Inflating Branes inside hyper-spherically symmetric defects

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Abstract

Static and inflating branes residing at the center of a hyper-spherical symmetric defect are considered in $4+n$-dimensions with a non zero bulk cosmological constant. Several vacuum solutions can be constructed explicitely when the bulk cosmological constant and cosmological constant on the brane are non zero. We consider as hyper-spherical symmetric defect a global monopole for generic values of n and a local monopole for $n = 3$. We construct new solutions which are regular or periodic in the bulk radial coordinate.

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I. INTRODUCTION

The basic idea of brane world models [1] is that our universe is represented by a (3+1)-dimensional subspace (a three-brane) embedded in a higher dimensional space-time (the bulk). While string theory has compact extra dimensions which are typically of the order of the Planck scale, brane world models can have large or even infinite extra dimensions [2]. The reason is that all matter fields are assumed to be confined to the brane, while only gravity lives in all dimensions. Since Newton's law, however, is well tested down to the mm-scale, gravity has to be localised "well enough" to the brane.

One possibility discussed extensively in recent years [3] assumes a topological defect to reside in the bulk and the brane to be localised at the center of this defect. In that case the cosmological constant in the bulk $\Lambda_{4+n} \neq 0$ and the localisation of gravity on the brane is achieved by a fine tuning of Λ_{4+n} with the other coupling constants appearing in the theory. Similar results have been developed recently [4] without a cosmological constant but including both a global and a local monopole. For large direct interaction of the two sectors of the theory, the global monopole disappears from the system and "leaves behind" a negative cosmological constant. In all the above mentioned cases, the brane is a Minkowski brane.

In [5, 6, 7] brane worlds have been considered from another point of view: the bulk cosmological constant is zero, while the branes are inflating, i.e. possess a physical cosmological constant. This is of interest since experimental data points to the fact that our universe contains a (small) cosmological constant.

In this paper, we extend the ideas of [5, 6, 7] by allowing the bulk cosmological constant to be non-zero. We thus have two independant cosmological constants, one in the bulk and one on the (3+1)-dimensional brane. The underlying field theory describing the topological defect is considered to be the Goldstone model (a global monopole) for n extra dimensions and the SO(3) gauge-Higgs theory describing non-abelian monopoles in the case of three extra dimensions. We find that the existing solutions of [5, 6, 7] are smoothly deformed by the bulk cosmological constant. In addition, we show numerically that new types of regular solutions exist. These solutions have periodic matter and metric functions and require a fine tuning of the two cosmological constants and, to our knowledge, provide a new class of regular space-times where a compactification of the extra dimensions occurs naturally.

The model and the equations are presented is Section II. Explicit vacuum solutions are considered in Section III. Melvin-type solutions are discussed in Section IV. Section V and VI, respectively, describes the solutions when a global monopole and local monopole is present in the bulk. We show that regular solutions, periodic with respect to the radial coordinate associated to the bulk exist. This contrasts with the solutions obtained in [8] where theories involving two extra-dimensions are investigated and an Abelian-Higgs model leads to vortex located in the bulk. In this theory, solutions become periodic only far enough from the vortex core where the matter fields have reached their vacuum values.

II. ACTION PRINCIPLE FOR AN n-DIMENSIONAL BRANE

The $(4+n)$-dimensional action reads:

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g}(R - 2\kappa^2 \Lambda_{n+4}) d^{4+n}x + S_{top} \tag{1}$$

where S_{top} is the action describing the topological defect residing in the bulk, it depends on the specific model studied and will be given in explicit form later in the paper. $\kappa^2 = \frac{1}{M_*^{2+n}}$ with M_* the $4+n$-dimensional Planck mass and Λ_{n+4} denotes the bulk cosmological constant.

The general form of the non-factorisable metric that we consider in this paper is given by

$$ds^2 = M(r)^2 ds_4^2 + dr^2 + L(r)^2 d\Omega_{n-1}^2 \tag{2}$$

where ds_4^2 describes the metric of the brane. The transverse space has rotational invariance, r describes the bulk radial coordinate, while $d\Omega_{n-1}$ denotes the line element associated with the $n-1$ angles θ_i, $i = 1, .., n-1$ of the transverse space.

The corresponding Einstein equations read [5]:

$$\begin{aligned} G_\mu^\mu &= -\frac{1}{4}\frac{R^{(4)}}{M^2} + 3\frac{M''}{M} + 3\frac{M'^2}{M^2} + 3(n-1)\frac{M'L'}{ML} \\ &+ (n-1)\frac{L''}{L} + \frac{(n-2)(n-1)}{2}\left(\frac{L'^2}{L^2} - \frac{1}{L^2}\right) = \beta(T_\mu^\mu - \gamma) \quad , \quad \mu = 0, \dots, 3 \end{aligned} \tag{3}$$

$$G_r^r = -\frac{1}{2}\frac{R^{(4)}}{M^2} + 6\frac{M'^2}{M^2} + 4(n-1)\frac{M'L'}{ML} + \frac{(n-2)(n-1)}{2}\left(\frac{L'^2}{L^2} - \frac{1}{L^2}\right) = \beta(T_r^r - \gamma) \tag{4}$$

$$\begin{aligned} G_{\theta_i}^{\theta_i} &= -\frac{1}{2}\frac{R^{(4)}}{M^2} + 4\frac{M''}{M} + 6\frac{M'^2}{M^2} + 4(n-2)\frac{M'L'}{ML} + (n-2)\frac{L''}{L} \\ &+ \frac{(n-2)(n-3)}{2}\left(\frac{L'^2}{L^2} - \frac{1}{L^2}\right) = \beta(T_{\theta_i}^{\theta_i} - \gamma) \quad , \quad i = 1, \dots, n-1 \end{aligned} \tag{5}$$

where $\Lambda_{n+4} \equiv \gamma$ and $\kappa^2 \equiv \beta$. The energy momentum tensor is given by

$$T_{MN} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{top}}{\delta g^{MN}} \ , \quad M, N = 0, .., n+4 \tag{6}$$

and $R^{(4)}$ denotes the Ricci scalar corresponding to the metric on the brane, which we choose to be

$$ds_4^2 = -dt^2 + e^{2Ht}((dx^1)^2 + (dx^2)^2 + (dx^3)^2) \tag{7}$$

such that $R^{(4)} = 12H^2$ with $H = \sqrt{\frac{\Lambda_{phys}}{3}}$, where Λ_{phys} denotes the physical cosmological constant on the brane.

III. VACUUM SOLUTIONS

Vacuum solutions correspond to $T_M^N = 0$. According to the sign of the $4+n$-dimensional cosmological constant they have positive, null or negative curvature. The result is very similar to the 4-dimensional case. In the present case we emphasize a more general situation where 4-dimensional slices of the space have an (anti)-de Sitter geometry. They can induce angular deficits in the n-dimensional subspace. Because the equations do not explicitely depend on the radial variable r the solutions given below can be arbitrality translated in r.

A. $\Lambda_{n+4} > 0$

The Einstein equations above possess explicit solutions in terms of trigonometric functions where L, M have the same period. The solution reads:

$$M(r) = \frac{H}{\omega}\sqrt{\frac{3}{n+2}}\sin\omega r \quad , \quad L(r) = \frac{1}{\omega}\sqrt{\frac{n-2}{n+2}}\sin\omega r \quad , \quad \omega^2 = \frac{2\beta\gamma}{12+(n-1)(n+6)} \tag{8}$$

Note that this solution depends crucially on the two cosmological constants and that the solution exists only for $n > 2$. Indeed, in the 6-dimensional case where $n = 2$, the equation for M decouples [8] and leads to $L \propto M'$, which is not compatible with the solution above where the metric functions M, L are characterized by the same phase. In the case $n = 1$, the function $L(r)$ is not defined.

Note that the solution is not regular since for $r = k\pi/\omega$ (k integer) we have $M(r) = L(r) = 0$ and the Ricci scalar

$$R = -12\frac{H^2}{M^2} + 12\frac{M'^2}{M^2} + 8\frac{M''}{M} + 8(n-1)\frac{M'L'}{ML} + 2(n-1)\frac{L''}{L} + (n-1)(n-2)\left(\frac{L'^2}{L^2} - \frac{1}{L^2}\right) \tag{9}$$

is obvioulsly singular for these values of r.

Let us finally emphasize that the warp factor M is directly proportional to the parameter H, both are related to the brane.

The solution has a natural geometric interpretation : they describe the surface of an $n+1$ dimensional sphere in the extra dimensions and presents an angular deficit relative to the angles θ_i. More precisely, with the line element in the extra dimensions

$$ds_n^2 = dr^2 + \frac{n-2}{n+2}\frac{1}{\omega^2}\sin^2(\omega r)d\Omega_{n-1}^2 \tag{10}$$

and setting $\Theta = \omega r$, we find

$$ds_n = \frac{1}{\omega^2}\left(d\Theta^2 + \frac{n-2}{n+2}\sin^2(\Theta)d\Omega_{n-1}^2\right) \ . \tag{11}$$

The solid angle corresponding to a fixed value of Θ presents an angular deficit

$$\sin(\Theta)\int\sqrt{\frac{n-2}{n+2}}d\Omega_{n-1} \tag{12}$$

although $\sin(\Theta)\int d\Omega_{n-1}$ is expected. It represents indeed a deficit since

$$\frac{n-2}{n+2} < 1 \ \ , \ \ \forall\, n \ . \tag{13}$$

In addition, the radius of compactification is found to be the parameter $1/\omega$ defined in (8).

B. $\Lambda_{4+n} = 0$

In the case of vanishing bulk cosmological constant, the solutions turn out to be linear functions or r, given by

$$M(r) = \sqrt{\frac{3H^2}{n+2}}\, r \ \ , \ \ L(r) = \sqrt{\frac{n-2}{n+2}}\, r \ . \tag{14}$$

This solution does not fulfill the regularity condition at $r=0$ and is singular at the origin (e.g. the scalar curvature is singular). Moreover, this solution describes a flat bulk with an angular deficit in the extra dimensions.

C. $\Lambda_{n+4} < 0$

Finally, the case of a negative bulk cosmological constant we find:

$$M(r) = \frac{H}{\tilde\omega}\sqrt{\frac{3}{n+2}}\sinh\tilde\omega r \ \ , \ \ L(r) = \frac{1}{\tilde\omega}\sqrt{\frac{n-2}{n+2}}\sinh\tilde\omega r \ \ , \ \ \tilde\omega^2 \equiv \frac{-2\beta\gamma}{12+(n-1)(n+6)} \tag{15}$$

This solution can also be obtained by a suitable analytic continuation of (8).

IV. MELVIN-TYPE UNIVERSE IN n DIMENSIONS

In this section, we look for the higher dimensional analogues of the well know 4-dimensional Melvin space-time [9], i.e. we look for solutions of the form:

$$M(r) = M_0(r-r_0)^\mu \quad , \quad L(r) = L_0(r-r_0)^\lambda \tag{16}$$

where M_0, L_0, r_0, μ, λ are constants to be determined.
In the following, we will distinguish the cases where solutions develop the above behaviour for $r \to \infty$ (asymptotic solution) and the case where the solutions develop a singularity in the neighbourhood of $r = r_0$ (see [5] for the $n = 3$ case), respectively.

A. Asymptotic solution

Because of the occurence of non-homogeneous terms (e.g. $1/L^2$ and Λ_{n+4}/M^2) in the Einstein equations, power-like solutions of the form above cannot be exact for generic values of H, Λ_{n+4}, n. However, we will see that they exist as asymptotic solutions and appear as "critical" solutions in the case of global and local monopoles (see Section VI). Let us for a moment neglect the non-homogeneous terms in the Einstein equations. Then inserting the power law decay above leads to the following conditions for the exponants μ, λ:

$$3\mu(\mu-1) + 3\mu^2 + 3(n-1)\mu\lambda + (n-1)\lambda(\lambda-1) + \frac{(n-1)(n-2)}{2}\lambda^2 = 0 \tag{17}$$

$$6\mu^2 + 4(n-1)\mu\lambda + \frac{(n-1)(n-2)}{2}\lambda^2 = 0 \tag{18}$$

$$4\mu(\mu-1) + 6\mu^2 + 4(n-2)\mu\lambda + (n-2)\lambda(\lambda-1) + \frac{(n-2)(n-3)}{2}\lambda^2 = 0 \ . \tag{19}$$

The solutions then read:

$$\mu = \frac{2 \pm \sqrt{(n+2)(n-1)}}{2(n+3)} \quad , \quad \lambda = \frac{(n-1) \mp 2\sqrt{(n+2)(n-1)}}{(n-1)(n+3)} \tag{20}$$

Note, however, that with these exponents, it is not justified to neglect the inhomogeneous terms $\propto 1/L^2$ and $\propto 1/M^2$ except is the particular case of the critical solutions (see Section V).

B. Singular solutions

If we want to interpret the functions (16),(20) as the dominant terms of a solution of the vacuum Einstein equations which are singular in the limit $r \to r_0$, the exponents should be such that $\mu < 1$, $\lambda < 1$. It turns out that these conditions are fulfilled for *both* values of the sign $\pm$ in (20).

The standard Kasner conditions read:

$$4\mu^2 + (n-1)\lambda^2 = 1 \quad , \quad 4\mu + (n-1)\lambda = 1 \ . \tag{21}$$

It turns out that only the *linear* relation is fulfilled. This means that

$$g^m_{n+4} = r^{4-2n} g^p_{n+4} \tag{22}$$

where g^m_{n+4} is the determinant of the metric corresponding to an n-dimensional Melvin space while g^p_{n+4} represents the determinant corresponding to the n-dimensional flat space.

V. GLOBAL MONOPOLES

Global monopole solutions occurs in the Goldstone model describing n scalar fields in a field theory globally invariant under the $O(n)$ transformations. The symmetry is spontaneously broken by a potential. In the present context, the Goldstone model and the corresponding scalar fields are formulated with respect to the n extra dimensions :

$$S_{top} = \int \left((\partial_A \Phi)^\dagger (\partial^A \Phi) - \frac{\alpha}{4}(\Phi^\dagger \Phi - v^2)^2 \right) d^7x \tag{23}$$

where the n scalar fields $\Phi = (\phi^1, \ldots, \phi^n)$ form a fundamental representation of the group $O(n)$. α is the self-coupling of the potential, v the vacuum expectation value of the scalar field.

We use the Ansatz:

$$\phi^i = \phi(r)\xi^i/r \ , \tag{24}$$

where the ξ^i denote the cartesian coordinates representing the extra dimensions. Correspondingly, the energy momentum tensor has only diagonal components given by ([7]):

$$T^\mu_\mu = \phi'^2 + \frac{(n-1)\phi^2}{2L^2} + \frac{\alpha}{4}(\phi^2 - v^2)^2 \tag{25}$$

$$T_r^r = -\phi'^2 + \frac{(n-1)\phi^2}{2L^2} + \frac{\alpha}{4}(\phi^2 - v^2)^2 \tag{26}$$

$$T_{\theta_i}^{\theta_i} = \phi'^2 + \frac{(n-3)\phi^2}{2L^2} + \frac{\alpha}{4}(\phi^2 - v^2)^2 \tag{27}$$

where $\phi' \equiv d\phi/dr$. The equation corresponding to the Goldstone field reads:

$$\phi'' + (n-1)(\frac{L'}{L} + \frac{4}{n-1}\frac{M'}{M})\phi' - \frac{1}{L^2}\phi = \alpha\phi(\phi^2 - v^2) \tag{28}$$

The appropriate boundary conditions read:

$$M(0) = 1 \quad , \quad M'(0) = 0 \quad , \quad L(0) = 0 \quad , \quad L'(0) = 1 \tag{29}$$

for the metric functions. In the case when the radial variable can be extended to $r = \infty$, the usual boundary conditions for the scalar field function are

$$\phi(0) = 0 \quad , \quad \phi(\infty) = v \ . \tag{30}$$

However, the presence of a cosmological constant can lead to a cosmological horizon at $r = r_c$. In such case, an appropriate boundary condition for $\phi(r_c)$ has to be imposed; this will be discussed in due course.

The expressions of the energy momentum tensor contain terms of the form $1/L^2$ which also appear in the Einstein tensor. If the gravitational constant κ is chosen such that

$$\kappa v = \sqrt{n-2} \ \ , \tag{31}$$

the two inhomogeneous terms cancel. This value determines the so-called "critical monopole". In this case, (and assuming $H = \Lambda_{n+4} = 0$ in addition) the asymptotic Melvin solutions are just solutions of the Einstein equations [5, 7].

A. Sub-critical monopoles

The case $\kappa\eta < \sqrt{n-2}$ corresponds to the case of subcritical monopoles [7]. The vacuum solutions for which the functions M, L go asymptotically to infinity (i.e. corresponding to $\Lambda_{n+4} \leq 0$) are such that the term $\phi/L^2 \to 0$. The metric function $M(r)$ remains the same irrespectively of the presence of a global monopole. The function $L(r)$ must be renormalized according to

$$L(r)_m = \frac{L(r)_v}{\sqrt{1 - \frac{(\kappa v)^2}{n-2}}} \tag{32}$$

Here L_v corresponds to the function of the vacuum solution while L_m corresponds to the solution in the presence of the monopole.

In the case $\Lambda_{n+4} > 0$ the arguments above do not apply because the terms ϕ/L^2 cannot be neglected. However, the profiles of the metric functions M, L remain very close to those of the vacuum solution. For larger values of r, the metric become singular at some finite value of r. The singularity is of Melvin type and is of the same nature as in the case of local monopoles (see discussion below).

B. Mirror symmetric solutions

The coupled system of equations possesses several symmetries, namely invariance under translations of the radial variable r and invariance under the reflexions $r \to -r$ and $\phi \to -\phi$. These symmetries suggest that solutions which are invariant under suitable combinations of the symmetries could exist . In the case of vacuum solutions, the solutions corresponding to $\Lambda_{n+4} > 0$ possess such a symmetry. The most natural combination of the symmetries suggests to look in particular for solutions with the following properties

$$L(r_0 - r) = L(r_0 + r) \quad , \quad M(r_0 - r) = M(r_0 + r) \quad , \quad \phi(r_0 - r) = \epsilon\phi(r_0 + r) \quad , \quad \epsilon = \pm 1 \quad (33)$$

where the reflexion point r_0 depends on the various coupling constants. The existence of solutions presenting one of the above symmetries can be analysed by solving the equations supplemented by the following boundary conditions at $r = r_0$:

$$L'(r_0) = 0 \ , \ M'(r_0) = 0 \ , \ \phi(r_0) = 0 \ \text{ if } \epsilon = -1 \quad , \quad \phi'(r_0) = 0 \ \text{ if } \epsilon = 1 \ . \qquad (34)$$

These conditions complete the ones given above for $r = 0$ and allow for a numerical study of the equations (no explicit solution was found for $\phi \neq 0$). Our numerical analysis suggests that (i) solutions corresponding to $\phi(r_0 - r) = \phi(r_0 + r)$ do not seem to exist. In fact we were able to construct numerically such configurations but they do not persist when increasing the accurancy, (ii) solutions obeying $\phi(r_0 - r) = -\phi(r_0 + r)$ do indeed exist for peculiar values of the coupling constants.

The existence of these "odd" solutions can be related to the fact that, in the neighbourhood of the symmetric point r_0, the Goldstone field equation can be put into the form

$$\phi'' - \alpha\phi^3 + (\alpha v^2 - \frac{1}{L(r_0)^2})\phi \sim 0 \qquad (35)$$

where we used the fact that $M'(r_0) = L'(r_0) = 0$. This simplified version of the Goldstone equation is identical to the kink equation provided $L(r_0)v > 1$ which turns out to hold true in the region of parameters that we have explored. Kink-like solutions can therefore be expected.

Two solutions of this type are presented in Figs. 1 and 2. These solutions are similar to the so called "Bag of Gold"- solutions discovered in the 4-dimensional Einstein-Yang-Mills equations with a positive cosmological constant [10]. Similar phenomena in 4-dimensional space-time were observed in [11] and more recently in [12]. However, to our knowledge, it is the first time that compact solutions relative to the spatial extra dimensions are constructed.

It has to be stressed that these type of solutions exist only for peculiar values of the cosmological constants Λ_{n+4}, H once the coupling constants α, β are fixed. Setting $\alpha = 0.2$, $\beta = 0.3$, the relations between Λ_{n+4}, H and r_c allowing for mirror symmetric solutions are given in Fig. 3. Our results suggest that several branches of solutions could exist. We were able to find two non trivially different branches, see Fig. 3. The construction of the second branch is however involved numerically.

It is clear that these solutions can be extended symetrically for $r \in [r_c, 2r_c]$ and further for $r \in [2r_c, 4r_c]$. This leads naturally to periodic solutions on $[0, 4r_c]$. The corresponding space-times correspond to hyperspheres with angular deficit.

VI. LOCAL MONOPOLE FOR $n = 3$

It is well know that monopoles exist in some spontaneously broken non-abelian gauge theories. The simplest case is the Georgi-Glashow model with gauge group $SO(3)$ and a Higgs triplet. In the present context, the action density reads

$$S_{top} = \int \left((D_A\Phi^a)(D^A\Phi^a) - \frac{1}{4}F^a_{AB}F^{a,AB} - \frac{\alpha}{4e^2}(\Phi^a\Phi^a - v^2)^2 \right) d^7x \qquad (36)$$

where $A, B = 0, 1, \ldots, 6$, $a = 1, 2, 3$ and using the usual definitions for the covariant derivative and fields strenghts :

$$D_M\Phi^a = \partial_M\Phi^a + e\epsilon^{abc}W^b_M\Phi^c \quad , \quad F^a_{MN} = \partial_M W^a_N - \partial_N W^a_M + e\epsilon^{abc}W^b_M W^c_N \qquad (37)$$

Along with [6], we use a spherically symmetric ansatz for the gauge and Higgs fields :

$$W_\mu^a = 0 \quad , \quad W_i^a = (1-w(r))\epsilon_{a\ (i-3)(j-3)}\frac{x^j}{er^2} \tag{38}$$

$$\Phi^a = \phi(r)\frac{x^{3+a}}{r} \quad , \quad a = 1,2,3 \tag{39}$$

leading to a energy momentum tensor with non-vanishing components:

$$\begin{aligned} T_\mu^\mu &= \frac{(\phi')^2}{2} + \frac{\phi^2 w^2}{L^2} + \frac{1}{e^2L^2}(\frac{(1-w^2)^2}{2L^2} + (w')^2) + \frac{\alpha}{4}(\phi^2 - v^2)^2 \\ T_r^r &= -\frac{(\phi')^2}{2} + \frac{\phi^2 w^2}{L^2} + \frac{1}{e^2L^2}(\frac{(1-w^2)^2}{2L^2} - (w')^2) + \frac{\alpha}{4}(\phi^2 - 1)^2 \\ T_{\theta_i}^{\theta_i} &= \frac{(\phi')^2}{2} - \frac{(1-w^2)^2}{2e^2L^4} + \frac{\alpha}{4}(\phi^2 - v^2)^2 \ . \end{aligned} \tag{40}$$

The classical equations can be obtained easily by substitution of the expressions above in (3)-(5). In the absence of a bulk comological constant, this model was studied in [6]. In this paper, the equations are solved with the usual boundary conditions for the functions $w(r), \phi(r)$: $\phi(0) = 0$, $w(0) = 1$ for regularity at the origin and $\phi(r \to \infty) = v$, $w(r \to \infty) = 0$ outside the monopole core.

Here we will analyze the influence of the bulk cosmological constant on the solutions. Again, the presence of cosmological constant will force us to impose conditions for the matter fields at an intermediate value $r = r_c$. In the discussion and the figures the following rescaling of the parameters: $\beta = e^2v^2\kappa^2$, $\gamma = \Lambda_{4+n}/e^2v^2$, $H/ev \to H$ are used.

In general, the three types of vacuum solutions presented in Section IV are deformed by the presence of the local monopole on the brane. The principle deformation of the metric fields for $\Lambda_{n+4} = 0, \pm 0.01$ are sketched in Fig.4 with $H = 0.15$. We will now discuss the different solutions more qualitatively.

A. $\Lambda_{n+4} = 0$

In this case, the solutions have been analyzed in details in [6]. Far from the monopole core, the metric functions behave linearly:

$$M(x) = \sqrt{\frac{3}{5}}Hr + m_0 + \frac{m_1}{r} \quad , \quad L(x) = \sqrt{\frac{1}{5}}r + l_0 + \frac{l_1}{r} \ . \tag{41}$$

These approach the corresponding vacuum solutions of Section 4 for $r \to 0$.

B. $\Lambda_{n+4} < 0$

For $\Lambda_{n+4} < 0$ our numerical analysis reveals that there are regular solutions which obey asymptotically the form

$$M(r) = M_0 \sinh \mu r \quad , \quad L(r) = L_0 \sinh \mu r \quad , \quad \mu = \sqrt{\frac{-\beta\gamma}{15}} \ . \tag{42}$$

These solutions have thus exponential behaviour. Two comments are in order here:

- These solutions make sense only with the positive square root of μ. Indeed, the non-homogeneous terms $1/L^2$ and $1/L^4$ occuring in the equations can then be neglected asymptotically. With the negative square root, these non-homogeneous terms cannot be neglected.

- These solutions persist in the $\Lambda_{4+n} = 0$ limit, forming another branch of solutions with respect to the one studied in [3], where the emphasis is set on warped solutions. The profile of such a solution is presented in Fig.5 for $H = 0$ and $\Lambda_{n+4} = -0.01$. In particular, we can see the deviation of the functions M, L from their asymptotic behaviour.

C. $\Lambda_{n+4} > 0$

We studied the solution for $\Lambda_{n+4} > 0$ and found qualitatively how the vacuum trigonometric solutions are deformed by the matter fields. This is illustrated in Fig.6. As expected, we observe that the monopole regularizes the singular vacuum solutions such that the metric fields become regular on the brane $r = 0$. The functions M, L reach their maximum value when the matter fields have already reached their asymptotic values. The mirror symmetry of the vacuum solutions is broken, in particular the numerical analysis reveals that the two functions M, L do not reach their maximum exactly at the same value of r. It is likely that mirror symmetric solution exist in this case as well for tuned values of the constants H, Λ_{n+4}. Mirror symmetric solutions will be discussed in the next section.

The challenging question is to understand how these solutions (regularized at the origin) behave in the neighbourhood of the first period of the associated periodic solution of the vacuum equation. This is illustrated in Fig. 7. This clearly indicates that the solution develops a singularity at some $r = r_c$, for $\gamma = 0.07, \Lambda_{n+4} = 0.005$ we have $r_c \approx 291.7$. A detailed analysis of the numerical solution for $r \propto r_c$ reveals that the singularity is of Melvin type discussed in the previous section. For generic values of the inflating parameters and positive bulk cosmological constant, the local monopole solutions present a singularity at some finite value of the radial variable relative to the extra dimensions.

D. Mirror symmetric solutions

In the previous sections we reported solutions obtained for generic values of the different coupling constants. However, it is likely that for specific values of H, Λ_{n+4} regular solutions exist. Mirror symmetric solutions, like the ones obtained in the context of global monopoles are good candidates. It turns out that also in the case of local monopoles, it is possible to tune H and Λ_{n+4} in such a way that a mirror symmetric solutions exist. In order to construct such solution , we imposed the constraints $\phi = 0$, $w = 0$ $M' = 0$, $L' = 0$ as supplementary boundary conditions at some finite value $r = r_c$; then we tried to determine numerically if the values of H, γ can be adjusted for all constaints to be fullfilled. It turns out to be possible. For fine tuned values of the two cosmological constants, indeed, our numerical integration of the equations exhibits solutions such that the function $\phi(r)$ bends backwards and reaches the value $\phi = 0$ at $r = r_c$ after developping a plateau where $\phi \sim 1$ for $0 < r < r_c$. At the same time the function $w(r)$ and the derivatives of the metric functions L and M, L', M' approach zero for $r \rightarrow r_0$.

An example of such a solution is given in Figs. 8 and 9. It corresponds to $r_c = 100$ which fixes $\gamma \approx -0.003$, $H \approx 0.187$. We constructed a branch of solutions for lower values of r_c and obtained , e.g. for $r_c = 50$ the values $\gamma \approx 0.018$, $H \approx 0.217$. The sign of the bulk cosmological constant Λ_{4+n} leading to these kind of solutions seems to be negative for $r_c > 82$ and positive for $r_c < 82$. The value H is positive and does not vary significantly with r_c.

The properly speaking mirror symmetric solutions can then be obtained by extending the numerical solution discussed above by mirror symmetry for $r \in [0, 2r_c]$ $(M, L, w$ are symmetric under the reflexion $r \rightarrow 2r_c - r$ while ϕ is antisymmetric). This is possible by using of the discrete symmetries of the equations as discussed above. These mirror symmetric solutions can further be extended periodically on the interval $r \in [0, 2pr_c]$ for p integer. The periodicity of the function $L(r)$ suggest that the geometry of these solutions in the extra dimension consists of the surface of a deformed 3-sphere. We can then interpret the radial coordinate r as the "colatitude" and the solution looks like a magnet (with non-abelian fields) whose positive and negative poles lay respectively on the north ($r = 0$) and south ($r = 2r_c$) poles of the sphere. In the region of the equator ($r = r_c$) the Higgs field form a domain wall which separates these two poles. This is illustated by fig. 10 where the Ricci scalar R (see Eq. (9)) , the full energy density T_0^0 and the contribution to the energy of the Yang-Mills fields $T_{0,ym}^0$ of the solution of Figs. 8,9 are represented in the north-pole region and in the equator region. In particular, we see that only the Yang-Mills energy is zero in the equator region, so that the energy content in this region is due to the Goldstone boson.

VII. CONCLUSIONS

In this paper, we have reconsidered a brane world model in $(4+n)$ dimensions. We have considered the cases of a bulk without matter, a global monopole and a local monopole, respectively. The difference with respect to existing literature is the inclusion of both a bulk cosmological constant and a physical cosmological constant on the brane. For generic values of these constants, the existing solution get smoothly deformed and leads to a pattern similar to the case of standard cosmological solutions of 4-dimensional space-time, largely determined by the sign of Λ_{4+n}.

The presence of two constants allows for new types of solutions, namely solutions which are mirror symmetric under the reflexion $r \to 2r_0 - r$, where r_0 is fune tuned with respect to the cosmologival constants γ and Λ_{4+n}. These mirror symmetric solution can eventually be extended into periodic solutions in the radial coordinate associated with the extra dimensions; this provides a natural compactification of the transverse space.

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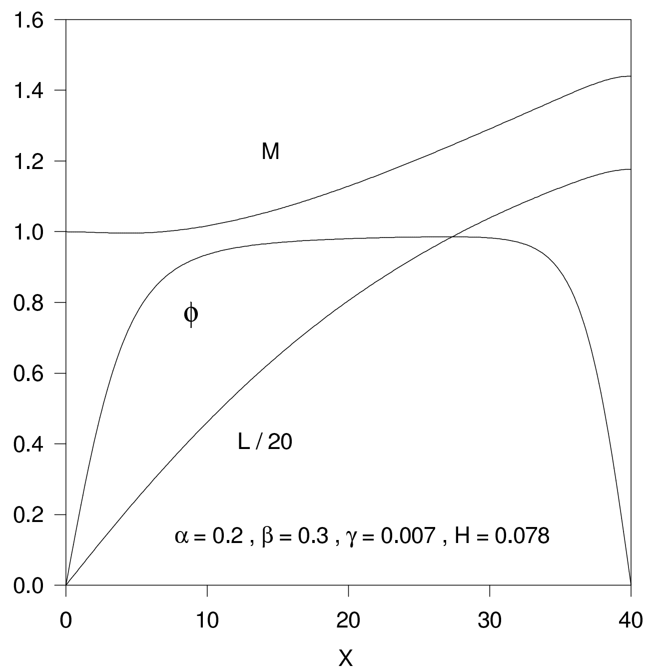

FIG. 1: The profiles of the metric and scalar functions for the miror symmetric solution corresponding to $x_c = 40$. This is a solution on the first branch.

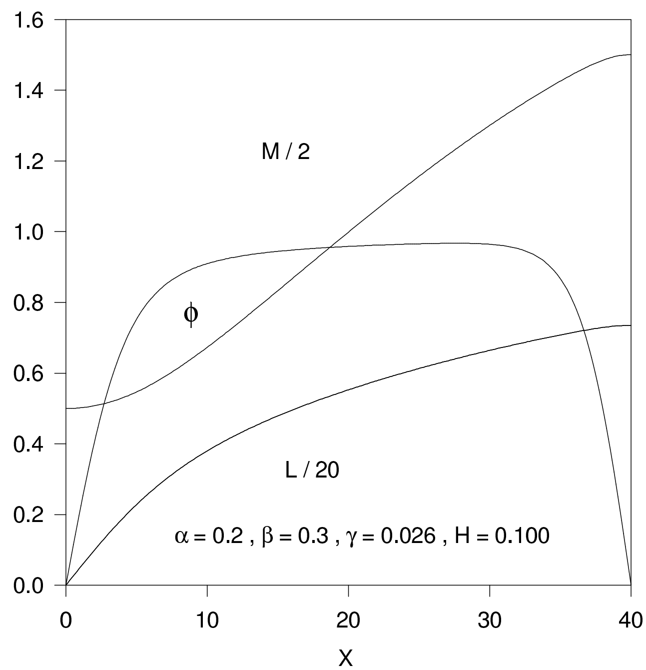

FIG. 2: The profiles of the metric and scalar functions for the mirror symmetric solution corresponding to $x_c = 40$. This is a solution on the second branch.

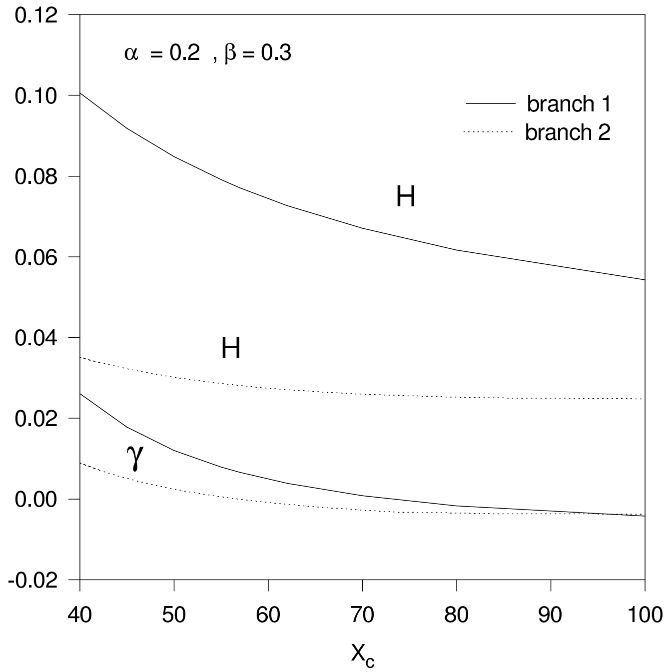

FIG. 3: The relations between the parameters Λ_{n+4}, H, r_0 allowing mirror symmetric solutions.

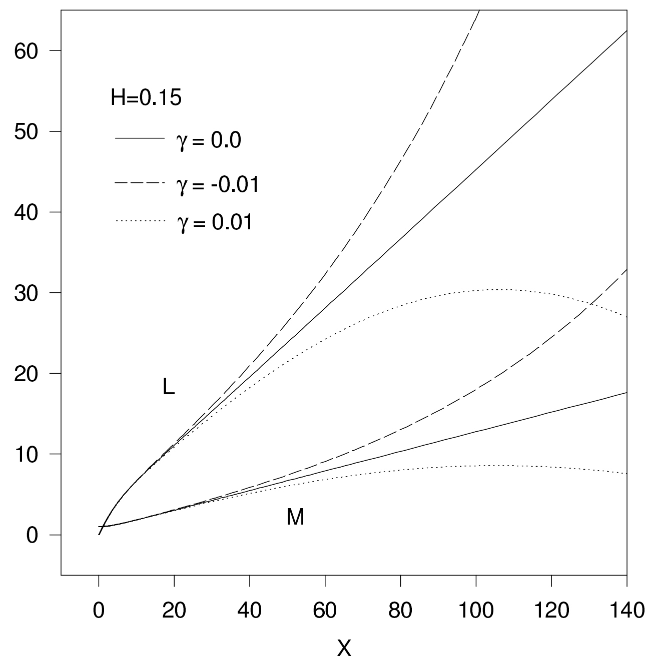

FIG. 4: The profiles of the metric functions L, M corresponding to an inflating brane with $H = 0.15$ regularized by a monopole are given for $\Lambda_{n+4} = 0, \pm 0.01$.

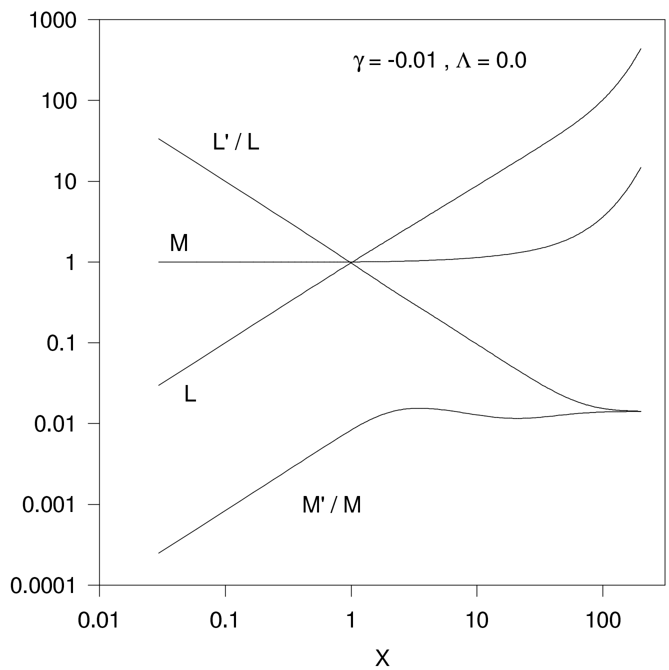

FIG. 5: The profiles of the metric functions L, M and the ratio $L'/L, M'/M$ corresponding to a static brane are shown for $\Lambda_{n+4} = -0.01$

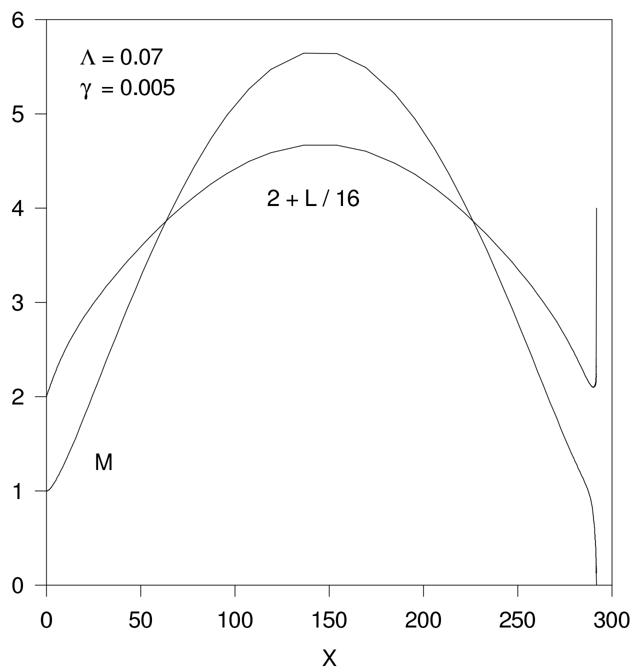

FIG. 6: The profiles of the metric functions of an inflating brane solution with $H = 0.07$ and positive cosmological constant $\Lambda_{n+4} = 0.005$ are shown.

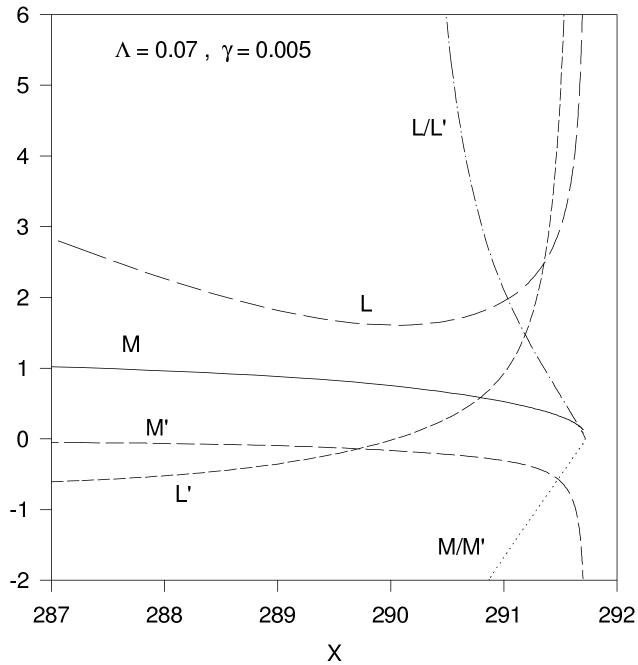

FIG. 7: The details of Fig.6 in the region of the singular point $r \approx 291.7$.

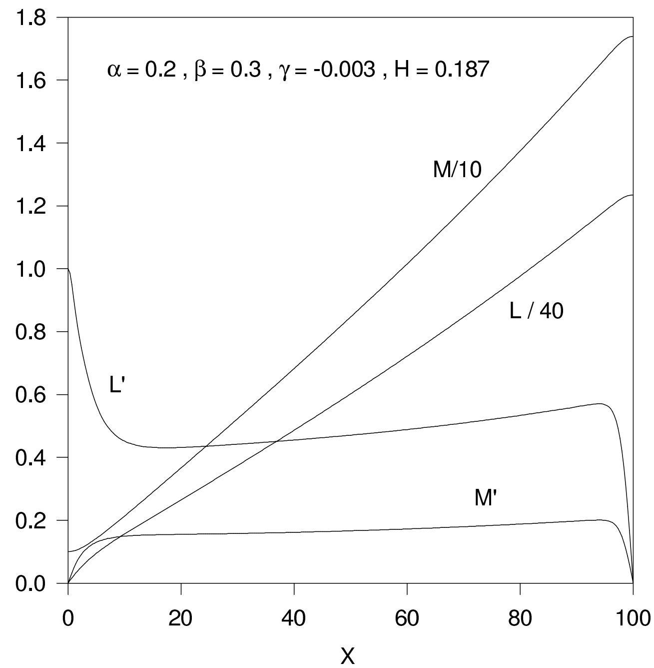

FIG. 8: The profiles of the metric functions M, L and their derivatives for a periodic monopole solution.

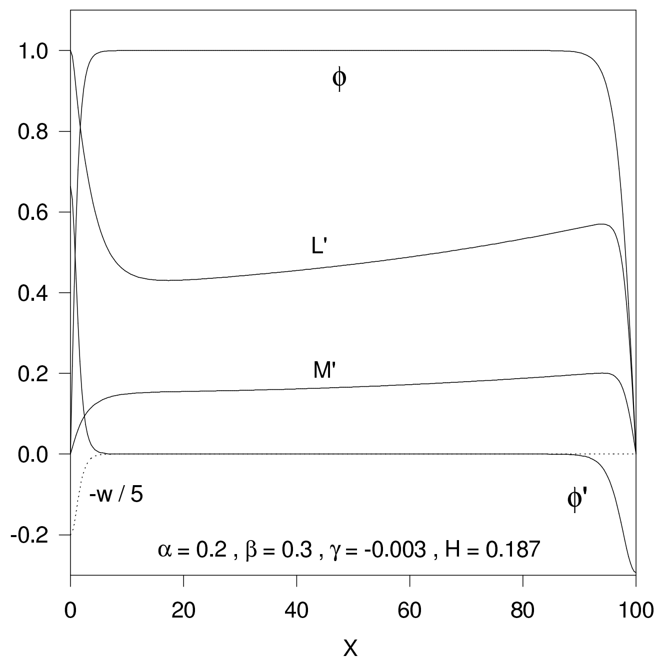

FIG. 9: The profiles of the matter functions w, ϕ, ϕ' and the derivatives M', L' for a periodic monopole solution.

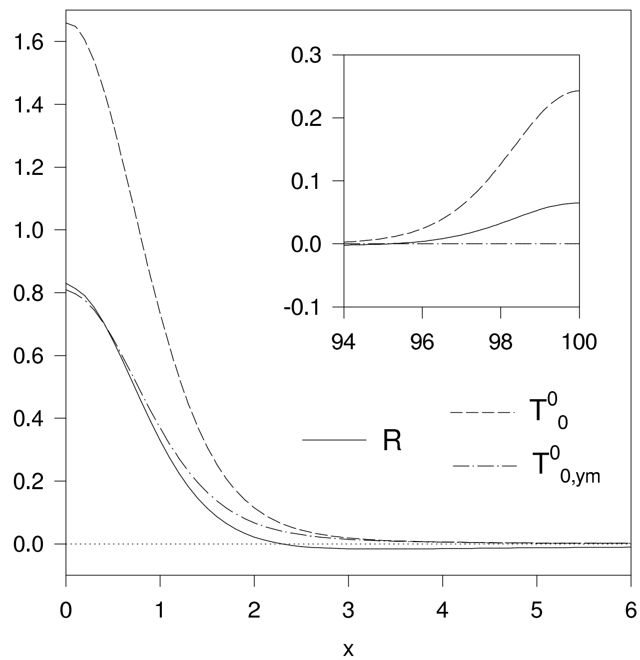

FIG. 10: The Ricci scalar R, the energy momentum tensor T_0^0 and the Yang-Mills contribution to the energy $T_{0,ym}^0$ corresponding to the periodic monopole of Figs. 8,9.