}_{\tilde{K}[X]_\sigma}$ and the corresponding set of zeros in $\tilde{K}^n$ by $V(\tilde{K})$.

The difference variety V is *irreducible* (over F) if $\Phi_{V,F} = \Phi_F(V)$ is prime and it is *absolutely irreducible* (over $\tilde{K}$) if $\Phi_{\tilde{K}}(V(\tilde{K}))$ is a σ-ideal of $\tilde{K}[X]_\sigma$ which is prime.

Finally, we will say that a variety $V \subset F^n$ is *defined over* K if I_V can be generated by a subset of $K[X]_\sigma$.

A tuple $\bar{c} \in \tilde{K}$ with $\bar{c} \in V(\tilde{K})$ is *σ-generic* (with respect to $\tilde{K}$) if $\Phi(V(\tilde{K}))$ is equal to $\Phi(\{\bar{c}\})$ (we will also say that $\bar{c}$ is a σ-$\tilde{K}$-generic point).

A (difference) variety V defined over K has a σ-$\tilde{K}$-generic point $\bar{c}$ in some intermediate field $K \subset F \subset \tilde{K}$ if $\bar{c} \subset F$ and $\Phi(V(\tilde{K}))$ is equal to $\Phi(\{\bar{c}\})$.

Let V be a difference variety defined over K and let S be a finite subset of $K[X]_\sigma$ whose perfect closure is equal to $\Phi(V)$. For sake of simplicity assume that X is a single variable. Let m be the maximal effective order of the elements of S. Let π_1 be the projection from K^{m+1} onto K, sending a tuple to its first component. Let S^* be the set of ordinary polynomials in $m+1$ variables such that if $p(X) \in S$, then $p(X) = p^*(X, X^\sigma, \ldots, X^{\sigma^m}) \in S^*$. Let $V^* := \{(x, x^\sigma, \ldots, x^{\sigma^m}) : x \in V\}$. We embed $\tilde{V}$ in K^{2m} by adding to the equations $X^\sigma = X_1, \ldots, X_{m-1}^\sigma = X_m$ and so setting $Y := (X, X_1, \ldots, X_{m-1})$ and re-writing $p^*(X, X_1, \ldots, X_m)$ as $p^{**}(Y, Y^\sigma)$ we get that $\pi_1(\tilde{V}) = V$, where

$$\tilde{V} := \{(Y, Y^\sigma) : p^{**}(Y, Y^\sigma) = 0\,;\, X^\sigma = X_1, \ldots, X_{m-1}^\sigma = X_m\,;\, p \in S\}.$$

The axiom scheme ACFA tells us that whenever there exists an absolutely irreducible (algebraic) variety $U \subset K^m$ into which $\tilde{V}$ projects generically, then V has a point in $\tilde{K}$.

1.2 Virtual points

In this section $\mathcal{K} := (K, +, -, \cdot, <, \sigma, 0, 1)$ will always denote a totally ordered real-closed difference field of cardinality κ.

There are two extensions of σ to $K^{\mathrm{ac}} = K(i)$ with $i^2 = -1$, one sending the element i to itself and the other to $-i$. We will still denote by σ the first extension and we will denote the second one by σ_-. Then we embed $(K^{\mathrm{ac}}, \sigma)$ (respectively $(K^{\mathrm{ac}}, \sigma_-)$) into a model $(\tilde{K}, \sigma)$ (respectively $(\tilde{K}, \sigma_-)$) of ACFA. We will distinguish those two cases by saying in the first case that $(\tilde{K}, \sigma)$ satisfies ACFA_+ and in the second one that $(\tilde{K}, \sigma_-)$ satifies ACFA_-.

Recall that the extension L of K is called *regular* if K is relatively algebraically closed in L and L is separable over K.

Remark 1.2 Let $(\tilde{K}_1, \sigma_1)$ and $(\tilde{K}_2, \sigma_2)$ be two κ^+-saturated models of ACFA_0 in which $(K, <, \sigma)$ embeds. Then $\tilde{K}_1 \equiv_K \tilde{K}_2$ if and only if $\sigma_1(i) = \sigma_2(i)$.

Proof. Indeed, the algebraic closure of K in $\tilde{K}_1$ is equal to $K(i)$. Either $\sigma_1(i) = \sigma_2(i)$ in which case $\tilde{K}_1 \equiv_{K(i)} \tilde{K}_2$ (see [5, Theorem 1.3]), or $\sigma_1(i) = -\sigma_2(i)$ in which case $\tilde{K}_1 \not\equiv_K \tilde{K}_2$ (indeed, the sentence $\exists x\, (x^2 = -1\ \&\ \sigma(x) = -x)$ distinguishes them). □

Lemma 1.3 *Assume that $(\tilde{K}, \sigma)$ is a κ^+-saturated model of* ACFA_+ *in which $(K^{\mathrm{ac}}, \sigma)$ embeds as a difference field. Then there is a unique, up to K-isomorphism, maximal difference totally ordered real-closed subfield $(L, <, \sigma)$ of cardinality κ in $(\tilde{K}, \sigma)$ extending $(K, <, \sigma)$.*

Proof. Note that the condition that $(L, <)$ is an ordered field extension of $(K, <)$ is equivalent to $\neg(-1 = \sum_j a_j \cdot \alpha_j^2)$ with $a_j \in K^+$ and $\alpha_j \in L$.

By Zorn's lemma, there is a maximal difference totally ordered real-closed subfield of cardinality κ in $(\tilde{K}, \tilde{\sigma})$ extending $(K, <, \sigma)$.

Now, given two ordered real-closed difference field extensions of cardinality κ, $(K_1, <_1, \sigma_1)$ and $(K_2, <_2, \sigma_2)$ of $(K, <, \sigma)$ in $\tilde{K}$, we will show that we can embed them in a third one by a K-isomorphism. Since K is relatively algebraically closed in K_1, K_1 is a regular extension of K. So, $K_1 \otimes_K K_2$ is an integral domain and its field of fractions L_0 is a regular extension of K_2. We endow $K_1 \otimes_K K_2$ with an order extending the order of K_1 and K_2. Indeed, the subset $\{\sum_j (k_{1j} \otimes k_{2j}) \cdot y_j^2 : k_{1j} \in K_1^+, k_{2j} \in K_2^+, y_j \in K_1 \otimes K_2\}$ is a preorder extending the orders on respectively K_1 and K_2 (see [22, (0.5)]). So, this preorder extends to an order and this order to the field of fractions L_0.

Then we show that L_0 is a difference field extension of K. Given a typical element of $K_1 \otimes_K K_2$, we define $\sigma_3(k_1 \otimes k_2) = \sigma_1(k_1) \otimes \sigma_2(k_2)$. Finally, one extends σ_3 on L_0 and then to its real closure L_0^{rc}. Since $\tilde{K} \models \mathrm{ACFA}_+$, we further extend σ_3 on the algebraic closure of $L_0^{\mathrm{ac}} = L_0^{\mathrm{rc}}(i)$ by setting $\sigma_3(i) = i$. We then embed L_0^{ac} in a κ^+-saturated model $\tilde{L}$ of ACFA_+. Since $\tilde{L} \equiv_{K^{\mathrm{ac}}} \tilde{K}$ and $\tilde{K}$ is κ^+-saturated, we may embed L_0^{rc} inside $\tilde{K}$, using a K-isomorphism f. Finally we embed K_1 and K_2 inside $f(L_0)$ using a K-isomorphism. □

Corollary 1.4 *The subfield* $\mathrm{Fix}_L(\sigma)$ *is a proper real-closed subfield of* $\mathrm{Fix}_{\tilde{K}}(\sigma)$.

Proof. First, notice that $\mathrm{Fix}_L(\sigma)$ is a proper subfield of $\mathrm{Fix}_{\tilde{K}}(\sigma)$. By [5, Proposition 1.2], $\mathrm{Fix}_{\tilde{K}}(\sigma)$ is a pseudo-finite field, in particular it is a PAC field and so every element is the sum of two squares, so $\mathrm{Fix}_{\tilde{K}}(\sigma)$ is never formally real ([12, Theorem 10.12]).

Now, let us show that $\mathrm{Fix}_L(\sigma)$ is real-closed. Let $P[X] \in \mathrm{Fix}_L(\sigma)[X]$ and suppose it has a root $b \in L$. Then $\sigma(b)$ is also a root of $P[X]$ and so whenever $\sigma(b) \neq b$, the polynomial $P[X]$ would have infinitely many roots. Therefore $\sigma(b) = b$. □

Remark 1.5 Note that the above corollary implies that $L(i)$, which is a difference algebraically closed subfield of $\tilde{K}$, is not a model of ACFA. (Its fixed subfield is algebraically closed if $\tilde{K} \models \mathrm{ACFA}_+$ and real-closed if $\tilde{K} \models \mathrm{ACFA}_-$).

Proof. Let $a + b \cdot i \in \mathrm{Fix}_{L(i)}(\tilde{\sigma})$. If $\sigma(i) = i$, then $a,\ b \in \mathrm{Fix}_L$. If $\sigma(i) = -i$, then $a \in \mathrm{Fix}(\sigma)$ and $\sigma(b) = -b$, which implies that $b = 0$ since L is an ordered field. So, $\mathrm{Fix}_{L(i)}(\sigma) = \mathrm{Fix}_L(\sigma)$. □

Let $(K, <, \sigma) \subset (L_1, <, \sigma)$ and $(K, <, \sigma) \subset (L_2, <, \sigma)$. If any existential formula with parameters in K which holds in L_1, holds in L_2 and conversely, we will use the notation: $(L_1, <, \sigma) \equiv_{\exists, K} (L_2, <, \sigma)$.

Proposition 1.6 *Let* $\tilde{K}_1,\ \tilde{K}_2$ *be two* κ^+*-saturated models of* ACFA_+ *containing* $\mathcal{K}$*. Let* L_1 (*respectively* L_2) *be a maximal difference totally ordered real-closed subfield of* K_1 (*respectively* K_2) *of cardinality* κ *containing* K *as an ordered difference subfield. Then* $(L_1, <, \sigma) \equiv_{\exists, K} (L_2, <, \sigma)$.

Proof. Let $\varphi(\bar{y}, \bar{x})$ be a quantifier-free $\mathcal{L}_{<,\sigma}$-formula and $\psi(\bar{x}) := \exists \bar{y}\, \varphi(\bar{y}, \bar{x})$ be an existential formula. Let $\bar{a}$ be parameters in K and suppose that $\psi(\bar{a})$ holds in L_1. In difference real-closed fields, the formula $\varphi(\bar{x}, \bar{y})$ is equivalent to an existential $\mathcal{L}_\sigma$-formula, say $\theta(\bar{x}, \bar{y})$, replacing atomic formulas of the form $t(\bar{x}, \bar{y}) \geq 0$ by $\exists u\, t(\bar{x}, \bar{y}) = u^2$.

Let $\bar{c} \in L_1$ such that $L_1 \models \theta(\bar{c}, \bar{a})$. Let $\mathrm{tp}(\bar{c}/K)$ be the $\mathcal{L}_\sigma$-type of $\bar{c}$ over K in L_1. This type is finitely satisfiable in $\tilde{K}_2$ since $\tilde{K}_1 \equiv_{K(i)} \tilde{K}_2$. Since $\tilde{K}_2$ is κ-saturated, there is a tuple $\bar{d} \in \tilde{K}_2$ realizing this type. So, $K\langle \bar{d} \rangle_\sigma$ is formally real and $\tilde{K}_2 \models \theta(\bar{d}, \bar{a})$. So, by the proof of Lemma 1.1, there is a K-isomorphism f sending $K\langle \bar{d} \rangle_\sigma$ in L_2 fixing K. Therefore, $L_2 \models \theta(f(\bar{d}), \bar{a})$, or equivalently, $L_2 \models \varphi(f(\bar{d}), \bar{a})$ and so $L_2 \models \psi(\bar{a})$. □

Definition 1.7 Let $(L, <, \sigma)$ be a maximal ordered real-closed field extension of $(K, <, \sigma)$ of cardinality κ with $(L, \sigma) \subset (\tilde{K}, \sigma)$. Let V be a difference variety defined over K and let S be a finite subset of $K[X]_\sigma$ whose perfect closure is equal to $\Phi_K(V)$. We will say that V has a *virtual point* if $V(L) \neq \emptyset$, equivalently, if $K\langle \bar{c} \rangle_\sigma$ is formally real, for some generic point $\bar{c}$ in $V(\tilde{K})$.

Namely, V has a virtual difference point if there is a generic point $\bar{c}$ in $V(\tilde{K})$ such that the difference subfield generated by K and this tuple can be endowed with an ordering extending the ordering of K. We will abbreviate the formula $\bigwedge_{s \in S} s(\bar{x}) = 0$ by $S(\bar{x}) = 0$. We can express that property by the following infinite conjunction:

$$\tilde{K} \models \exists \bar{c}\, [S(\bar{c}) = 0 \ \&\ \bigwedge_{a_j \in K^+ \text{ and } p_j, q \in K[\bar{X}]_{\tilde{\sigma}}} (q(\bar{c}) \neq 0 \to -1 \neq \textstyle\sum_j a_j \cdot \dfrac{p_j(\bar{c})^2}{q(\bar{c})^2})].$$

Since K is real-closed, this is equivalent to:

$$\tilde{K} \models \exists \bar{c}\, [S(\bar{c}) = 0 \ \&\ \bigwedge_{p_j, q \in K[\bar{X}]_\sigma} (q(\bar{c}) \neq 0 \to q(\bar{c})^2 + \textstyle\sum_j p_j(\bar{c})^2 \neq 0)].$$

Since $\tilde{K}$ is κ^+-saturated, this is equivalent to require that any finite system in $\bar{x}$ of the form

$$S(\bar{x}) = 0 \ \& \ \bigwedge_{i\in I}(q_i(\bar{x}) \neq 0 \rightarrow q_i(\bar{x})^2 + \sum_{j\in J_i} p_{ij}(\bar{x})^2 \neq 0),$$

has a solution in $\tilde{K}$, where I, J_i are finite and $p_{ij}, q_i \in K[\bar{X}]_\sigma$,

Remark 1.8 Let $(\tilde{K}_1, \sigma_1)$ and $(\tilde{K}_2, \sigma_2)$ be two κ^+-saturated models of ACFA$_+$ containing $(K, <, \sigma)$. Let $\bar{c}_1$ be a virtual point of V in $\tilde{K}_1$. By Remark 1.2, any finite subset of formulas satisfied in $\tilde{K}_1$, is also satisfied in $\tilde{K}_2$. So the above type is finitely satisfiable in $\tilde{K}_2$ and since $\tilde{K}_2$ is κ^+-saturated it is satisfied in $\tilde{K}_2$. So, the variety V has a virtual point $\bar{c}_2$ in $\tilde{K}_2$.

Remark 1.9 Assume the difference variety V defined over K has a virtual point $\bar{c}$. Then, using model-completeness of RCF, $K \vDash \exists(\bar{a}, \bar{a}_1, \ldots, \bar{a}_n) \in V^* \cap (O \times O^\sigma \times \cdots \times O^{\sigma^n})$ for any open set O defined over K containing $\bar{c}$, where $(\bar{a}, \bar{a}_1, \ldots, \bar{a}_n)$ is a non singular point of V^*.

Definition 1.10 Let $\mathcal{C}_{\rm pra}$ be the class of totally-ordered commutative difference fields $(K, <, \sigma)$ satisfying the following properties:

1. $K \vDash$ RCF.

2. σ is an automorphism of K.

3. For every absolutely irreducible (algebraic) variety $U \subset (\Omega^{\rm ac})^n$ defined over K, where Ω is a model of RCF containing K and κ^+-big, and for every absolutely irreducible algebraic variety V defined over K with $V \subseteq U \times U^\sigma$ projecting generically onto U and onto U^σ, the following holds: Assume that for any finite index set I and $p_i[X, Y] \in K[X, Y]$, $i \in I$, $X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_n)$, we have that

$$(*) \qquad \sum_{i\in I} p_i^2 \in I_K(V) \rightarrow \bigwedge_{i\in I} p_i \in I_K(V).$$

Then there exists an element $\bar{r}$ in K such that $(\bar{r}, \bar{r}^\sigma) \in V$.

Note that the condition on V is expressed by an infinite conjunction, since each condition $(*)$ is an elementary statement.

Lemma 1.11 *Any real-closed difference field embeds in an element of* $\mathcal{C}_{\rm pra}$*. Moreover, given any two elements* $\mathcal{K}_1$*,* $\mathcal{K}_2$ *of* $\mathcal{C}_{\rm pra}$ *with* $\mathcal{K}_1 \subset \mathcal{K}_2$*, then* $\mathcal{K}_1 \subset_{\rm ec} \mathcal{K}_2$*.*

Proof. First, let $\mathcal{K} := (K, <, \sigma)$ be a real-closed difference field, we will show that it embeds in an element of $\mathcal{C}_{\rm pra}$. Since we can embed K in an existentially closed real-closed difference field containing K, w.l.o.g. we may assume that K is itself existentially closed and we will show that then K belongs to $\mathcal{C}_{\rm pra}$.

So, let V be an absolutely irreducible variety defined over K satisfying the condition stated in scheme 3 and let us show that it has a point in K. This condition we put on V implies that the fraction field of $K[X, Y]/I_K(V)$ is formally real. So there exists a generic point $(\bar{a}, \bar{b})$ of V in Ω. Since V projects generically on U and on U^σ, $\bar{a}$ (respectively $\bar{b}$) is a generic point of U (respectively U^σ). Since $\mathrm{Frac}(K[X, Y]/I_K(V))$ is formally real, we have also that $\mathrm{Frac}(K[X]/I_K(U))$ is formally real, so $K(\bar{a})$ can be endowed with an ordering extending the ordering $<$ on K (see [22, (0.4)]). Since σ is an automorphism of K, we similarly get that if $\bar{b}$ is a generic point in $U^\sigma(\Omega)$, then $K(\bar{b})$ can be endowed with an ordering extending the ordering $<$ on K. Moreover, we can choose the ordering in such a way that the $\mathcal{L}_<$-type of $\bar{a}$ over K is equal to the $\mathcal{L}_<$-type of $\bar{b}$ in K. We have a partial isomorphism of Ω extending σ and sending $\bar{a}$ to $\bar{b}$ and preserving the order on $K(\bar{a})$, respectively $K(\bar{b})$. Since Ω is κ^+-big and so κ^+-strongly homogeneous ([14, p. 487]) and since $(\Omega, K(\bar{a})) \equiv (\Omega, K^\sigma(\bar{b}))$, there is an automorphism of Ω extending σ and taking $\bar{a}$ to $\bar{b}$. Let $K\langle\bar{a}\rangle_\sigma$ be the difference ordered subfield of Ω generated by K and $\bar{a}$, where V has a point of the form $(\bar{a}, \bar{a}^\sigma)$. Finally, we extend σ to the real-closure of $K\langle\bar{a}\rangle_\sigma$, and since K is existentially closed, V has also a point in K of the form $(\bar{c}, \bar{c}^\sigma)$.

Second, let $K_1 \subset K_2 \in \mathcal{C}_{\rm pra}$ and let us show that $K_1 \subset_{\rm ec} K_2$. Let $\varphi(x_1, \ldots, x_n)$ be a quantifier-free formula with parameters in K_1 satisfied by a tuple $\bar{a} \subset K_2$. Then there exists $k \in \mathbb{N}$ such that $\varphi(\bar{x})$ is a finite disjunction over I of formulas of the form

$$\varphi_i(\bar{x}) := f(\bar{x}, \sigma(\bar{x}), \ldots, \sigma^k(\bar{x})) = 0 \ \& \ f_1(\bar{x}, \sigma(\bar{x}), \ldots, \sigma^k(\bar{x})) > 0 \ \& \cdots \& \ f_s(\bar{x}, \sigma(\bar{x}), \ldots, \sigma^k(\bar{x})) > 0,$$

with $f(X), f_j(X) \in K_1[X]_\sigma$, $i \in I$.

Note that if $k = 0$, we simply apply the model-completeness of the theory RCF. In RCF, each formula $\varphi_i(\bar{x})$ is equivalent to a finite disjunction of existential formulas of the form

$$\begin{aligned}\exists\bar{y}\,\exists\bar{z}\,\psi_j(\bar{x},\bar{y},\bar{z}) :=\ &\exists\bar{y}\,\exists\bar{z}\; f(\bar{x},\sigma(\bar{x}),\dots,\sigma^k(\bar{x})) = 0\\ &\&\ f_1(\bar{x},\sigma(\bar{x}),\dots,\sigma^k(\bar{x})) = y_1^2\ \&\ \cdots\ \&\ f_s(\bar{x},\sigma(\bar{x}),\dots,\sigma^k(\bar{x})) = y_s^2\\ &\&\ \textstyle\bigwedge_{j=1}^{s}\ f_j(\bar{x},\sigma(\bar{x}),\dots,\sigma^k(\bar{x}))\cdot z_j - 1 = 0.\end{aligned}$$

Assume that $\varphi_j(\bar{a})$ holds in K_2. So, there exists $\bar{c}$, $\bar{d}$ such that $\psi(\bar{a},\bar{c},\bar{d})$ holds in K_2. Let us put this tuple in the form $(\bar{b},\bar{b}^\sigma)$. Let U be an absolutely irreducible (algebraic) variety whose $\bar{b}$ is a generic point and let V be an absolutely irreducible variety whose $(\bar{b},\bar{b}^\sigma)$ is a generic point. Since $\langle K_1,\bar{b}\rangle_\sigma \subset K_2 \models \mathrm{RCF}$, we have that the fraction field of $K_1[X,Y]/I_{K_1}(V)$ is formally real. The difference variety V projects generically on $U \times U^\sigma$. So, since K_1 satisfies scheme 3, there is an element $(\bar{r},\bar{r}^\sigma) \in V(K_1)$. So, there exists $\bar{r}_0 \subset \bar{r}$ such that $K_1 \models \varphi(\bar{r}_0)$. □

1.3 Properly preordered fields

(We will follow the presentation of [23]). Let (K,S) be a field with a preordering S, i.e., with a subset S satisfying $S+S \subset S,\ S\cdot S \subset S,\ K^2 \subset S$. Note that such a subset S is closed under $^{-1}$ since $s^{-1} = s\cdot s^{-2}$. Let $K^\times$ (respectively $S^\times$) denote the multiplicative group of non-zero elements of K (respectively S).

The preordering S is proper if $-1 \notin S$, in which case K is formally real. A preordered field L extending K is said to be *totally real* if all orderings of K extend to L.

Let $X_K(S)$ be the set of orderings P extending S, i.e., the proper preorderings P containing S such that $K = P \cup (-P)$. The set $X_K(S)$ can be endowed with an Hausdorff topology generated by the sets $H(a) := \{P : a \in P\}$; we will denote the corresponding topological space by $\mathcal{X}_K(S)$.

A preordered field (K,S) is called SAP if for any a,b, there is an element c such that $H(a)\cap H(b) = H(c)$.

Let $\mathcal{L}_S := \mathcal{L}\cup\{S\}$ be the language of preordered rings, where S is a unary predicate and $\mathcal{L}_{S,\sigma}$ its expansion to the language of difference rings.

Definition 1.12 (See [23]). Let T_n, $n \in \mathbb{N}\cup\{+\infty\}$, be the following theories:

(1) $S = K^2$;

(2) K does not admit a totally real algebraic extension;

(3) K is *pseudo-real-closed*, i.e., every absolutely irreducible variety V defined over K which has a simple point in the real closure of $(K,P)^{\mathrm{rc}}$ with respect to any ordering P, has a K-rational point;

$(4)_n$ $|K^\times/S^\times| = 2^n$, where $n \in \mathbb{N}$, or $(4)_\infty$ $\mathcal{X}_K(S)$ is non-empty and has no isolated points.

For $n \in \mathbb{N}$, a model of T_n has exactly n orderings and is the model companion of the theory of preordered fields (K,S) such that $|K^\times/S^\times| = 2^n$ and such that for all $a,b \in K^\times$ there exists $c \in K^\times$ such that $H(a)\cap H(b) = H(c)$ ([23]).

This extends the result of van den Dries that T_n is the model-companion of the theory of rings with exactly n orderings (in the language of rings expanded with n unary predicates) ([9]). Whereas T_∞ is the model companion of the theory of properly preordered fields (namely preordered fields (K,S) where S is a proper preorder) ([23, Theorem 2]).

Now, we will consider a preordered difference field (K,S,σ). Let K^{ac} be the algebraic closure of K and denote by σ_1 an extension of σ to K^{ac}. We fix a $|K|^+$-saturated model $(\tilde{K},\tilde{\sigma})$ of ACFA into which we fix an embedding of K^{ac}.

Lemma 1.13 *Let (K,S,σ) be a preordered difference field; assume that whenever $[K^\times : (S^2)^\times] = 2^n$, K has exactly n orderings and that K has no totally real algebraic extension. Then there is a unique, up to K-isomorphism, maximal difference preordered subfield $(L,\tilde{S},\sigma) \models T_n$ of cardinality κ in $(\tilde{K},\tilde{\sigma})$ extending (K,S,σ), $n \in \mathbb{N}\cup\{+\infty\}$.*

Proof. First, we note that if (K^*,S^*,σ) is an existentially closed extension of (K,S,σ), which we may assume to be of cardinality κ, with $|K^\times/S^\times| = 2^n$ if this index is finite and it is a model of T_n, or of T_∞ if this index is infinite. This verification is analogous to the proof of [23, Lemma 2]. We check that (K^*,S^*,σ) satisfies properties (1) – (3) and either $(4)_n$ or $(4)_\infty$ of Definition 1.12.

Indeed, one considers either (i) algebraic extensions of K^* and so the automorphism σ extends in a natural way, or (ii) extensions where one adds points to absolutely irreducible varieties defined over K^*, or (iii) extensions where one adds a transcendental element over K^*. First all these extensions can take place in $\tilde{K}$. In the second case, we have to check that the difference field extension is still a formally real field, and in the third case – in order to show that $\mathcal{X}_{K^*}$ has no isolated point – that we can choose a transcendental element over K^* in $\mathrm{Fix}(\sigma)$.

Let us examine more closely the second case. Let $f(X_1, \dots, X_m, Y) \in K^*[\bar{X}, Y]$ be absolutely irreducible and monic in Y such that for each $P \in \mathcal{X}_{K^*}$ there exists $(\bar{x}, y) \in \overline{(K^*, P)}$ with $f(\bar{x}, y) = 0$ and $\dfrac{\partial f}{\partial Y}(\bar{x}, y) \neq 0$. Then each P extends to the fraction field of $K^*[\bar{X}, Y]/(f)$. Equivalently, the fraction field of $K^*[\bar{X}, Y]/(f)$ is formally real. So, if we choose a generic point $\bar{b}$ of $f = 0$ in $\tilde{K}$, the field $K^*(\bar{b})$ is formally real and is a regular extension of K^*. Since σ is an automorphism of K^*, the same property holds for the polynomial f^σ and $\bar{b}^\sigma$ is the corresponding generic point of $f^\sigma = 0$. The extension $K^*(\bar{b}) \otimes K^*(\bar{b}^\sigma)$ of K^* is a domain and it is formally real. Iterating the same reasoning we get a chain of formally real fields of the form $\bigotimes_{-n \le i \le n} K^*(\bar{b}^{\sigma^i})$. Taking the union we have a formally real difference field extension of K^*, where $f = 0$ has a generic point. Since K^* is existentially closed, $f = 0$ has a point in K^*.

Then, by Zorn's lemma, there is a maximal difference preordered model of T_n of cardinality κ, $(L, \tilde{S}, \sigma)$ in $(\tilde{K}, \tilde{\sigma})$ extending (K, S, σ).

Given two preordered difference field extensions of (K, S, σ) of cardinality κ in $\tilde{K}$, models of T_n, say (K_1, S_1, σ_1) and (K_2, S_2, σ_2), which we may assume to be linearly disjoint over K, we form $K_1 \otimes_K K_2$. This latter ring is an integral domain since K_1 is a regular extension of K and so it has a field of fractions L_0, which is a regular extension of K_1 and K_2. Using the same reasoning as in Lemma 1.3, we may assume that L_0 is a subfield of $\tilde{K}$.

For each preorder S_1 on K_1, respectively S_2 on K_2, one shows that one can endow $K_1 \otimes_K K_2$ with a preorder T extending the preorder S_1 of K_1 and the preorder S_2 of K_2. Indeed, one shows that the subset $\{\sum_j (k_{1j} \otimes k_{2j}) \cdot y_j^2 : k_{1j} \in S_1, k_{2j} \in S_2, y_j \in K_1 \otimes K_2\}$ is a preorder extending the preorders S_1 on K_1 and S_2 on K_2 ([22, (0.5)]). Finally, one extends this preorder to the field of fractions L_0.

Then one has to show that it is a difference field extension of K. Given a typical element of $K_1 \otimes_K K_2$, one defines $\sigma_3(k_1 \otimes k_2) = \sigma_1(k_1) \otimes \sigma_2(k_2)$ and one extends σ_3 on L_0.

Then we consider the existential closure of L_0 inside $\tilde{K}$ and so we get a difference preordered field extension of K of cardinality κ which is a model of T_n (see [9, Theorem 1.2], [23] and the above) into which both K_1 and K_2 embed by an endomorphism fixing K. □

Definition 1.14 Let $\mathcal{C}_{\mathrm{pra}_n}$, $n \in \mathbb{N} \cup \{+\infty\}$, be the class of preordered commutative difference fields (K, S, σ) satisfying the following properties:

1. $K \vDash T_n$.
2. σ is an automorphism of K.
3. For every absolutely irreducible variety U defined over K and every absolutely irreducible variety V defined over K with $V \subseteq U \times U^\sigma \subset \tilde{K}^{2n}$, where $\tilde{K} \vDash \mathrm{ACFA}$, projecting generically onto U and onto U^σ, the following holds:

If for all $q[X, Y] \notin I_K(V)$ and any finite number of difference polynomials $p_i[X, Y]$, $i \in I$, we have that $q^2 + \sum_{i \in I} p_i^2 \notin I_K(V)$, then there exists an element $\bar{r}$ in K such that $(\bar{r}, \bar{r}^\sigma) \in V$.

Lemma 1.15 *Let $\mathcal{C}$ be the class of preordered difference fields. Then any element of $\mathcal{C}$ embeds in an element of $\mathcal{C}_{\mathrm{pra}_\infty}$ and given two elements of $\mathcal{C}_{\mathrm{pra}_\infty}$, $K_1 \subset K_2$, then $K_1 \subset_{\mathrm{ec}} K_2$.*

Proof. First, let (K, S, σ) be a preordered difference field of cardinality κ and (K^*, S^*, σ) an existentially closed extension of the same cardinality inside a κ^+-saturated extension of (K, σ), model of ACFA_+.

Let us show that (K^*, S^*, σ) embeds in an element of $\mathcal{C}_{\mathrm{pra}_\infty}$.

By the proof of Lemma 1.13, we have that (K^*, S^*, σ) is a model of T_∞. So, $S^* = (K^*)^2$.

Let V be an absolutely irreducible variety defined over K^*. Assume that for all $q[X, Y] \notin I_K(V)$ and all finite set of $p_i[X, Y]$, $i \in I$, we have that $q^2 + \sum_{i \in I} p_i^2 \notin I_K(V)$. Namely -1 is not a sum of squares in the fraction field of $K^*[X, Y]/I_{K^*}(V)$, so it is formally real. Since V is absolutely irreducible, the fraction field of $K^*[X,Y]/I_{K^*}(V)$ is a regular extension of K^* which is a SAP field. So all orderings of K^* extends to it (Proposition in [23, Section 2]). Similarly as in the proof of Lemma 1.11, we wish to get a difference preordered field extension of K^*, where V has a point of the form $(\bar{b}, \bar{b}^\sigma)$. We proceed as follows. Let Ω is a model of RCF containing K^* and κ^+-big. Let $(\bar{a}, \bar{b})$ be a generic point of V in Ω. Since V projects generically on U and on U^σ, $\bar{a}$ (respectively $\bar{b}$) is a generic point of U (respectively U^σ) in Ω. So these two tuples have the same $\mathcal{L}$-type over K^*. Since $\mathrm{Frac}(K^*[X,Y]/I_K(V))$ is formally real, we have also that $\mathrm{Frac}(K^*[X]/I_K(U))$ is formally real. Let S_1 (similarly S_2) be the set of squares in $\mathrm{Frac}(K^*[X]/I_K(U))$ (similarly in $\mathrm{Frac}(K^*[X]/I_K(U^\sigma))$).

We have a partial isomorphism of Ω extending σ and sending $\bar{a}$ to $\bar{b}$ and S_1 to S_2. Since Ω is κ^+-big and so κ^+-strongly homogeneous ([14, p. 487]) and since $(\Omega, K^*(\bar{a}), S_1) \equiv (\Omega, (K^*)^\sigma(\bar{b}), S_2)$, there is an automorphism of Ω extending σ and taking $\bar{a}$ to $\bar{b}$. Let $(K^*\langle\bar{a}\rangle_\sigma, S)$ be the difference preordered subfield of Ω generated by K^* and $\bar{a}$, where S is the set of squares and V has a point of the form $(\bar{a}, \bar{a}^\sigma)$. Since (K^*, S^*) is existentially closed, V has also a point in K^* of the form $(\bar{c}, \bar{c}^\sigma)$.

Second, let $(K_1, S_1) \subset (K_2, S_2) \in \mathcal{C}_{\mathrm{pra}_\infty}$ and let us show that $(K_1, S_1) \subset_{\mathrm{ec}} (K_2, S_2)$. So, we have to show that any existential formula $\varphi(\bar{b}) := \exists \bar{x}\theta(\bar{x}, \bar{b})$, where $\bar{b} \subset K_1$ and θ is a conjunction of basic formulas in the language $\mathcal{L}_{S,\sigma}$ satisfied in K_2, is already satisfied in K_1.

Let $\theta(x_1, \ldots, x_n, \bar{b})$ be of the form

$$\begin{aligned} f(\bar{x}, \sigma(\bar{x}), \ldots, \sigma^k(\bar{x})) = 0 \ \&\ & g(\bar{x}, \sigma(\bar{x}), \ldots, \sigma^k(\bar{x})) \neq 0 \\ \&\ & f_1(\bar{x}, \sigma(\bar{x}), \ldots, \sigma^k(\bar{x})) \in S \ \&\ \cdots\ \&\ f_s(\bar{x}, \sigma(\bar{x}), \ldots, \sigma^k(\bar{x})) \in S \\ \&\ & g_1(\bar{x}, \sigma(\bar{x}), \ldots, \sigma^k(\bar{x})) \notin S \ \&\ \cdots\ \&\ g_t(\bar{x}, \sigma(\bar{x}), \ldots, \sigma^k(\bar{x})) \notin S, \end{aligned}$$

where f, g, f_i, g_j are polynomials with coefficients in K_1, $i \in I,\ \ j \in J$.

As in the proof of [23, Lemma 3], we replace the basic subformulas $f_i(\bar{x}, \bar{y}) \in S$ by $\exists z_i\, f_i(\bar{x}, \bar{y}) = z_i^2$, and $g_j(\bar{x}, \bar{y}) \notin S$ by $\exists z_j\, g_j(\bar{x}, \bar{y}) \cdot z_j^2 = c_j$, where c_j is some element of K_1 such that $-c_j \in K_1 - K_1^2$. We call the obtained (difference field) formula $\tilde{\theta}(\bar{x}, \bar{z}, \bar{b})$. To check the equivalence between $\varphi(\bar{b})$ and $\exists\bar{x}\exists\bar{z}\tilde{\theta}(\bar{x}, \bar{z}, \bar{b})$ one considers a finitely generated subfield of K_2 containing K_1 and algebraic over it.

Suppose the formula $\tilde{\theta}$ is satisfied by a tuple $\bar{d}$ in K_2; put it in the form $(\bar{d}_0, \bar{d}_0^\sigma)$. Let U be an absolutely irreducible (algebraic) variety whose $\bar{d}_0$ is a generic point and let V be an absolutely irreducible variety whose $(\bar{d}_0, \bar{d}_0^\sigma)$ is a generic point. Let $S' := S_2 \cap \langle K_1, \bar{d}_0\rangle_\sigma$ of K_2. Then S' is proper and so the fraction field of $K_1[X]_\sigma/I_{K_1}(V)$ is formally real. The variety V projects generically on $U \times U^\sigma$. So, since K_1 satisfies scheme 3, there is an element $(\bar{r}, \bar{r}^\sigma) \in V(K_1)$. So, there exists $\bar{r}_0 \subset \bar{r}$ such that $K_1 \vDash \varphi(\bar{r}_0)$.

Note that $g \notin I(V)$, whereas $g_j(\bar{x}, \bar{b}) \cdot z_j^2 - c_j \in I(V)$, $j \in J$, and $f_i(\bar{x}, \bar{b}) - z_i^2 \in I(V)$.

Therefore, the variety V satisfies the hypothesis of scheme 3 and so we may find a tuple $\bar{a}$ in K_1 in V and satisfying $\theta(\bar{a}, \bar{b})$. Therefore, $K_1 \vDash \varphi(\bar{b})$. □

2 Ordered difference valued fields

In this section, we will consider ordered difference fields where the distinguished automorphism is ω-increasing. Expanding the language with a valuation, will allow us to first-order axiomatize a class of existentially closed such ordered difference valued fields (we will put the additional hypothesis that the fixed field is dense in the set of elements of valuation zero). In view of Baur's result on pairs of real-closed fields recalled in the introduction, such hypothesis may be reasonable.

A field $(K, +, -, \cdot, <, v, 0, 1)$ is called a *valued ordered field* ([7, 1.2]) if

1. $(K, +, -, \cdot, <, 0, 1)$ is an ordered field,
2. $(K, +, -, \cdot, v, 0, 1)$ is a valued field,
3. the following compatibity relation holds between the valuation and the order:

$$(\forall a > 0)\, \forall b\, (0 < b < a \to v(b) \geq v(a)).$$

As it was recalled in the introduction, any totally ordered field K embeds in a power series field of the form $k((G))$, where $k \subseteq \mathbb{R}$ and G is the set of archimedean classes of K^+. We define an order by setting that $a := \sum_{i \in \mathrm{supp}(a)} k_i \cdot x^{g_i} > 0$, where the support $\mathrm{supp}(a)$ of a is a well-ordered subset of G, g_{i_0} is the smallest element of $\mathrm{supp}(a)$ and $k_{i_0} > 0$. One endows this power series field with a valuation sending an element to the smallest element of its support and so we get in this way a valued ordered field as defined above.

More generally, we will consider power series fields $k((G))$, where k is not necessarily archimedean.

Denote by $\bar{K}$ the residue field of (K, v), by $\mathcal{O}_K$ the elements of positive value, by $\mathcal{M}_K$ the maximal ideal of $\mathcal{O}_K$ and by Γ_K the value group of v. We will use here the notation $\bar{a}$ to denote the image of the element $a \in K$ in the residue field $\bar{K}$. Let Γ_K^+ denote the strictly positive elements of Γ_K.

Recall that $\mathcal{O}_K$ is convex in K, $\mathcal{M}_K$ is bounded by ± 1. If K is a real-closed field, then $\bar{K}$ is a real-closed field, Γ_K is a divisible group and (K, v) is Henselian (see [7, Lemmas 4, 5 and Theorem 3]).

Definition 2.1 We will say that $(K, +, -, \cdot, <, v, \sigma, 0, 1)$ is a *valued ordered difference field* if σ is an automorphism of the structure $(K, +, -, \cdot, <, \mathrm{div}, 0, 1)$, where

$$a \,\mathrm{div}\, b \quad \text{if and only if} \quad v(a) \leq v(b).$$

We will denote by $\mathcal{L}_{<,\mathrm{div}}$ (respectively by $\mathcal{L}_{<,\mathrm{div},\sigma}$) the language $\mathcal{L}_<$ (respectively $\mathcal{L}_{<,\sigma}$) augmented by the relation symbol div and we will consider the theories of valued ordered (difference) fields in these languages.

Therefore, in a valued ordered difference field $(K, +, -, \cdot, <, v, \sigma, 0, 1)$, σ induces an endomorphism $\tilde{\sigma}$ on the value group $(\Gamma_K, +, 0, <)$; indeed the map $\tilde{\sigma}(v(a)) := v(\sigma(a))$ is well-defined and

$$v(\sigma(a \cdot b)) = v(\sigma(a) \cdot \sigma(b)) = v(\sigma(a)) + v(\sigma(b)).$$

So, Γ_K is endowed with a structure of a $\mathbb{Z}[t]$-module, the action of t being defined by $v(a) \cdot t = v(a^\sigma)$ and extended by linearity on $\mathbb{Z}[t]$. Let $q(t) = \sum_{j=0}^n z_j \cdot t^j$ with $z_j \in \mathbb{Z}$, then

$$v(a) \cdot q(t) = \textstyle\sum_{j=0}^n v(a) \cdot z_j \cdot t^j = \sum_{j=0}^n v((a^{z_j})^{\sigma^j}) = v(\prod_{j=0}^n (a^{z_j})^{\sigma^j}).$$

For $a \in K$, we will use the notation $a^{q(\sigma)} := \prod_{j=0}^n (a^{z_j})^{\sigma^j}$.

Recall that the model-completion of the theory of torsion-free $\mathbb{Z}[t]$-modules is the theory of divisible torsion-free $\mathbb{Z}[t]$-modules. Further, we will endow $\mathbb{Z}[t]$ with the following order extending the order on $\mathbb{Z}$: Let $\sum_{i=0}^n z_i \cdot t^i$ be in $\mathbb{Z}[t]$, then

$$\textstyle\sum_{i=0}^n z_i \cdot t^i > 0 \quad \text{if and only if} \quad z_n > 0.$$

Denote by $\mathbb{Z}[t]^+$ the set of strictly positive elements of $\mathbb{Z}[t]$.

Recall that the theory T_{do} of divisible ordered torsion-free $\mathbb{Z}[t]$-modules admits quantifier elimination and is the model-companion of the theory of the ordered torsion-free $\mathbb{Z}[t]$-modules satisfying in addition the following axiom scheme:

$$(\forall m > 0)\,(m \cdot p(t) > 0), \text{ for } p(t) \in \mathbb{Z}[t]^+.$$

Moreover, T_{do} is an o-minimal theory (see [6], [10]).

Definition 2.2 Let T_{vod} be the $\mathcal{L}_{<,\mathrm{div},\sigma}$-theory of valued ordered difference fields. Let $T_{\mathrm{vod,inc}}$ be the theory T_{vod} together with:

1. σ is strictly increasing on the set of strictly positive elements of the value group, i.e.,

$$\forall a\,(v(a) > 0 \rightarrow v(\sigma(a)) > v(a)).$$

2. $(\forall a \in K)\,(\exists b \in \mathrm{Fix}(\sigma))\,(v(a) = 0 \rightarrow v(a - b) > 0)$.

Let $T_{\mathrm{vod},\omega\mathrm{inc}}$ be the theory $T_{\mathrm{vod,inc}}$ together with the scheme of axioms

3. $\forall a\,(v(a) > 0 \rightarrow v(\sigma(a)) > n \cdot v(a))$, for each $n \in \mathbb{N}, n \geq 2$.

Note that (for clarity sake) we have written the above axioms in a language with the valuation v, but this can easily be translated into the $\mathcal{L}_{<,\mathrm{div},\sigma}$-language.

Note that in any model K of $T_{\mathrm{vod,inc}}$, σ induces the identity automorphism on the residue field, i.e.,

$$\forall a\,(v(a) = 0 \rightarrow v(a - a^\sigma) > 0).$$

Indeed, if $a \in K$ with $v(a) = 0$, then by axiom 2 there exists $b \in \mathrm{Fix}(\sigma)$ such that $v(a - b) > 0$. Applying axiom 1, we get $v((a - b)^\sigma) > v(a - b)$, and so since $v(a - a^\sigma) \geq \min\{v(a - b), v(a^\sigma - b)\}$, we get the result.

Notation 2.3 For $a, b \in K$ with $1 < a < b$ (respectively $a < b < 1$) we will denote by $a \ll b$ that for any positive natural number n one has $a^n < b$ (respectively $a < b^n$). We will say that σ is *ω-increasing* if for any $a \in \mathcal{M}_K - \{0\}$, $\sigma(a) \ll a$.

Lemma 2.4 *Assume that K is a model of $T_{\mathrm{vod,inc}}$. Then $\mathrm{Fix}_K(\sigma) \cong \bar{K}$.*

Proof. First, note that by axiom 1, we have that $\mathrm{Fix}(\sigma) \subset \{x \in K : v(x) = 0\}$. Indeed, if a is an element of $\mathrm{Fix}(\sigma) - \{0\}$ and $v(a) \neq 0$, then either $v(a) > 0$ or $v(a^{-1}) > 0$. W.l.o.g. let $v(a) > 0$, which implies by axiom 1, $v(\sigma(a)) > v(a)$ which contradicts the fact that $\sigma(a) = a$.

Conversely, let $\bar{a} \in \bar{K}$ and let $a \in \mathcal{O}_K$ with $a + \mathcal{M}_K = \bar{a}$. By axiom 2, there exists $b \in \mathrm{Fix}(\sigma)$ such that $v(a - b) > 0$, and such element b is unique. Indeed, suppose that $v(a - b_1) > 0$ and $v(a - b_2) > 0$, with $b_1, b_2 \in \mathrm{Fix}(\sigma)$. Then $v(b_1 - b_2) > 0$ with $b_1 - b_2 \in \mathrm{Fix}(\sigma)$. So, by the first part, $b_1 = b_2$. □

Remark 2.5 Let $(K, v, \sigma) \subset_{\mathrm{ec}} (L, v, \sigma)$ be two valued ordered difference fields models of $T_{\mathrm{vod,inc}}$. Then $\bar{K} \subset_{\mathrm{ec}} \bar{L}$.

Proof. Indeed, $\mathrm{Fix}_K(\sigma) \subset_{\mathrm{ec}} \mathrm{Fix}_L(\sigma)$ since these are quantifier-free definable in K respectively L. □

Lemma 2.6 *Let K be a model of $T_{\mathrm{vod},\omega\mathrm{inc}}$. Then Γ_K is a torsion-free $\mathbb{Z}[t]$-module. If K is real-closed, then Γ_K is $\mathbb{Q}$-divisible.*

Proof. Since σ is increasing on the elements of positive values, $\mathcal{M}_K^\sigma \subseteq \mathcal{M}_K$. By induction on the degree of $f(t) \in \mathbb{Z}[t]^+$, we show that $(\forall \gamma \in \Gamma_K^+)(\gamma \cdot f(t) > 0)$.

Indeed, it is the content of axiom scheme 3 for $f(t)$ monic of degree 1, and it is easily seen that it also holds for all polynomials of degree 1.

Now, let us assume that for elements $g(t)$ of $\mathbb{Z}[t]^+$ of degree less than or equal to $m \geq 1$ and all $\gamma \in \Gamma_K^+$, $\gamma \cdot g(t) > n.\gamma$, for every $n \in \mathbb{N}$. Let us prove it for elements $f(t)$ of degree $m + 1$ of $\mathbb{Z}[t]^+$.

Write $f(t) = t \cdot (z_{m+1} \cdot t^m + \sum_{i=1}^m z_i \cdot t^{i-1}) + z_0)$ with $z_{m+1} \in \mathbb{N} - \{0\}$ and $z_i \in \mathbb{Z}$ for $i = 0, \ldots, m$. For $\gamma \in \Gamma_K^+$ we have $\gamma \cdot t \in \Gamma_K^+$ and by induction hypothesis we have $(\gamma \cdot t) \cdot (z_{m+1} \cdot t^m + \sum_{i=1}^m z_i \cdot t^{i-1}) > (\gamma \cdot t) \cdot n$ for all $n \in \mathbb{N}$. By the case $m = 1$ of the induction, we have that $\gamma \cdot t \cdot n > \gamma \cdot (n' - z_0)$, for any $n' \in \mathbb{N}$. Therefore, for any $n' \in \mathbb{N}$ we get $\gamma \cdot (f(t) - z_0) > \gamma \cdot (n' - z_0)$.

In particular, Γ_K is a torsion-free $\mathbb{Z}[t]$-module. Moreover, since K is real-closed, Γ_K is divisible as a $\mathbb{Z}$-module and so it can be endowed with a structure of $\mathbb{Q}$-module. □

Notation 2.7 For $p(X) \in \mathcal{O}_K[X]_\sigma$ we denote by $\bar{p}(X)$ the σ-polynomial where the coefficients of $p(X)$ have been replaced by their images in the residue field of K. For $b \in \mathcal{O}_K$ let $\bar{b}$ be the image of b in $\bar{K}$.

Definition 2.8 Let $K \vDash T_{\mathrm{vod}}$. The valued ordered difference field K *satisfies the σ-Hensel Lemma* if, for any difference polynomial $p(X) \in \mathcal{O}_K[X]_\sigma$ of effective order n such that $\bar{p}(X) \neq 0$ and there exists $b \in \mathcal{O}_K$ with $\bar{p}(\bar{b}) = 0$ and $\overline{\dfrac{\partial p^*}{\partial X_0}(b, b^\sigma, \ldots, b^{\sigma^n})} \neq 0$, there exists $a \in \mathcal{O}_K$ such that $p(a) = 0$ and $v(a-b) = v(p(b)) > 0$.

Lemma 2.9 *Let $K \vDash T_{\mathrm{vod}}$ satisfying axiom 1 in Definition 2.2. Assume that K is a complete valued field. Then K satisfies the σ-Hensel Lemma.*

Proof. We will prove a slightly stronger version of σ-Hensel Lemma. Namely we will replace the hypothesis that the derivative with respect to X_0 is of valuation zero by $(\star)$ below. Let $p(X) \in \mathcal{O}_K[X]_\sigma$ of effective order n, let $a_0, \ldots, a_n, \eta_0, \ldots, \eta_n \in K$ and write

$$p^*(a_0 + \eta_0, a_1 + \eta_1, \ldots, a_n + \eta_n) = p^*(a_0, a_1, \ldots, a_n) + \sum_{j=0}^n \frac{\partial p^*}{\partial X_j}(a_0, a_1, \ldots, a_n) \cdot \eta_j + \mathrm{O}(\|(\eta_0, \ldots, \eta_n)\|^2).$$

By hypothesis, there is an element $b \in \mathcal{O}_K$ such that $v(p(b)) > 0$ and $v(\dfrac{\partial p^*}{\partial X_i}(\boldsymbol{b})) = 0$ for some $0 \leq i \leq n$, where $\boldsymbol{b} := (b, b^{\sigma}, \ldots, b^{\sigma^n})$, and moreover

$(\star)$ there exists only one j_0 such that $v((\dfrac{\frac{\partial p^*}{\partial X_{j_0}}(\boldsymbol{b})}{p(b)})^{\sigma^{-j_0}}) = \min\{v((\dfrac{\frac{\partial p^*}{\partial X_j}(\boldsymbol{b})}{p(b)})^{\sigma^{-j}}) : 0 \leq j \leq n\}$

(note that $j_0 \leq i$).

We will build a Cauchy sequence indexed by ordinal numbers, starting with $a_0 := b$. Suppose we have constructed a_β for all $\beta < \alpha$ with the following properties: for all $\gamma < \beta$ we have that $v(p(a_\beta)) > v(a_\beta - a_\gamma) > 0$ and $v(a_{\gamma+1} - a_\gamma) \geq v(p(a_\gamma)^{\sigma^{-n}})$.

Assume that α is a successor ordinal, i.e. of the form $\beta + 1$. Then, by assumption,

$$v((\frac{p(a_\beta)}{\frac{\partial p^*}{\partial X_{j_0}}(\boldsymbol{b})})^{\sigma^{-j_0}}) \geq v((p(a_\beta))^{\sigma^{-i}}),$$

and so it is strictly positive. Let

$$\varepsilon := (\frac{p(a_\beta)}{\frac{\partial p^*}{\partial X_{j_0}}(\boldsymbol{a}_\beta)})^{\sigma^{-j_0}} \in \mathcal{M}_K \text{ and } \boldsymbol{\varepsilon} := (\varepsilon, \varepsilon^{\sigma}, \ldots, \varepsilon^{\sigma^n}).$$

Note that $v(\varepsilon) \geq v(p(a_\beta)^{\sigma^{-j}}) > 0$ for any $0 \leq j \leq n$. Evaluate $p(a_\alpha - \varepsilon) = p^*(\boldsymbol{a}_\alpha - \boldsymbol{\varepsilon})$. We get

$$\begin{aligned} p(a_\beta - \varepsilon) &= p(a_\beta) - \textstyle\sum_{j=0}^{n} \dfrac{\partial p^*}{\partial X_j}(\boldsymbol{a}_\beta) \cdot (\varepsilon^{\sigma^j}) + \mathrm{O}(\|(\varepsilon, \varepsilon^{\sigma}, \ldots, \varepsilon^{\sigma^n})\|^2) \\ &= p(a_\beta) \cdot (- \textstyle\sum_{j=0,\, j \neq j_0}^{n} \dfrac{\frac{\partial p^*}{\partial X_j}(\boldsymbol{a}_\beta)}{p(a_\beta)} \cdot (\varepsilon^{\sigma^j}) + \dfrac{1}{p(a_\beta)} \cdot \mathrm{O}(\|(\varepsilon, \varepsilon^{\sigma}, \ldots, \varepsilon^{\sigma^n})\|^2). \end{aligned}$$

So,

$$v(p(a_\beta - \varepsilon)) = v(p(a_\beta)) + v(\textstyle\sum_{j=0,\, j \neq j_0}^{n} \dfrac{\frac{\partial p^*}{\partial X_j}(\boldsymbol{a}_\beta)}{p(a_\beta)} \cdot (\varepsilon^{\sigma^j}) + \dfrac{1}{p(a_\beta)} \cdot \mathrm{O}(\|(\varepsilon, \varepsilon^{\sigma}, \ldots, \varepsilon^{\sigma^n})\|^2)) > v(p(a_\beta)).$$

Assume now that α is a limit ordinal. Since K is a complete valued field, there exists a_α such that for all $\beta_1 < \beta_2 < \alpha$ we have $v(a_\alpha - a_{\beta_1}) < v(a_\alpha - a_{\beta_2})$. By replacing in the above equation ε by $a_\alpha - a_\beta$ with $\beta < \alpha$, we obtain that $v(p(a_\alpha)) \geq \min\{v(p(a_\beta)), v(a_\alpha - a_\beta)\}$. By induction hypothesis we have that $v(p(a_\beta)) > v(a_\beta - a_\delta)$ for all $\delta < \beta$. Moreover, we have $v(a_\beta - a_\delta) = v(a_\alpha - a_\delta)$. So, we get that $v(p(a_\alpha)) > v(a_\alpha - a_\delta)$.

Let $a := \lim_\alpha a_\alpha$. □

Lemma 2.10 *Let K be a model of T_{vod} satisfying the σ-Hensel Lemma. Then for each irreducible polynomial $q(t) \in \mathbb{Z}[t]$ and $u \in \mathcal{O}_K$ with $\bar{u} = 1 \in \bar{K}$ there exists $a \in \mathcal{O}_K - \mathcal{M}_K$ such that $a^{q(\sigma)} = u$.*

Proof. Write $q(t) = n \cdot (p_1(t) - p_2(t))$ with $p_1(t), p_2(t) \in \mathbb{N}[t]$ and such that the gcd of the coefficients of both polynomials $p_1(t)$, $p_2(t)$ is equal to 1 and, for each $i \in \mathbb{N}$, t^i occurs in at most one of them. For $i = 1, 2$, set $p_i(t) = \sum_{j=0}^{n_i} m_j \cdot t^j$. Then, for $z \in K$, $z^{q(\sigma)} = z^{np_1(\sigma)}/z^{np_2(\sigma)}$ and we look for such an element for which $z^{q(\sigma)} = u$, e.g. $z^{np_1(\sigma)} = u \cdot z^{np_2(\sigma)}$. So, we apply the σ-Hensel Lemma to the σ-polynomial $\tilde{q}(X) := \prod_{j=0}^{n_1} (X^{m_j})^{\sigma^j} - u \cdot \prod_{j=0}^{n_2} (X^{m_j})^{\sigma^j}$. We have that $\tilde{q}^*(X_0, \ldots, X_n) = \prod_{j=0}^{n_1} (X_j^{m_j}) - u \cdot \prod_{j=0}^{n_2} (X_j^{m_j})$. The element 1 is a residual root of $\tilde{q}(X) = 0$ and for $i \in \{0, \ldots, \max\{n_1, n_2\}\}$ with $m_i \neq 0$, $\dfrac{\partial \tilde{q}^*(X_0, \ldots, X_n)}{\partial X_i}$ is equal to $m_i \cdot X_i^{m_i - 1} \cdot \prod_{j=0,\, j \neq i}^{n_1} (X_j^{m_j})$ or to $-u \cdot m_i \cdot X_i^{m_i - 1} \cdot \prod_{j=0,\, j \neq i}^{n_2} (X_j^{m_j - 1})$, and so when it is evaluated at 1, it is non zero. □

Notation 2.11 Let $T_{\mathrm{vod},\omega\mathrm{inc,h}}$ be the theory $T_{\mathrm{vod,inc}}$ together with RCF and σ-Hensel Lemma.

Given an element $a \in K$ satisfying a σ-polynomial belonging to $K[X]_\sigma$. We choose among all the polynomials $p(X, X^\sigma, \dots, X^{\sigma^n})$ that it satisfies the ones with minimal effective order n (which we assume to coincide with the order), and among these, that we can write as $\sum_{j=0}^{d} (X^{\sigma^n})^j \cdot q_j^*(X, \dots, X^{\sigma^{(n-1)}})$ with $q_j \in K[X]_\sigma - \{0\}$, we choose the ones with d minimal. We will call such polynomial a *minimal σ-polynomial* satisfied by a and of effective order n and degree d.

We would like an Ax-Kochen-Ershov result for the models of $T_{\mathrm{vod},\omega\mathrm{inc,h}}$, analogous to the classical AKE-Theorem (see for instance [18, Appendix]).

Definition 2.12 ([16]) Recall that a well-ordered subset of elements $a_\varrho \in K$, $\varrho \in \mathrm{On}$, without a last element is said to be *pseudo-convergent* (p.c.) if $v(a_{\varrho_1} - a_{\varrho_2}) < v(a_{\varrho_2} - a_{\varrho_3})$ with $\varrho_1 < \varrho_2 < \varrho_3$.

Recall that if (a_ϱ) is p.c., then either $v(a_{\varrho_1}) < v(a_{\varrho_2})$ for all $\varrho_1 < \varrho_2$, or for some μ we have that for all $\mu < \varrho_1 < \varrho_2$ it holds $v(a_{\varrho_1}) = v(a_{\varrho_2})$ (see [16, Lemma 1]). Moreover, $v(a_\varrho - a_{\varrho_1}) = v(a_{\varrho_1+1} - a_{\varrho_1})$ for any $\varrho > \varrho_1$ (see [16, Lemma 2]).

Definition 2.13 An element a is a *limit* of the p.c. set (a_ϱ) if $v(a - a_{\varrho_1}) = v(a_{\varrho_1} - a_{\varrho_2})$ with $\varrho_1 < \varrho_2$.

Note that, since σ is a valued field automorphism, if (a_ϱ) is p.c., then (a_ϱ^σ) is also p.c. Further if a is a limit of a p.c. set (a_ϱ), then a^{σ^m} is a limit of the p.c. set $(a_\varrho^{\sigma^m})$, $m \in \mathbb{Z}$.

Lemma 2.14 (See [16, Theorem 1]) *If L is an immediate ordered field extension of K, then any element $a \in L - K$ is a limit of a p.c. strictly monotone sequence $(a_\varrho) \subset K$ without a limit in K.*

P r o o f. For convenience of the reader, we reproduce the proof of this Lemma below. Let $a \in L - K$ and let $S = \{v(a-k) : k \in K\}$. This set S does not contain $+\infty$ and it has no greatest element since L is an immediate extension of K. From S we select a well-ordered set of cofinal elements γ_ϱ and we choose elements a_ϱ in K with $v(a - a_\varrho) = \gamma_\varrho$. Suppose that $a_0 > a$, then we may choose $a < a_1 < a_0$ and $v(a - a_1) = \gamma_1$. By induction on ϱ, we may assume that this sequence a_ϱ is decreasing to a. The other case when $a_0 < a$ is similar and we obtain an increasing sequence.

Denote by a_{K_+} (respectively a_{K_-}) the set of elements of K which are bigger (respectively smaller) than a in L We claim that (a_ϱ) is cofinal in a_{K_+}. Suppose not, namely that there is an element c in a_{K_+} which is strictly smaller than (a_ϱ). By construction, $v(a - c) = v(a - a_\mu)$ for some μ. So, $v(a - a_{\mu+1}) > v(a - c)$; however $0 < c - a < a_{\mu+1} - a$, so by the compatibility relation between v and $<$, we get that $v(c - a) \geq v(a_{\mu+1} - a)$, which is a contradiction. □

In [16, p. 306], Kaplansky defines p.c. sets of algebraic and transcendental types, and as in [5], we will adapt the definitions for σ-polynomials, using the fact (adapted to σ-polynomials and proved by A. Ostrowski for ordinary polynomials) that if $p[X] \in K[X]_\sigma$, then there exists an index $\mu \in \mathrm{On}$ such that $(p(a_\varrho))_{\varrho > \mu}$ is p.c.

Definition 2.15 We will say that a p.c. sequence (a_ϱ) is of

1. *σ-transcendental type* with respect to a field K, if for all $p[X] \in K[X]_\sigma$ there exists μ such that $v(p(a_{\varrho_1})) = v(p(a_{\varrho_2}))$ for all $\mu < \varrho_1 < \varrho_2$;

2. *σ-n-algebraic type* with respect to a field K, if for some $p[X] \in K[X]_\sigma$ of order n there exists μ such that $v(p(a_{\varrho_1})) < v(p(a_{\varrho_2}))$ for all $\mu < \varrho_1 < \varrho_2$.

Proposition 2.16 *Let (K, v, σ), (L, v, σ) be two valued ordered difference fields models of $T_{\mathrm{vod},\omega\mathrm{inc,h}}$. Suppose that L is an $|K|^+$-saturated $\mathcal{L}_{<,\mathrm{div},\sigma}$-extension of K, that $\bar{K} \subset_{\mathrm{ec}} \bar{L}$ as ordered fields, and that $\Gamma_K \subset_{\mathrm{ec}} \Gamma_L$ as $\mathbb{Z}[t]$-modules. Then $K \subset_{\mathrm{ec}} L$ in $\mathcal{L}_{<,\mathrm{div},\sigma}$.*

P r o o f. Using the classical Ax-Kochen-Ershov Theorem (see for instance [18]), the hypotheses imply that $K \subset_{\mathrm{ec}} L$ in $\mathcal{L}_{<,\mathrm{div}}$. In particular K is a relatively algebraically closed subfield of L. We use Frayne's Lemma and so we can embed L into a non principal ultrapower K^* of K, that we may choose to be $|L|^+$-saturated and such that this embedding respects the $\mathcal{L}_{<,\mathrm{div}}$-structures and is fixed on K. In particular, the induced embedding sending Γ_L in Γ_{K^*} is the identity on Γ_K; we have that $\Gamma_{K^*} \cong \Gamma_K^*$ as $\mathbb{Z}$-modules.

Note that since $\Gamma_K \subset_{\mathrm{ec}} \Gamma_L$ as $\mathbb{Z}[t]$-modules, it implies that Γ_L/Γ_K is a torsion-free $\mathbb{Z}[t]$-module. Indeed, let $\gamma \in \Gamma_L$ and suppose there exists $q(t) \in \mathbb{Z}[t]$ such that $\gamma \cdot q(t) \in \Gamma_K$. So, $\Gamma_L \vDash \exists x\, x \cdot q(t) = \gamma_0$ with $\gamma_0 \in \Gamma_K$. Since $\Gamma_K \subset_{\mathrm{ec}} \Gamma_L$, $\Gamma_K \vDash \exists x\,(x \cdot q(t) = \gamma_0)$. But Γ_L is torsion-free (see Lemma 2.6), so $\gamma \in \Gamma_K$.

Throughout the proof, we will use the following fact:

($\star$) Let a be an element of L which is σ-algebraic over K of σ-degree n, then its value $v(a)$ belongs to Γ_K.

Indeed, let $p[X] \in K[X]_\sigma$ and assume that $p(a) = 0$. Then for some $q(t) \in \mathbb{Z}[t]$ we have $v(a) \cdot q(t) \in \Gamma_K$ with degree of $q(t)$ less than or equal to n (one expresses that the values of two σ-monomials in a with coefficients in K are equal). By the above, $v(a) \in \Gamma_K$.

We will denote in the same way σ (respectively the valuation v) and its extension to K^*.

(1) The first step consists in showing that any maximal difference subfield L_0 of L which embeds in K^* with $\Gamma_{L_0} = \Gamma_K$ and $\mathrm{Fix}_K(\sigma) \subseteq \mathrm{Fix}_{L_0}(\sigma) \subseteq \mathrm{Fix}_{K^*}(\sigma)$ has the property that $\bar{L}_0 \cong \bar{L}$.

(2) Second, we consider, more generally, maximal difference subfields L_0 of L which embeds in K^*, with the two properties that $\bar{L} = \bar{L}_0$ and Γ_L/Γ_{L_0} is a torsion-free $\mathbb{Z}[t]$-module, and we show that this implies that $\Gamma_L = \Gamma_{L_0}$.

(3) Finally, we show that any maximal difference subfield L_0 of L with the property that $\Gamma_{L_0} = \Gamma_L$ and $\bar{L}_0 = \bar{L}$ which embeds in K^* is equal to L.

(1) Note that $\mathrm{Fix}_L(\sigma) \cong \bar{L}$. Suppose that there exists $a \in \mathrm{Fix}_L(\sigma) - \mathrm{Fix}_{L_0}(\sigma)$. First, assume that a is algebraic over $\mathrm{Fix}_{L_0}(\sigma)$. Let $f[X] \in \mathrm{Fix}_{L_0}(\sigma)[X]$ be such that $f[X]$ is the minimal polynomial of a over $\mathrm{Fix}_{L_0}(\sigma)$. By the classical Hensel's Lemma, there exists $a^* \in \mathcal{O}_{K^*}$ with $f(a^*) = 0$. Since $f^\sigma = f$ and $\mathrm{Fix}_{K^*}(\sigma)$ is totally ordered, we obtain $a^* \in \mathrm{Fix}_{K^*}(\sigma)$.

Moreover, since $\mathrm{Fix}_L(\sigma)$ is ordered, for some positive integer i less than the degree of f, the element a is the i^{th} root of f, it belongs to certain cut with respect to $\mathrm{Fix}_{L_0}(\sigma)$ and the polynomial f changes of signs within this cut. So, since $\mathrm{Fix}_{L_0}(\sigma)$ embeds in $\mathrm{Fix}_{K^*}(\sigma)$, we may choose $a^* \in \mathrm{Fix}_{K^*}(\sigma)$ in the same cut as a is.

Suppose now that $a \in \mathrm{Fix}_L(\sigma)$ is transcendental over $\mathrm{Fix}_{L_0}(\sigma)$. We will choose an element in $\mathrm{Fix}_{K^*}(\sigma)$ which is transcendental over $\mathrm{Fix}_{L_0}(\sigma)$ and is in the same cut with respect to L_0 than a. We use the $|L|^+$-saturation of K^* and the fact that $\mathrm{Fix}_K(\sigma) \subset \mathrm{Fix}_{L_0}(\sigma) \subset \mathrm{Fix}_{K^*}(\sigma)$.

Therefore in neither cases, the subfield L_0 is maximal with these properties.

(2) Now, we want to extend the embedding to a difference subfield with the same value group as L. Let L_0 be a maximal difference subfield of L which embeds in K^*, such that $\bar{L}_0 \cong \bar{L}$ and Γ_L/Γ_{L_0} is a torsion-free $\mathbb{Z}[t]$-module. If $\gamma \in \Gamma_L - \Gamma_{L_0}$, then $\Gamma_{L_0} \cap \gamma \cdot \mathbb{Z}[t] = \{0\}$. W.l.o.g., we will assume that $\gamma > 0$. Let $a \in L^+$ be such that $v(a) = \gamma$. Note that since $v(a) \notin \Gamma_{L_0}$, it determines the cut of a with respect to L_0. By ($\star$) above, a is not σ-algebraic over L_0. By [3, Proposition 1, paragraph 10.1], we define in this way a unique valued field extension of L_0 to $L_{0,0} := L_0(a)$. Then we proceed by induction extending the valuation first from $L_{0,0}$ to $L_{0,1} := L_{0,0}(a^{\sigma^{\pm 1}})$ by setting $v(a^{\sigma^{\pm 1}}) := \gamma \cdot t^{\pm 1}$, and more generally from $L_{0,n}$ to $L_{0,n+1} := L_{0,n}(a^{\sigma^{\pm n}})$ setting $v(a^{\sigma^{\pm n}}) := \gamma \cdot t^{\pm n}$. Set $L_0(a)_\sigma := \bigcup_{n \in \mathbb{N}} L_{0,n}$; it is a valued field extension of L_0. Then let L_1 be the real-closure of $L_0(a)_\sigma$ inside L.

Now, if we take an element $\tilde{a} \in K^*$ with $v(\tilde{a}) = \gamma$, then as ordered difference valued fields, $L_0(a)_\sigma$ and $L_0(\tilde{a})_\sigma$ are isomorphic, as well as their real-closures.

Then we have to extend this embedding to a σ-algebraic difference extension of L_1 inside L in such a way that Γ_L/Γ_{L_1} is $\mathbb{Z}[t]$-torsion-free, which will contradict the maximality of L_0.

Suppose that there is $\delta \in \Gamma_L - \Gamma_{L_1}$ such that $\delta \cdot q(t) \in \Gamma_{L_1}$ for some $q(t) \in \mathbb{Z}[t]$. We may assume w.l.o.g. that $q(t)$ is irreducible, of minimal degree and such that the gcd of its coefficients is equal to 1.

Let $b \in L$ be such that $v(b) = \delta$. First, we show that we may choose b such that $b^{q(\sigma)} \in L_1$ (see Definition 2.1).

Let $c \in L_1$ such that $v(c) = \delta \cdot q(t)$, so $v(b^{q(\sigma)} \cdot c^{-1}) = 0$. Since $\bar{L} = \bar{L}_0$, there exists $e \in L_0$ with $v(e) = 0$ such that $b^{q(\sigma)} \cdot c^{-1} \cdot e^{-1} \equiv 1$ (modulo $\mathcal{M}_L$). We use the σ-Hensel Lemma (see Lemma 2.10) in order to find $z \in L$ such that $b^{q(\sigma)} \cdot c^{-1} \cdot e^{-1} = z^{q(\sigma)}$ and with $v(z) = 0$. Therefore, $(b \cdot z^{-1})^{q(\sigma)} = c \cdot e \in L_1$. Set $b_0 := b \cdot z^{-1}$ and consider the extension $L_1(b_0)_\sigma$, we have that $v(b_0) = \delta$. W.l.o.g. we may assume that $b_0 > 0$.

Second, let d be the degree of $q(t)$ and write it as $q_1(t) - q_2(t)$ with $q_1(t), q_2(t) \in \mathbb{N}[t]$ and for any $0 \leq m \leq d$, the coefficient of t^m is non-zero in at most one of $q_1(t), q_2(t)$. Since σ is an automorphism, we may assume that $q(0) \neq 0$; let n_d be the coefficient of t^d and let $n_0 \neq 0$ be the constant term.

The extension $L_1(b_0)_\sigma$ is included in the real-closure of $L_1(b_0, \ldots, b_0^{\sigma^{d-1}})$.

The valuation on $L_1(b_0, \ldots, b_0^{\sigma^{d-1}})$ is completely determined by $v(b_0)$ (see [3, Proposition 1, paragraph 10.1]) and the cut b_0 belongs to with respect to L_1 is determined by $v(b_0)$ since it belongs to $\Gamma_L - \Gamma_{L_1}$.

Note that $(b_0^{\sigma^d})^{n_d} \in L_1(b_0, \ldots, b_0^{\sigma^{d-1}})$ and further it is of degree n_d over $L_1(b_0, \ldots, b_0^{\sigma^{d-1}})$. Indeed, suppose it is of degree smaller than n_d, then we will contradict the minimality of $q(t)$. Similarly, $b_0^{\sigma^{d+1}}$ is of degree n_d over $L_1(b_0^{\sigma}, \ldots, b_0^{\sigma^d})$. Also, $(b_0^{\sigma^{-1}})^{n_0} \in L_1(b_0, \ldots, b_0^{\sigma^{d-1}})$ and n_0 is the degree of ${b_0}^{\sigma^{-1}}$ over that subfield (if not this would contradict the minimality of $q(t)$).

Since in an ordered field a positive element has only one positive n^{th}-root, the order type of $b_0^{\sigma^d}$ is determined over $L_1(b_0, \ldots, b_0^{\sigma^{d-1}})$.

First, we embed $L_0(a)_\sigma$ in K^* sending a to $\tilde{a}$; then L_1 in the real-closure $\tilde{L}_1$ of $L_0(\tilde{a})_\sigma$ in K^*. Note that $\delta \in \Gamma_{K^*}$, and so there exists $b' \in K^*$ with $v(b') = \delta$. Since K^* satisfies the σ-Hensel Lemma, as before we may assume that b' is such that $b'^{q(\sigma)} \in L_1$. Then we claim that the field $\tilde{L}_1(b')_\sigma$ is isomorphic as a difference ordered valued field to $L_1(b_0)_\sigma$. Again, we take the real-closure of both ordered difference fields and we iterate this construction.

(3) Finally, let L_0 be a maximal difference subfield of L with the property that $\Gamma_{L_0} = \Gamma_L$ and $\bar{L}_0 \cong \bar{L}$ and which embeds in K^*. We will denote by $\tilde{L}_0$ its image in K^*. Note that by Frayne's Lemma we have an embedding of $\mathbb{Z}[t]$-modules of Γ_L in Γ_{K^*}.

Let $a \in L - L_0$, w.l.o.g. we may assume that $v(a) = 0$ (since $\Gamma_L = \Gamma_{L_0}$). Moreover, there exists an $\alpha \in \mathrm{Fix}_{L_0}(\sigma)$ such that $v(a - \alpha) > 0$.

By [16, Theorem 1] and Lemma 2.14, a is a limit of a p.c. monotone sequence $(a_\varrho) \subset L_0$ without a limit in L_0.

1. Suppose that this sequence is of σ-n-algebraic type with a minimal such n. This implies that for every $q[X] \in L_0[X]$ of order $< n$, there exists μ such that for any $\mu < \varrho_1 < \varrho_2$ we have $v(q(a_{\varrho_1})) = v(q(a_{\varrho_2}))$. So, the valued field structure of $L_0(a, a^\sigma, \ldots, a^{\sigma^{n-1}})$ is determined, since each a^{σ^i}, $0 < i < n$, is transcendental over $L_0(a, \ldots, a^{\sigma^{i-1}})$ (see [16, Theorem 2]).

We have that the image $(\tilde{a}_\varrho^\sigma)$ in K^* of sequence (a_ϱ^σ) has the same properties and so, since K^* is $|L|^+$-saturated, it contains a maximal immediate extension of $\tilde{L}_0$ and this sequence has a pseudo-limit $\tilde{a}_1$ in that extension, and $\tilde{a}_m$ is a pseudo-limit of $(a_\varrho^{\sigma^m})$ for $1 \leq m \leq n$. Note that $\tilde{a}_m$ will be in the same cut with respect to $\tilde{L}_0$ than it was a^{σ^m} with respect to L_0. By [16, Theorem 2], we have an isomorphism of valued fields between $L_2 = L_0(a^\sigma, \ldots, a^{\sigma^{n-1}})$ and $\tilde{L}_2 := \tilde{L}_0(\tilde{a}_1, \ldots, \tilde{a}_{n-1})$ and by construction this isomorphism also respects the order.

Now, a is algebraic over L_2. This situation is described in [16, Theorem 3]. Let $p[X] \in L_2[X]$ be a polynomial of minimal degree d with $p(a) = 0$. The roots of this polynomial are separated by elements of L_2 and belong to the real-closure of L_2 inside L. Denote by $\tilde{p}[X]$ the image of $p[X]$ in $\tilde{L}_2[X]$.

Note that $\bar{p}[X] = X - \alpha$. So, $\bar{\tilde{p}}[X] = X - \alpha$ and we may apply Hensel's Lemma, i.e., there is a unique element a^* in O_{K^*} with $\tilde{p}(a^*) = 0$ and $v(a^* - \tilde{\alpha}) > 0$. Moreover the sequence $(\tilde{a}_\varrho)$ is pseudo-convergent to a^*.

There is an isomorphism f of valued fields between $\tilde{L}_0(a^*, a_1, \ldots, a_{n-1})$ and $\tilde{L}_0(a_1, \ldots, a_n)$ sending a^* to a_1 and a_i to a_{i+1}, $1 \leq i \leq n-1$. We can extend f to the real-closure of both fields. But these real-closures coincide, so f extends to an automorphism τ of these real-closures. Then by induction on $i \leq 0$, one shows that $\tau^i(a^*)$ is the unique root of $\tilde{p}^{\tau^i}[X]$ satisfying $v(\tau^i(a^*) - \tilde{\alpha}) > 0$. Moreover one shows that the sequence $(\tilde{a}_\varrho^{\sigma^i})$ is pseudo-convergent to $\tau^i(a^*)$. Then the isomorphism type of the valued field $\tilde{L}_0(a^*)_\tau^{\mathrm{rc}}$ is uniquely determined by the fact that $(\tilde{a}_\varrho^{\sigma^i})$ is pseudo-convergent to $\tau^i(a^*)$ and that $p^{\tau^i}(\tau^i(a^*)) = 0$ for $i \in \mathbb{Z}$.

2. Assume now that this sequence is of σ-transcendental type over L_0. Again we use [16, Theorem 2] and so the valued field structure of the extension $L_0(a)_\sigma$ is determined. We consider the partial type

$$\mathrm{tp}(x) := \{v(x - \tilde{d}) = v(a - d) :\ d \in L_0\} \cup \{\tilde{d}_1 < x < \tilde{d}_2 :\ d_1,\ d_2 \in L_0 \text{ and } d_1 < a < d_2\}.$$

The map sending a to a realization $a^* \in K^*$ of that type, extends to a map from $L_0(a)$ to $\tilde{L}_0(a^*)$ and the value group of that extension is still equal to Γ_L (see [18, p. 194]). Then we have to show that $\sigma(a^*)$ satisfies the type $\{v(x - \tilde{d}) = v(a^\sigma - d) : d \in L_0\} \cup \{\tilde{d}_1^\sigma < x < \tilde{d}_2^\sigma : d_1, d_2 \in L_0 \text{ and } d_1 < a < d_2\}$.

The same reasoning can be applied to $L_0(a)(a^\sigma)$ and iterating this procedure, we obtain a difference field which is a proper extension of L_0 with the same properties, contradicting the maximality of L_0. □

Notation 2.17 Let $q(t) \in \mathbb{Q}[t]$, we may assume that it is in the form $q(t) = \frac{1}{n} \cdot p(t)$, where $p(t) \in \mathbb{Z}[t]$ and $n \in \mathbb{N}$, then write $p(t) = p_1(t) - p_2(t)$, where both $p_1(t), p_2(t) \in \mathbb{N}[t]$. We denote by $P_{q(t)}(x)$ the predicate defined by $\exists y\, \exists z\, (z^\sigma = z \;\&\; \frac{y^{p_1(\sigma)}}{y^{p_2(\sigma)}} = (x \cdot z)^n)$.

Definition 2.18 Let $T^{\mathrm{ec}}_{\mathrm{vod},\omega\mathrm{inc,h}}$ be the theory $T_{\mathrm{vod},\omega\mathrm{inc,h}} \cup \mathrm{RCF}$ plus the following scheme of axioms:

For each $q(t) \in \mathbb{Q}[t] - \{0\}$ of the form $q(t) = \frac{1}{n} \cdot (p_1(t) - p_2(t))$, where $p_1(t), p_2(t) \in \mathbb{N}[t] - \{0\}$ and $n \in \mathbb{N} - \{0\}$, we add the axiom $\forall x\, P_{q(t)}(x)$.

In particular, a model K of $T_{\mathrm{vod},\omega\mathrm{inc,h}} \cup \mathrm{RCF}$ is such that $\bar{K} \cong \mathrm{Fix}(\sigma)$ is a model of RCF and Γ_K is a $\mathbb{Q}[t]$-divisible module. (Note that the scheme of axioms we added, is equivalent to the fact that Γ_K is a divisible $\mathbb{Q}(t)$-module, see the proof of case (2) in the above Proposition).

Corollary 2.19 *The theory $T^{\mathrm{ec}}_{\mathrm{vod},\omega\mathrm{inc,h}}$ is the model-companion of $T_{\mathrm{vod},\omega\mathrm{inc,h}}$.*

P r o o f. We embed K in the power series field $\bar{K}((\Gamma_K))$ and we do it by sending $\mathrm{Fix}_K(\sigma)$ to $\bar{K}$. Let $\gamma \in \Gamma_K$. Note that there exists $k \in K$ such that $v(k) = \gamma$; the action of t on $v(k)$ was defined as $v(k^\sigma) := v(k) \cdot t$. So, we extend σ on $\bar{K}((\Gamma_K))$ by defining $\sigma(\sum_\gamma k_\gamma \cdot x^\gamma) := \sum_\gamma k_\gamma \cdot x^{\gamma \cdot t}$. Let us denote by Γ_K^{div} the divisible closure of Γ_K as a $\mathbb{Q}(t)$-module.

Then we embed $\bar{K}((\Gamma_K))$ into $\bar{K}^{\mathrm{rc}}((\Gamma_K^{\mathrm{div}}))$, which is a complete valued field and so a model of σ-Hensel Lemma by Lemma 2.9. By the preceding Proposition, this ordered valued difference field is existentially closed.

The fact that $T^{\mathrm{ec}}_{\mathrm{vod},\omega\mathrm{inc,h}}$ is model-complete follows from Robinson's criterium for model-completeness, the preceding Proposition and the facts that T_{do} admits quantifier elimination as well as RCF. □

3 Difference lattice-ordered commutative rings

First, we will recall a few facts on lattice-ordered commutative rings (in short ℓ-rings) ([2]). An ℓ-ring R is a commutative ring with two additional operations $\wedge$ and $\vee$ such that

1. $(R, \wedge, \vee)$ is a lattice,
2. $\forall a\, \forall b\, \forall c\, (a \leq b \rightarrow a + c \leq b + c)$,
3. $\forall a\, \forall b\, \forall c\, (a \leq b \;\&\; c \geq 0 \rightarrow a \cdot c \leq b \cdot c)$,

where $\leq$ is the lattice order, i.e., $a \leq b$ if and only if $a \wedge b = a$. In this section, R will always denote such a ring. Let $\mathcal{L}_\ell = \mathcal{L}_{\mathrm{rings}} \cup \{\wedge, \vee\}$ the language of ℓ-rings.

An ℓ-ideal I of R is a (ring) ideal which has the following property: $(\forall a \in I)(\forall x \in R)(|x| \leq |a| \rightarrow x \in I)$. In an ℓ-ring any finitely generated ℓ-ideal is principal (see [2, Corollary 8.2.9]). An ℓ-ideal I of R is *irreducible* if whenever $a, b \in R$ are such that $\langle a \rangle \cap \langle b \rangle \subset I$, then $a \in I$ or $b \in I$.

An *f-ring* is an ℓ-ring where $(\forall a, b, c > 0)\, (a \wedge b = 0 \rightarrow a \wedge (b \cdot c) = 0 \;\&\; a \wedge (c \cdot b) = 0)$. If R is an f-ring, then $\mathrm{Spec}_\ell(R)$ denotes the set of irreducible ℓ-ideals of R with the spectral topology, i.e. an open set is the set of ideals which do not contain a given element ([2, Chapter 10]). An f-ring without nilpotent elements can be represented as a subdirect product of totally ordered integral domains (see [2, Corollary 9.2.5]), and in von Neumann regular f-rings any irreducible ideal contains no non trivial idempotents and so the quotient of such a ring to an irreducible ℓ-ideal is a field (see [2, Chapter 10]).

Finally, a *real-closed* commutative von Neumann regular f-ring is a von Neumann commutative regular f-ring where every monic polynomial of odd order has a root and every positive element is a square.

A.J. Macintyre proved that the theory T_{f} of commutative f-rings with no nonzero nilpotent elements has a model-companion T_{vrc}, namely the theory of commutative real-closed von Neumann regular f-rings without minimal idempotents (see [19]).

Let us consider now the difference ℓ-rings. Recall that we have obtained undecidability results for any difference ℓ-ring (R, σ) when the automorphism σ has an infinite orbit on the set of its maximal ℓ-ideals ([15, Corollary 8.1]).

From now on, (R, σ) will denote a von Neumann regular difference f-ring where the automorphism σ fixes $\mathrm{Spec}_\ell(R)$. In particular, σ induces an automorphism on each quotient of the form $R_x := R/x$, where x is in $\mathrm{Spec}_\ell(R)$ and x is left invariant by σ.

In a commutative von Neumann regular ring, it is often convenient to add a new unary function *, a pseudo-inverse, sending an element a to the element b such that $a \cdot (a \cdot b) = a$ and $b \cdot (b \cdot a) = b$; notice that $a \cdot b$ is an idempotent, and we will call it the *support of* a. Let $\mathcal{L}_{\ell,*} := \mathcal{L}_\ell \cup \{^*\}$ and $\mathcal{L}_{\text{rings},*} := \mathcal{L}_{\text{rings}} \cup \{^*\}$.

In the class of difference von Neumann regular f-rings, we look for a result analogous to the following one which holds in the class of commutative von Neumann regular rings. In [15, Proposition 6.8] we showed that any such difference ring can be embedded in a model of $T_{\text{atm},1,\sigma}$, where $T_{\text{atm},1,\sigma}$ is the $\mathcal{L}_{\text{rings},\sigma}$-theory expressing the following properties of a model (R, σ):

1. R is a commutative von Neumann regular difference ring without minimal idempotents where any monic polynomial has a root.

2. The Boolean algebra of idempotents is included in the set of fixed points of σ.

3. For each idempotent e, for every absolutely irreducible variety U on e, for every variety $V \subset U \times \sigma(U)$ projecting generically onto U and $\sigma(U)$, and for every algebraic set W properly contained in V, there is $a \in U(R)$ such that $(a, \sigma(a)) \in V - W$.

Definition 3.1 Let R be a commutative von Neumann regular difference f-ring and let $S \models T_{\text{atm},1,\sigma}$ extending R as a $\mathcal{L}_\sigma \cup \{^*\}$-structure and which is $|R|^+$-saturated. Let $X = \text{Spec}_\ell(R)$. A subset U of R^n is said to be an *algebraic variety on an idempotent* e if it is the set of all solutions of a finite conjunction of polynomial equations where the support of each non zero coefficient is equal to the idempotent e.

We will denote by $U(x)$ the subset of elements $\bar{s}$ in ${R_x}^n$ such that there exists $\bar{r} \in R^n \cap U$ such that $\bar{s} = \bar{r}(x)$. Recall that the property for a variety U for being irreducible (respectively absolutely irreducible) is a first-order property of the set of coefficients, which can be expressed by a quantifier-free formula (see [9]). We define the property of being *irreducible* (respectively *absolutely irreducible*) for a variety U on an idempotent e as the property that $U(x)$ is irreducible (respectively absolutely irreducible) for each $x \in e$. This last property can be expressed in $\mathcal{L}_\sigma$ by a quantifier-free formula in the coefficients and the idempotent e. We will denote by $\sigma(U)$ the set $\{\sigma(\bar{r}) \cdot e : \bar{r} \in U\}$. Let U be an irreducible variety on e and let V be a variety included in $U \times \sigma(U)$. Then V projects generically onto U if, for every $x \in e$, $V(x)$ projects generically onto $U(x)$.

Let V be a variety defined on e. We will denote by $I_R(V)$ the set of difference polynomials in coefficients in R which annihilate every tuple in $V(R)$.

Now, we will try to describe a class of von Neumann regular ℓ-rings such that each quotient to a maximal ℓ-ideal belongs to $\mathcal{C}_{\text{pra}}$ (see Definition 1.10).

Definition 3.2 Let $\mathcal{C}_{\text{vrca}}$ be the class of difference ℓ-ring (R, σ) such that $R \models T_{\text{vrc}}$ and for each idempotent $e \in R$, for every absolutely irreducible variety U on e, and for every variety $V \subset U \times \sigma(U)$ projecting generically onto U and $\sigma(U)$ such that for any finite set I and $p_i[X, Y] \in R[X, Y]$, $i \in I$, we have that

$$\sum p_i^2 \in I_R(V) \to \bigwedge_{i \in I} p_i \in I_R(V).$$

Then there is $a \in U(R)$ such that $(a, \sigma(a)) \in V$.

Using the construction of bounded Boolean powers, one can exhibit elements of $\mathcal{C}_{\text{vrca}}$ (see [4, p. 274]). Let X_0 be a Cantor space, i.e., a Boolean space without isolated points. Let (F, σ) be an element of $\mathcal{C}_{\text{pra}}$, let $\Gamma_a(X_0, F)$ be the set of locally constant functions from X_0 to F. Any element $\Gamma_a(X_0, F)$ is of the form $\sum_{i \in I} e_i \cdot f_i$, where I is a finite set, $f_i \in F$ and e_i is a characteristic function of a clopen subset of X_0. Then $\Gamma_a(X_0, F)$ belongs to $\mathcal{C}_{\text{vrca}}$.

Proposition 3.3 *Let* R, S *be elements of* $\mathcal{C}_{\text{vrca}}$ *such that* $R \subset S$*, then* $R \subset_{\text{ec}} S$.

Proof. Let $\exists x_1 \dots \exists x_n \, \varphi(x_1, \dots, x_n, \bar{a})$ be an existential formula with parameters $\bar{a} \subset R$, where φ is a conjunction of basic formulas. As commutative von Neumann regular f-rings, R and S are Boolean products of totally ordered fields. So, $S \models \varphi(\bar{u}, \bar{a})$ if and only if a conjunction of the form below holds, letting $\bar{u}_\sigma := (\bar{u}, \dots, \bar{u}^{\sigma^n})$:

$$(\forall x \in X(S)) \;\; S_x \models \bigwedge_i f_i(\bar{u}_\sigma, \bar{a}(x)) \geq 0 \;\&\; g(\bar{u}_\sigma, \bar{a}(x)) = 0,$$
$$\bigwedge_{j \in J} (\exists y_j \in X(S)) \;\; S_{y_j} \models \bigwedge_i f_i(\bar{u}_\sigma, \bar{a}(y_j)) \geq 0 \;\&\; g(\bar{u}_\sigma, \bar{a}(y_j)) = 0 \;\&\; \bigwedge_{k \in I_j} h_k(\bar{u}_\sigma, \bar{a}(y_j)) \neq 0,$$

where J is finite, $f_i, g, h_k \in \mathbb{Z}[X, Y]_\sigma$.

Then we replace that system by a disjunction of systems (of the same form), where we may assume that all the points y_j are distinct. By [4, Lemma 9.6 (c)], given $y_j \in X(S)$, there exists $x_j \in X(R)$ such that R_{x_j} embeds in S_{y_j}. By Lemma 1.11, since both belong to $\mathcal{C}_{\rm pra}$, $R_{x_j} \subset_{\rm ec} S_{y_j}$. So, we may find $\bar{u}_j \in R$ with disjoint supports e_j such that $\prod_{k\in I_j} h_k(\bar{u}_{j_\sigma}, \bar{a})^* \cdot e_j = e_j$ and $\bigwedge_i f_i(\bar{u}_{j_\sigma}, \bar{a})^* \cdot e_j \geq 0$ & $g(\bar{u}_{j_\sigma}, \bar{a})^* \cdot e_j = 0$. Then we consider the elements $x \in X' := X(R) - (\bigcup_{j\in J} e_j)$ and we use the fact that for any $x \in \mathrm{Spec}_\ell(R)$, there is an element y of $\mathrm{Spec}_\ell(S)$ such that R_x embeds in S_y ([4, Lemma 9.10]). But both R_x and S_y belong to $\mathcal{C}_{\rm pra}$ and so $R_x \subset_{\rm ec} S_y$. So, for each $x \in X'$ there will an idempotent e_x disjoint from $\bigcup_{j\in J} e_j$ and a tuple $\bar{u}_x \in R$ such that $\bigwedge_i f_i(\bar{u}_{x_\sigma}, \bar{a})^* \cdot e_x \geq 0$ & $g(\bar{u}_{x_\sigma}, \bar{a})^* \cdot e_x = 0$. From the covering of the space $X(R)$ with the idempotents e_x and e_j, $j \in J$, we extract a finite disjoint subcovering e_j, $j \in J$, and e'_x, $x \in X_0$ with $e'_x \cdot e_x = e'_x$ and consider the tuple $\bar{r} := \sum_j \bar{u}_j \cdot e_j + \sum_{x\in X_0} \bar{u}_x \cdot e'_x$. It belongs to R and satisfies the formula $\varphi(\bar{r}, \bar{a})$. □

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