

ASYMPTOTIC THEORY OF MODULES OF SEPARABLY CLOSED FIELDS

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Abstract. We consider the reduct to the module language of certain theories of fields with a non surjective endomorphism. We show in some cases the existence of a model companion. We apply our results for axiomatizing the reduct to the theory of modules of non principal ultraproducts of separably closed fields of fixed but non zero imperfection degree.

§1. Introduction. In [5], we described the theory of a separably closed field of characteristic p and imperfection degree e (with e either finite or infinite) viewed as a module over a skew polynomial ring where the action of $''t''$ was interpreted as the Frobenius map. We showed that in the reduct of the field language consisting of an expansion by definition of the language of modules (we added the analog of the p -components functions) the theory is still complete and recursively axiomatisable.

Now, we would like to describe the “asymptotic” theory in the module language described above of the classes of separably closed fields either of characteristic p_n with $p_n \in \mathcal{P}$ (\mathcal{P} denotes the set of prime numbers), or of characteristic p with the powers of the Frobenius maps $x \rightarrow x^{p^n}$, fixing the imperfection degree. By asymptotic theory, we mean that we want to identify the theory of non principal ultraproducts of elements of that class. Note that the languages of different structures in the class we are considering are in general different, which is usually the case when working with modules.

Anyway, we will consider ultraproducts $K := \prod_U K_n$ of separably closed fields with K_n of characteristic p_n and imperfection degree e , $p_n \in \mathcal{P}$, $e \in \omega \cup \{\infty\}$. Either, $(p_n)_{n \in \omega}$ is a strictly increasing sequence and K is an algebraically closed field of characteristic zero. Or, K is a separably closed field of characteristic p , of fixed finite imperfection degree e . In each K_n , we have the Frobenius maps either $x \rightarrow x^{p^n}$ or $x \rightarrow x^{p^n}$ that give rise in the ultraproducts to non standard Frobenius maps. In the characteristic zero case, the theory T of such ultraproducts, in the field language augmented by a symbol for an endomorphism has been considered by Chatzidakis and Hrushowski [3]. They showed that some expansion by definition of T is model-complete.

Here, adopting the same point of view as in [5], we will consider those fields in a reduct of the field language, namely the language of modules over a skew polynomial ring of the form $K_0[t; \alpha]$, where K_0 is a subfield of K .

Received September 10, 2002; revised December 13, 2004.

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We will first place ourselves in a general setting and then we will specialize to the fields with non-standard Frobenius considered above.

We could have proceeded differently in considering those structures in a two-sorted module language (see for instance [10] chapter 9) i.e., one sort for the module and the other sort for the ring; so, one may quantify over the ring elements. One advantage in adopting this point of view, is that then the theory of the structures of the form $(\prod_U K_n, \prod_U R_n)$, where U is a non principal ultrafilter on ω , is equal to the set of sentences true in all but finitely many (K_n, R_n) , $n \in \omega$. In choosing the ring R_n , we may either take the skew polynomial ring $K_{n(0)}[t; \alpha_n]$, where $K_{n(0)}$ is a certain subfield of K_n and α_n an endomorphism of $K_{n(0)}$ or we may take the ring $K_{n(0)}[t; \alpha_n][\lambda]$ (see [1]). This last ring has the advantage that an ultraproduct of indecomposable pure-injective remains so and that for p.p. formulas whose only quantified variables are the module variables one still get positive quantifier elimination, the drawback is that its ring structure is quite complicated. While, the first ring has the advantage to be right Ore which is axiomatisable in the ring language. But in both cases, the Diophantine theory of the ring is undecidable. Indeed, one may define $\mathbb{F}_p[t]$ in $K_{n(0)}[t; \alpha_n]$ (respectively in $K_{n(0)}[t; \alpha_n][\lambda]$) by the atomic formula: $v \in \mathbb{F}_p[t]$ iff $v.t = t.v$; and the Diophantine theory of $\mathbb{F}_p[t]$ is undecidable (see [7] Theorem A).

§2. Axiomatization of $T(R)$. This section mainly consists in recalling facts, in this particular setting, that were already proved in [5] (infinite imperfection degree case) and in [6], section 4.

For S a ring, the language of right S -modules is $\mathcal{L}_S = \{+, -, 0, \cdot r; r \in S\}$, where for any $r \in S$ and x an element in a right S -module M , $(x) \cdot r := x \cdot r$ (scalar multiplication by the ring element r). Let T_S be the theory of all right S -modules in this language.

DEFINITION 2.1. Let K be an infinite field with an endomorphism α (note that it is necessarily injective). We generalize the notion of a p -basis as follows. An α -basis of K is a basis of K viewed as a (left) vector-space over the subfield K^α . Fix such a basis \mathcal{E} and we will always assume that it contains 1.

Let (K_0, α) be an infinite subfield of (K, α) (closed under α), containing \mathcal{E} and the subfield $Fix(\alpha)$ of elements of K fixed by α .

Consider the skew polynomial ring $R := K_0[t; \alpha]$ with the commutation rule $k.t = t.k^\alpha$, $k \in K_0$. Note that K is an R -module by interpreting the right multiplication by t as the action of α on K .

The ring R is a right Euclidean domain and so right principal and right Ore, but not left Ore unless α is an automorphism (see [4] Proposition 2.1.1 and Theorem 2.1.3). The center of R is the subfield $Fix(\alpha)$ of K_0 fixed by α (express that an element of the center commutes with t and with an element of $K_0 - K_0^\alpha$). In the case when K_0 is a non principal ultraproduct of fields of non zero characteristic and α is a non-standard Frobenius map, $Fix(\alpha)$ is a pseudo-finite field namely either $\prod_U \mathbb{F}_{p_n}$ or $\prod_U \mathbb{F}_{p^n}$.

NOTATION 2.1. We will call an element $q(t)$ of R , α -separable if $q(0) \neq 0$. Let X be the set of α -separable elements. Note that in writing down an element of R , we

will allow ourselves to either write it as q or $q(t)$ when stressing the fact that it is a polynomial in t .

In the case of α being the Frobenius map and interpreting multiplication by t by applying the Frobenius, this coincides with the usual notion of separable polynomials (see [5]).

PROPOSITION 2.1. *The set X forms a right denominator set in R i.e.,*

$$\forall r \in R \ \forall x \in X \ \exists s \in R \ \exists y \in X \ r \cdot y = x \cdot s.$$

PROOF. See [11] and also Lemma 5.7 in [5]. ⊢

COROLLARY 2.2. *Any right R -module M has a module of fractions $M \cdot X^{-1}$ with respect to the right denominator set X in which it embeds whenever it is X -torsion-free. Moreover $M \cdot X^{-1}$ is X -divisible (and X -torsion-free whenever M is).*

PROOF. See Proposition 9.12 and Theorem 9.13 in [8]. ⊢

Let M be an R -module and let M_{tor} be the set of elements of M annihilated by some non zero element of R ; it is a submodule since R is right Ore and also a $Fix(\alpha)$ vector-space. We will add new unary function symbols to generalize the fact that any element of K has a unique decomposition along a basis of K over its subfield K^α . These will be \mathcal{L}_R -definable in the theory T_R .

Set $\omega^* := \omega - \{0\}$. We enumerate all finite subsets of elements of \mathcal{E} , which are presented as finite tuples, denote this set by $\mathcal{E}^{(\omega)} = \bigcup_{n \in \omega^*} \mathcal{E}^n$. We will expand the usual module language by adding unary functions $\lambda_i^{\bar{c}}$, where $\bar{c} := (c_0, \dots, c_{n-1})$ with the convention that all elements in this tuple are distinct and $i \in n = \{0, \dots, n-1\}$.

DEFINITION 2.2. Let $\mathcal{L} = \mathcal{L}_R \cup \{\lambda_i^{\bar{c}} ; \bar{c} = (c_0, \dots, c_{n-1}) \in \mathcal{E}^{(\omega)}, i \in n, n \in \omega^*\}$, where the $\lambda_i^{\bar{c}}$'s are unary functions.

DEFINITION 2.3. Let $T(R)$ be the following \mathcal{L} -theory:

1. T_R the theory of all right R -modules,
2. $\forall x (\bigvee_{i \in n} \lambda_i^{\bar{c}}(x) \neq 0 \rightarrow x = \sum_{i \in n} \lambda_i^{\bar{c}}(x) \cdot t \cdot c_i)$, for each $\bar{c} \in \mathcal{E}^{(\omega)}$.
3. $\forall x \forall (x_i)_{i \in n} (x = \sum_{i \in n} x_i \cdot t \cdot c_i \rightarrow \bigwedge_{i \in n} x_i = \lambda_i^{\bar{c}}(x))$, for each $\bar{c} \in \mathcal{E}^n$ and $n \in \omega^*$.
4. $\exists x \neq 0 \ x \cdot q(t) = 0, q(t) \in X$.
5. $\forall x \exists y \ x = y \cdot q(t)$, where $q(t) \in X$.

Note that T_R is r.e. whenever K_0 is, as well as axiom schemes 4 and 5. Axiom schemes 2 and 3 are r.e. whenever \mathcal{E} is.

First, note that any model of $T(R)$ satisfies the following set of \mathcal{L}_R -sentences:

$$\forall y_0 \dots y_{n-1} \sum_{j \in n} y_j \cdot t \cdot c_j = 0 \rightarrow \bigwedge_{j \in n} y_j = 0.$$

Second, since our expansion is not an abelian structure which is not very common in the setting of modules, we will make a few comments on the functions $\lambda_i^{\bar{c}}, i \in n$ where $n = length(\bar{c})$. Those functions are defined in any model of $T(R)$ by the

following \mathcal{L}_R -formula:

$$\lambda_i^{\bar{c}}(x) = y \iff (\exists y_0 \dots y_{n-1} \ x = \sum_{j \in n} y_j.t.c_j \text{ and } y_i = y)$$

$$\text{or } (\forall y_0 \dots y_{n-1} \ x \neq \sum_{j \in n} y_j.t.c_j \text{ and } y = 0).$$

Let $M \models T(R)$ and fix a tuple $\bar{c} = (c_0, \dots, c_{n-1})$; set

$$E_{\bar{c}} := \{x \in M : \bigvee_{i \in n} \lambda_i^{\bar{c}}(x) \neq 0\} \cup \{0\}.$$

Then $E_{\bar{c}}$ is a K_0^α -vector subspace of M and the functions $\lambda_i^{\bar{c}}, i \in n$, are additive homomorphisms on this subspace, more precisely K_0^α -linear maps. Denote by E the subspace generated by the subspaces $E_{\bar{c}}$ with $\bar{c} \in \mathcal{E}^{(\omega)}$; the set of the subspaces $E_{\bar{c}}$ together with the maps $\lambda_i^{\bar{c}}, i \in n = \text{length}(\bar{c})$, forms a direct system and E can be viewed as its direct limit.

A calculation analogous to the one in [5] (Lemma 4.2) shows that E is an R -submodule of M . Indeed, let $k \in K$ and $x \in E_{\bar{c}}$, assume w.l.o.g. that $k = \sum_j k_j^\alpha.c_j$. Let $n = \text{length}(\bar{c})$. Express each product $c_i.c_j, 1 \leq i, j \leq n$ with respect to \mathcal{E} . Namely, let \bar{d} be a finite subset of \mathcal{E} such that each product $c_i.c_j, 1 \leq i, j \leq n$ belongs to the K^α -subspace generated by \bar{d} . So, $c_i.c_j = \sum_{\ell=0}^{n(i,j)} k_{ij}^\alpha.d_\ell$. We obtain

$$\left(\sum_i \lambda_i^{\bar{c}}(x).t.c_i\right). \left(\sum_j k_j^\alpha.c_j\right) = \sum_{i,j} \lambda_i^{\bar{c}}(x).k_j.t.c_i.c_j$$

$$= \sum_\ell \sum_{i,j} \lambda_i^{\bar{c}}(x).k_j.k_{ij}.t.d_\ell.$$

REMARK 1. Note that applying the right Euclidean algorithm, we have that if $m \neq 0$ and $m.q(t) = 0$ with $q(t)$ a prime polynomial, then $q(t)$ is minimal such that $m \cdot q(t) = 0$. We have that $m.R \cong m.K_0 \oplus m.t.K_0 \oplus \dots \oplus m.t^{n-1}.K_0$, where $n = \text{deg}(q(t))$.

NOTATION 2.2. Let T_α be the theory consisting of axioms schemes 1 up to 3, together with $\{\exists x (x \neq 0)\}$.

LEMMA 2.3. *The theory T_α is consistent.*

PROOF. Note that we have taken \mathcal{E} to be an α -basis of K . We consider the field K_0 (respectively K) as an R -module in interpreting (right) multiplication by t as the application of the endomorphism α , and the unary function $\lambda_i^{\bar{c}}(k)$ is defined on $k \in K_0$ (respectively K) by taking the value 0 if k does not belong to the K_0^α vector-space (respectively K^α) generated by the tuple $\bar{c} = (c_1, \dots, c_n)$, otherwise k can be written in a unique way as $\sum_i \alpha(k_i).c_i$ and we define $\lambda_i^{\bar{c}}(k) = k_i$. \dashv

REMARK 2. In any model M of T_α , the action of t is injective (by axiom 3) and therefore any model where t is non surjective, is non superstable (see [6] Proposition 3.2). Note that as soon as \mathcal{E} is not a singleton, the action of t is not surjective ($m_1.t.c = m_2.t.1$ implies that $m_1 = m_2 = 0$.) Now, in the descending chain of subgroups $M \supseteq M.t \supseteq M.t^2 \supseteq \dots$, the index of each subgroup is infinite in the preceding one. Take $n \in M - M.t$ and consider the set of elements $\{n.k^\alpha; k \in K_0\}$.

Then if $n.k_1^\alpha - n.k_2^\alpha \in M.t$ with $k_1 \neq k_2$, then $n \in M.t$, which is a contradiction. Since K_0 is infinite and α a field morphism, the index of $M.t$ in M is infinite as well as the index of $M.t^{k+1}$ in $M.t^k$ considering the elements $n.t^k \in M.t^{k+1}$, and from this it follows that the theory of M is not superstable (see for instance [14] Theorem 2.1 (3)).

The first point we want to make is that in models of T_α the torsion and the α -separable torsion is the same. It is convenient to begin by introducing the following notation.

NOTATION 2.3 (See Notation 3.2, Remark 2 and section 4 in [5]). Given $q \in R$, we will define $\sqrt[\alpha]{q}$ and q^α . First, for $k = \sum_i k_i^\alpha \cdot c_i \in K_0$, where the k_i 's belong to K_0 and c_i 's to \mathcal{C} , set $k^{1/\alpha} := \sum_i k_i \cdot c_i$. (Observe that $(k^\alpha)^{1/\alpha} = k$, but unless $k \in K_0^\alpha$, $(k^{1/\alpha})^\alpha$ and k are distinct.) Then, for $q = \sum_{i=0}^n t^i \cdot k_i \in R$ with $k_i \in K_0$, set $\sqrt[\alpha]{q} := \sum_{j=0}^n t^j \cdot k_j^{1/\alpha}$. We also define q^α as $\sum_{i=0}^n t^i \cdot k_i^\alpha$.

Iteration m times of $\sqrt[\alpha]{}$ is denoted $\sqrt[\alpha^m]{}$.

Given $q \in R$, we write it as $q = \sum_i q_i \cdot c_i$ with the property that $q_i \in K_0^\alpha[t; \alpha]$ e.g., $q_i = \sum_j t^j k_{ij}^\alpha$, with $k_{ij} \in K_0$. Therefore, we have that $\sqrt[\alpha]{q_i} = \sum_j t^j k_{ij}$, so

$$\sum_i \sqrt[\alpha]{q_i} \cdot t \cdot c_i = \sum_i \sum_j t^j k_{ij} \cdot t \cdot c_i = \sum_i \sum_j t^{j+1} k_{ij}^\alpha \cdot c_i = t \cdot q.$$

As in the commutative case, we have on the elements $q = \sum_{i=0}^n t^i \cdot k_i$ of R a function deg taking its value in \mathbb{N} and defined as $deg(q) = n$.

LEMMA 2.4 (See Proposition 3.5 and section 4 in [5]). *Let M be a model of T_α . Assume that $m \in M_{tor}$ with $m \cdot q = 0$, for some $q \in R - \{0\}$. Then there exists $q' \in X$ such that $m \cdot q' = 0$.*

PROOF. If q is not α -separable, we can write it as $q = t \cdot q'$ with $q' \in R$. With the above notation we have: $m \cdot t \cdot q' = 0 = \sum_i m \cdot \sqrt[\alpha]{q_i} \cdot t \cdot c_i$ with

$$deg(\sqrt[\alpha]{q_i}) \leq deg(q') < deg(q).$$

Applying axiom 3, we get that $\bigwedge_i m \cdot \sqrt[\alpha]{q_i} = 0$, if none of $\sqrt[\alpha]{q_i}$ is α -separable, we iterate the procedure. Note that it stops since the function deg takes its values in \mathbb{N} . ⊥

LEMMA 2.5. *Let M be a model of T_α . Then, M_{tor} is an \mathcal{L} -substructure.*

PROOF. Let $m \in M_{tor}$, by the above Lemma, there exists $q \in X$ such that $m \cdot q = 0$. Further we may assume that q is of the form $(1 - t \cdot q')$. Therefore, using the above notation, we get that $m = \sum_i m \cdot \sqrt[\alpha]{q_i} \cdot t \cdot c_i$. Since M is a model of T_α , we get that $\lambda_i^{\bar{c}}(m) = m \cdot \sqrt[\alpha]{q_i}$ and so it belongs to the R -submodule generated by m and so to M_{tor} . ⊥

NOTATION 2.4. Let T_α^{sep} be the theory consisting of T_α together with axiom scheme 5. Let $(T_\alpha^{sep})^{tf}$ be the theory of the class of the torsion-free models of T_α^{sep} . (It is axiomatized by adding to T_α^{sep} the scheme of axioms

$$\forall m (m \cdot q(t) = 0 \rightarrow m = 0),$$

for each $q(t) \in X$.

LEMMA 2.6. *Let M be a model of $T(R)$ and let M_1 be a pure R -submodule of M . Then, M_1 is an \mathcal{L} -substructure of M and a model of T_α^{sep} .*

PROOF. First note that once we have proved that M_1 is an \mathcal{L} -substructure of M , it will follow that it is a model of T_α^{sep} . Indeed, axiom schemes (2) and (3) are universal and we already know that axiom scheme (5) holds in M_1 since it is a pure \mathcal{L}_R -submodule of M .

To show that M_1 is an \mathcal{L} -substructure, we will proceed in two steps. Let \mathcal{U} be an $|R|$ -regular ultrafilter and consider the ultrapower $M^* := \prod_{\mathcal{U}} M$ of M (respectively M_1^* of M_1). Then since M_1^* is a pure-injective pure \mathcal{L}_R -submodule of M^* , it is a direct summand of M^* : $M^* = M_1^* \oplus N$ for some \mathcal{L}_R -submodule N of M^* . Let $x_1 \in M_1^*$. Then in M^* either $\lambda_i^{\bar{c}}(x_1) = 0$ or $x_1 = \sum_{i \in n} \lambda_i^{\bar{c}}(x_1).t.c_i$. In the second case, let us show that $\lambda_i^{\bar{c}}(x_1) \in M_1^*$. Write $\lambda_i^{\bar{c}}(x_1) = x_{i1} + x_{i2}$ with $x_{i1} \in M_1^*$ and $x_{i2} \in N$, $i \in n$. Then $\sum_{i \in n} x_{i2}.t.c_i = 0$, so by axiom scheme (3), each $x_{i2} = 0$, so $\lambda_i^{\bar{c}}(x_1) \in M_1^*$, $i \in n$.

Now, suppose that $x \in M_1$ and assume that in M , $\lambda_i^{\bar{c}}(x) \neq 0$. Then by the above, $M_1^* \models \exists x_0 \dots \exists x_{n-1} \ x = \sum_{i \in n} x_i.t.c_i$. Since M_1 is an \mathcal{L}_R -elementary substructure of M_1^* , the same formula holds in M_1 . So, by axiom scheme (3), $\lambda_i^{\bar{c}}(x) = x_i \in M_1$. ⊢

LEMMA 2.7. *Let M be a torsion-free right R -module, which is a model of T_α . Then, the corresponding module of fractions $M.X^{-1}$ is a model of $(T_\alpha^{sep})^{tf}$.*

PROOF. See Proposition 9.12 in [8] and Proposition 8.8, chapter 1 in [13] and [6] Proposition 4.3. The proof has two steps, first, one shows that one can define on $M.X^{-1}$ the functions $\lambda_i^{\bar{c}}$'s as in the above Lemma. Then, one has to show that there is only one way to define them. ⊢

PROPOSITION 2.8. *In T_α^{sep} , any positive primitive (p.p.) \mathcal{L}_R -formula is equivalent to a positive quantifier-free \mathcal{L} -formula.*

PROOF. See [5] Proposition 7.2. This is based on a proposition which can be found in [9] p. 176 in the commutative case and one has to check that it adapts to right Euclidean rings. ⊢

COROLLARY 2.9. *Let M be a model of T_α^{sep} . Then, M_{tor} is a pure submodule.*

PROPOSITION 2.10. *Given any two p.p. formulas $\psi \rightarrow \phi$ defining two distinct subgroups in T_R . Then either in any model of $(T_\alpha^{sep})^{tf}$ the index $[\phi : \psi]$ is infinite, or in such a model the index is equal to 1.*

PROOF. See [5] Lemma 7.3. ⊢

Let us state a corollary (see below) under the hypothesis that $T(R)$ is consistent (and $Fix(\alpha)$ infinite), which will be proved in the next section where we will show that one can embed any model of T_α in a model of $T(R)$.

COROLLARY 2.11. *If $T(R)$ is consistent and whenever $Fix(\alpha)$ is infinite, it is complete and it admits quantifier elimination.*

PROOF. Let $M_1, M_2 \models T(R)$. First, note that since the functions $\lambda_i^{\bar{c}}$ are \mathcal{L}_R -definable, if we show that M_1 is elementarily equivalent to M_2 as \mathcal{L}_R -structures, then they will be elementarily equivalent as \mathcal{L} -structures.

As an \mathcal{L}_R -structure, M_1 (respectively M_2) is elementarily equivalent to a direct sum N_1 (respectively N_2) of pure-injective indecomposable R -modules (see [14]

Corollary 6.9). Therefore, N_1 (respectively N_2) can be expanded to an \mathcal{L} -structure; we will stress it by denoting the expanded structure $N_{1\mathcal{L}}$ (respectively $N_{2\mathcal{L}}$) and as such is elementarily equivalent to M_1 (respectively M_2), in particular $N_{1\mathcal{L}}$ (respectively $N_{2\mathcal{L}}$) is a model of $T(R)$.

By Lemma 2.6, each \mathcal{L}_R -direct summand of $N_{1\mathcal{L}}$ (respectively $N_{2\mathcal{L}}$) is an \mathcal{L} -substructure of $N_{1\mathcal{L}}$ (respectively $N_{2\mathcal{L}}$) and satisfy T_α^{sep} . (Even though, in general, $N_{1\mathcal{L}}$ is not a direct sum of those considered as \mathcal{L} -substructures.)

We consider two cases, either such direct summand is torsion-free or contains non-trivial torsion.

By Proposition 2.10, each torsion-free \mathcal{L}_R -direct summand of $N_{1\mathcal{L}}$ is elementarily equivalent to a torsion-free \mathcal{L}_R -direct summand of $N_{2\mathcal{L}}$.

So, it remains to consider the case where those pure-injective indecomposable direct summands are of the form $H(t_0)$ where t_0 is the type of a torsion element and w.l.o.g. we assume that this element annihilates a prime (separable) polynomial. The same proof as in [6] Lemma 3.11 goes through showing that the isomorphism type of a pure-injective \mathcal{L}_R -indecomposable model of T_α^{sep} is determined by the fact that a non trivial element is annihilated by the same prime (separable) polynomial. Since the setting is slightly different, let us outline the argument. Suppose that one has two indecomposable types t_1 and t_2 which contains the formula $x.r(t) = 0$, with $r(t) \in R$. If $H(t_1)$ were non isomorphic to $H(t_2)$, then there would be a p.p. \mathcal{L}_R -formula strictly between $x = 0$ and $x.r(t) = 0$ which belongs to only one of t_1 or t_2 (see [14] Lemma 7.10). By Proposition 2.8, any p.p. \mathcal{L}_R -formula is equivalent to a positive quantifier-free \mathcal{L} -formula and the type of a torsion element is determined by the polynomial of minimal degree it annihilates. So for any prime separable polynomial $r(t)$ we get that any two pure-injective indecomposables which have non trivial $r(t)$ torsion are isomorphic.

Now given any pair of p.p. formulas (ψ, ϕ) with $T_R \models \psi \rightarrow \phi$, which is non trivial in a direct summand containing non-trivial torsion, we get that the index of the corresponding p.p. definable subgroups is infinite, since $Fix(\alpha)$ is infinite. Indeed, given any p.p. formula $\chi(x)$, we have for all $s \in Fix(\alpha) - \{0\}$ that $\chi(x) \leftrightarrow \chi(x.s)$.

The quantifier elimination result follows from the two preceding propositions and from the completeness result as in [5] Proposition 7.4. Let $\phi(\bar{x})$ be an \mathcal{L} -formula. Adding possibly new quantifiers and replacing the functions $\lambda_i^{\bar{c}}$ by their \mathcal{L}_R -definitions, we get an equivalent \mathcal{L}_R - formula $\psi(\bar{x})$. By the Baur-Monk quantifier elimination result, $\psi(\bar{x})$ is equivalent to a boolean combination of p.p. \mathcal{L}_R -formulas $\chi_i(\bar{x})$ (in any given complete theory of R -modules). By Proposition 2.8, each of these \mathcal{L}_R -formulas $\chi_i(\bar{x})$ is equivalent (in any model of $T(R)$) to a quantifier-free \mathcal{L} -formula. \dashv

COROLLARY 2.12. *Assume $T(R)$ is consistent and that the cardinality of the subfield $Fix(\alpha)$ of K is finite. Then, the different completions of $T(R)$ are obtained specifying the cardinalities of the annihilators of prime separable elements of R . Whether these are finite or not only depends on the field (K, α) and when finite they are equal to some multiple of a number which only depends on the field (K, α) .*

PROOF. We apply Lemma 3.14 in [6] and the proofs of Lemmas 7.3 and 6.8 in [5]. Let $\phi(x)$ and $\psi(x)$ be two p.p. \mathcal{L}_R -formulas defining two subgroups with one included in another in any model M of $T(R)$.

We will distinguish two cases. Either $\phi(M)$ is not included in an annihilator of an element of R and this case corresponds to whether the co-rank of the matrix associated to this p.p. formula is bigger than or equal to 1 (see Definition 8 and Lemma 7.1 in [5]). So, in this case we may proceed as in the proofs of Lemmas 6.8 and 7.3 in [5]; and we show that if the two subgroups $\phi(M)$ and $\psi(M)$ are distinct, then their index is infinite.

Or, $\phi(M)$ is included in M_{tor} (co-rank 0 case). Denote by $\text{ann}(q(t))$, where $q(t) \in X$, the set of elements b of M such that $b.q(t) = 0$. So, $\phi(x)$ is equivalent to $x.r(t) = 0$ and $\psi(x)$ is equivalent to $x.s(t) = 0$, for some $r(t), s(t) \in X$ and w.l.o.g. $r(t), s(t)$ are monic. Let $v \in \text{ann}(r(t)) \cap \text{ann}(s(t))$, then using the Euclidean algorithm and letting $f(t) := \text{gcd}(r(t), s(t))$, one has that $v \in \text{ann}(f(t))$. We may write $r(t) = f(t).r'(t)$ and $s(t) = f(t).s'(t)$. By assumption, we have that $\text{ann}(s(t)) \subseteq \text{ann}(r(t))$ in any model of $T(R)$ and so $\text{ann}(s(t)) = \text{ann}(f(t))$, but if $s'(t)$ has degree at least equal to 1, then taking a non zero element v in $\text{ann}(s'(t))$ and dividing it by $f(t)$ we get $v = w.f(t)$ and so $w \in \text{ann}(s(t)) - \text{ann}(f(t))$, a contradiction. Therefore since $s(t)$ is monic, we obtain that the following decomposition of $r(t)$ (respectively of $s(t)$) of the following form $r(t) := q_1(t) \cdots q_n(t)$ and $s(t) := q_1(t) \cdots q_m(t)$ with $m \leq n$ and $q_i(t), 1 \leq i \leq n$, prime separable in R .

The model M is elementarily equivalent to a direct sum of pure-injective indecomposable R -modules N . As in the proof of the previous corollary, we may expand this direct sum to an \mathcal{L} -structure and as such obtain a model of $T(R)$. The index $[\text{ann}(r(t)) : \text{ann}(s(t))]$ in M is equal to the product of the indices in each pure-injective indecomposable direct summands. So, to determine the different completions of $T(R)$, it suffices to determine the indices of such pairs in each indecomposable summand. Note that applying Lemma 2.6, we obtain that such N is an \mathcal{L} -substructure and satisfies T_α^{sep} . Let \mathcal{E} be the set of prime separable polynomials with non trivial annihilators in N . It remains to determine on one hand given an element of \mathcal{E} which are the other ones and on the other hand the size of the annihilator of one of its elements. We use now Lemma 3.14 in [6], replacing the prime subfield \mathbb{F}_p by $\text{Fix}(\alpha)$. There, we showed that the elements of \mathcal{E} are of the same degree say n and given $q \in \mathcal{E}$, we have that $q' \in \mathcal{E}$ iff there exist $q_1, q_2 \in R$ of degree less than or equal to $n - 1$ such that (*) $q.q_1 = q_2.q'$. Moreover, for $q, q' \in \mathcal{E}$, their annihilators in N are isomorphic as $\text{Fix}(\alpha)$ -vector spaces. (This result is proved using the positive \mathcal{L} -quantifier elimination result for p.p. \mathcal{L}_R -formulas and the fact that in any pure-injective indecomposable R -module two non zero elements are linked by a p.p. \mathcal{L}_R -formula.) We apply the above result to the case where $q(t) = q'(t)$, namely given an element $a \in \text{ann}(q(t)) - \{0\}$ and another element $b \in \text{ann}(q(t))$, there exist $q_1(t), q_2(t) \in R$ of degree less than or equal to $n - 1$ such that

$$(*) \quad q_1(t).q(t) = q(t).q_2(t)$$

and $b = a.q_1(t)$. Equation (*) determines all possible $q_1(t)$ in the following way. Set $q(t) = \sum_{i=0}^n t^i.a_i$ with $a_n, a_0 \neq 0$, $q_1(t) = \sum_{j=0}^d t^j.b_j$ and $q_2(t) = \sum_{k=0}^d t^k.c_k$ with $a_i, b_j, c_k \in K$. First, we determine the coefficients c_k in terms of the coefficients b_j and a_i and then we find a linear system of n equations that

$b_0, \dots, b_0^{\alpha^n}, \dots, b_d, \dots, b_d^{\alpha^n}$ satisfy.

$$\begin{pmatrix} b_0 & 0 & \dots & \dots & & & & \\ b_1 & b_0^\alpha & 0 & \dots & \dots & & & \\ & & & & & & & \\ b_d & b_{d-1}^\alpha & \dots & \dots & b_0^{\alpha^d} & 0 & & \\ 0 & b_d^\alpha & b_{d-1}^{\alpha^2} & \dots & \dots & b_0^{\alpha^{d+1}} & 0 & \\ \dots & \dots & \dots & \dots & \dots & \dots & b_0^{\alpha^n} & \\ \dots & \dots & \dots & \dots & \dots & \dots & b_1^{\alpha^n} & \\ 0 & \dots & \dots & 0 & \dots & \dots & b_d^{\alpha^n} & \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ a_d \\ a_{d+1} \\ a_{d+2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} = \begin{pmatrix} a_0 & 0 & \dots & \dots & 0 \\ a_1 & a_0^\alpha & \dots & \dots & 0 \\ \dots & \dots & \dots & a_0^{\alpha^{d-1}} & 0 \\ a_d & a_{d-1}^\alpha & \dots & a_1^{\alpha^{d-1}} & a_0^{\alpha^d} \\ a_{d+1} & a_d^\alpha & \dots & \dots & a_1^{\alpha^d} \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_{n-1}^\alpha & \dots & \dots & a_{n-d}^{\alpha^d} \\ 0 & a_n^\alpha & \dots & \dots & \dots \\ 0 & \dots & \dots & a_n^{\alpha^{d-1}} & a_{n-1}^{\alpha^d} \\ 0 & \dots & 0 & \dots & a_n^{\alpha^d} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \cdot \\ \cdot \\ c_d \end{pmatrix}$$

Rewriting the first $d + 1$ lines of this matrix equation, we get:

$$\begin{pmatrix} b_0 & 0 & \dots & 0 \\ b_1 & b_0^\alpha & \dots & 0 \\ & & & \\ & & & \\ b_d & b_{d-1}^\alpha & \dots & b_0^{\alpha^d} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ a_d \end{pmatrix} = \begin{pmatrix} a_0 & 0 & \dots & 0 \\ a_1 & a_0^\alpha & \dots & 0 \\ & & & \\ & & & \\ a_d & a_{d-1}^\alpha & \dots & a_0^{\alpha^d} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \cdot \\ \cdot \\ c_d \end{pmatrix}$$

Since the matrix in the right hand side is invertible ($a_0 \neq 0$), we can express the elements c_i 's in terms of the elements b_j 's and a_k 's. So, we get a system of n linear difference equations in the b_j , $0 \leq j \leq d$ of order n with coefficients in K . Its set of solutions will determine the structure of the annihilator of $q(t)$ in any pure-injective \mathcal{L}_R -indecomposable summand of a model of $T(R)$. −

§3. Consistency of $T(R)$. Let \mathcal{R} be the class consisting of R -modules where the action of t is injective.

From now on, for ease of notation, we will assume that $K = K_0$.

DEFINITION 3.1. Let $M \subseteq N$ belonging to \mathcal{R} .

Let $a \in N$. Then, a is α -separable over M (respectively α -algebraic over M) if there exists an α -separable (respectively a non zero) element $q(t)$ of R such that $a \cdot q(t) \in M$.

N is an α -separable extension of M if every $m \in N$ is α -separable over M .
 N is an α -algebraic extension of M if every element $m \in N$ is α -algebraic over M .

NOTATION 3.1. We will use the notation $n \equiv_M 0$ and the expression “ n is zero modulo M ” to mean that $n \in M$.

$M \subseteq_t N$ means that ($n.t \in M$ implies that $n \in M$), for any $n \in N$. Also, the condition t injective in \mathcal{R} means that $0 \subseteq_t M$, for any $M \in \mathcal{R}$.

LEMMA 3.1. Let $M \subseteq N$ belonging to \mathcal{R} and $a \in N$.

1. Suppose that N is a model of T_α and that M is an \mathcal{L} -substructure of N . Then, $M \subseteq_t N$.
2. Suppose that $a.q(t) \in M$, with $q(t) \in X$, then the R -submodule of N generated by a is a finite-dimensional K -vector-space modulo M . Moreover, if in addition $a.t \in M$, then $a \in M$. So, if N is α -separable extension of M , then $M \subseteq_t N$.
 Assume now that N is a model of T_α , then under the same hypothesis as above, the \mathcal{L} -substructure generated by a is a finite-dimensional K -vector-space, modulo M .
3. The set of elements of N which are α -separable over M forms a K -vector-space containing M .

PROOF.

1. Let $a.t = b \in M$. Since N is a model of T_α , by axiom scheme 3, $a = \lambda_0^1(b)$. Since M is an \mathcal{L} -substructure of N , we get that $\lambda_0^1(b) \in M$.

2. This is a straightforward verification (see Remark 1).

For the second assertion, we proceed as follows. Assume now that $N \models T_\alpha$. W.l.o.g., we may assume that $q(t) = (t.q'(t) - 1)$, so $a(t.q'(t) - 1) = b \in M$ and $b = \sum_i a.\sqrt[q_i]{t}.c_i - a$ (see Notation 2.3). Since N is a model of T_α , either there exist $n \in \omega$ and $\bar{c} \in \mathcal{C}^n$ such that $a = \sum_i \lambda_i^{\bar{c}}(a).t.c_i$, or $\forall n \in \omega$ and $\forall \bar{c} \in \mathcal{C}^n$ we have that $\lambda_i^{\bar{c}}(a) = 0$. In the last case, the \mathcal{L} -substructure generated by a is equal to the R -submodule generated by a and so we apply the first part of the statement. In the first case, we get that $\lambda_i^{\bar{c}}(b) = a.\sqrt[q_i]{t} - \lambda_i^{\bar{c}}(a)$ and so $\lambda_i^{\bar{c}}(a)$ belongs to the R -submodule generated by a modulo M and so the conclusion follows using the first part of the statement.

3. Let a_1, a_2 be two elements of N which are α -separable over M . By hypothesis, there exist $p_1(t), p_2(t) \in X$ such that $a_1.p_1(t), a_2.p_2(t) \in M$. Since X is a right denominator set of R , $p_1(t), p_2(t)$ have a right common multiple: $p(t) = p_1(t).p_3(t) = p_2(t).p_4(t)$ where $p_3(t), p_4(t) \in X$. So,

$$(a_1 + a_2).p(t) = (a_1.p_1(t)).p_3(t) + (a_2.p_2(t)).p_4(t) \in M.$$

Let $a \in N$ be α -separable over M and let $k \in K$, then $a.k$ is α -separable over M . By assumption on a , there exist $k_i \in K$ such that $a + \sum_{i>0} a.t^i.k_i \equiv_M 0$. So,

$$a.k(1 + \sum_{i>0} t^i.(k^{\alpha^i})^{-1}.k_i) \equiv_M 0. \quad \dashv$$

DEFINITION 3.2. A set of elements $(c_i)_{i \in I}$ of K is α -free in N over M , if for any finite subset I_0 of I

$$\forall m_i \in N \left(\sum_{i \in I_0} m_i.t.c_i \equiv_M 0 \implies \bigwedge_{i \in I_0} m_i \equiv_M 0 \right).$$

Note that since the action of t on K is given by the action of the endomorphism α , an α -basis of K is α -free in K over $\{0\}$. Also, if $M \subseteq_t N$, then 1 is α -free in N over M .

LEMMA 3.2. *The property: n_1, \dots, n_s are K -linearly independent over M implies that $n_{1.t}, \dots, n_{s.t}$ are K -linearly independent over M*

is equivalent to:

there is an α -basis of K is α -free in N over M .

PROOF.

(\rightarrow) Let (c_i) be an α -basis of K . Suppose that $\sum_{i=1}^l n_i.t.c_i \equiv_M 0$ with $n_i \in N$ and suppose that not all $n_i \equiv_M 0$. Extract a maximal K -linearly independent subset, w.l.o.g., we assume that it is: n_1, \dots, n_s and express the other elements in terms of these. So, for $l \geq j > s$, we have $n_j \equiv_M \sum_{i=1}^s n_i.k_{ij}$. We get $\sum_{i=1}^s n_i.t.(c_i + \sum_{j=s+1}^l k_{ij}^\alpha.c_j) \equiv_M 0$. By hypothesis on N , we get that $\bigwedge_{i=1}^s (c_i + \sum_{j=s+1}^l k_{ij}^\alpha.c_j) = 0$, which is a contradiction.

Note that in fact we have shown that any α -basis of K is α -free in N over M .

(\leftarrow) Let (c_i) be an α -basis of K which is α -free in N over M . Suppose that $\sum_i n_i.t.k_i \equiv_M 0$ with $k_i = \sum_j k_{ij}^\alpha.c_j$ and assume that not all $k_i \neq 0$ and so not all $k_{ij} \neq 0$. Then $\sum_i n_i.t.(\sum_j k_{ij}^\alpha.c_j) \equiv_M 0$, $n_i \in N$, $k_i \in K$, equivalently $\sum_i n_i.(\sum_j k_{ij}.t.c_j) = \sum_j (\sum_i n_i.k_{ij}).t.c_j \equiv_M 0$. So, $\bigwedge_j \sum_i n_i.k_{ij} \equiv_M 0$. \dashv

PROPOSITION 3.3. *Let $M \subseteq_t N$ belong to \mathcal{R} . Then,*

N is α -algebraic over M and there is an α -basis of K which remains α -free in N (or equivalently any α -basis of K remains α -free in N)

iff

N is an α -separable extension of M .

Therefore, if M is a model of T_α and if N is an α -separable extension of M , then N can be expanded to a model of T_α .

PROOF.

(\rightarrow) Let (c_i) be an α -basis which remains α -free in N over M . Let $b \in N - M$ be such that $b.t.p(t)$ belongs to M . Then, we have $\sum_i b.\sqrt[p_i]{t}.c_i \equiv_M 0$ (see Notation 2.3.) By assumption on (c_i) , it implies that $\bigwedge_i (b.\sqrt[p_i]{t} \equiv_M 0)$. If none of the $\sqrt[p_i]{t}$ are α -separable, we iterate the procedure which eventually stops since $\text{deg}(\sqrt[p_i]{t}) \leq \text{deg}(p)$.

(\leftarrow) In this direction, using the preceding lemma, we will show that if b_1, \dots, b_n are K -linearly independent elements of N over M , then the elements $b_{1.t}, \dots, b_{n.t}$ are also K -linearly independent over M . Let N_0 be the K -subspace of N containing M , closed under the action of t and containing $b_{1.t}, \dots, b_{n.t}$. Since each b_i is α -separable (α -algebraic), this subspace N_0 is finite-dimensional over M (Lemma 3.2 (2)). Complete the set b_1, \dots, b_n to get a K -basis of N_0 over M , say $(b_i)_{i \geq 1}$.

Let $b \in N_0$. So, $b \equiv_M \sum b_i.k_i$ and so

$$(*) \quad b.t^j \equiv_M \sum b_i.t^j.k_i^{\alpha^j}.$$

Since b is α -separable, it belongs to the K -subspace containing M , closed under t and generated by $b.t$, namely

$$(**) \quad b \equiv_M \sum_{h>0} b.t^h.k_h \text{ with } k_h \in K.$$

Now, $(b_i)_{i \geq 1}$ is a basis, so for each $i, j \geq 1$ there exist $r_{\ell j} \in K$ such that $b_i.t^j \equiv_M \sum_{\ell} b_{\ell}r_{\ell j}$ and so

$$(***) \quad b_i.t^{j+1} \equiv_M \sum_i b_i.t.r_{ij}^{\alpha}.$$

Using $(*)$, $(**)$ and $(***)$ for $j > 0$, we get that $b \equiv_M \sum_i b_i.t.k'_i$, for some $k'_i \in K$. Therefore, the set of $(b_i.t)_{i \geq 1}$ is a K -generating subset of N_0 modulo M and so it remains K -linearly independent over M .

Now, suppose that M is a model of T_{α} and let $b \in N - M$. By the above, there exists $q(t) \in X$ such that $b.q(t) = m \in M$. W.l.o.g., we may assume that $q(t) = 1 - t.q'(t)$. So, $b - \sum_i b.\sqrt[\alpha]{q'_i(t)}.t.c_i = m \in M$. Either, there exists $(c_i)_{i \in I_0} \subseteq (c_i)_{i \in I}$ with I_0 a finite subset of I such that $m = \sum_{i \in I_0} \lambda_i^{\bar{c}}(m).t.c_i$, and so we set $\lambda_i^{\bar{c}}(b) := b.\sqrt[\alpha]{q'_i(t)} + \lambda_i^{\bar{c}}(m)$, or no such subset exist and for all $i \in I$ $\lambda_i^{\bar{c}}(m) = 0$, in that latter case we set $\lambda_i^{\bar{c}}(b) = 0$.

We have to check that N is a model of T_{α} . Let $b \in N$ be such that $\lambda_i^{\bar{c}}(b) \neq 0$. Using the same notations as above, assume that $b.q(t) \in M$ and $q(t) \in X$. We defined $\lambda_i^{\bar{c}}(b) := b.\sqrt[\alpha]{q'_i(t)} + \lambda_i^{\bar{c}}(m)$ and we have that $b = \sum_i \lambda_i^{\bar{c}}(b).t.c_i$. Now suppose that $b = \sum_i d_i.t.c_i = 0$ for some elements $d_i \in N$. On the other hand we have that $b - \sum_i b.\sqrt[\alpha]{q'_i(t)}.t.c_i = m \in M$. Since (c_i) is α -free we have that $\bigwedge_i d_i - b.\sqrt[\alpha]{q'_i(t)} \equiv_M 0$. So, since M is a model of T_{α} , $\lambda_i^{\bar{c}}(m) = d_i - b.\sqrt[\alpha]{q'_i(t)}$. \dashv

In the following, before proving the main result of this section (Proposition 3.8), we will examine more closely this notion of α -freeness (Definition 3.2).

NOTATION 3.2. Let $K^{\alpha^{-1}}$ be a field extending K , with an endomorphism $\tilde{\alpha}$ equal to α on K and generated by the α roots of elements of K more precisely it is minimal with the property that $\forall k \in K \exists \ell \ell^{\alpha} = k$ (see [4] chapter 2, paragraph 5, Theorem 2).

LEMMA 3.4. Let L belong to \mathcal{R} . Then, there is a $K^{\alpha^{-1}}$ -vector space belonging to \mathcal{R} in which L embeds, namely $L.T^{-1}$.

PROOF. The subset $T := \{1, t, t^2, \dots\}$ of R and check that it is a right denominator set (we have for any $n \in \omega$, for any $q \in R$, $t^n.q^{\alpha^n} = q.t^n$) (see Notation 2.3). Then we form the ring of fractions $R.T^{-1}$ of R with respect to this right denominator set and consider the module of fractions $L.T^{-1}$ of L with respect T . Since the action of t is injective, L embeds in this $R.T^{-1}$ -module as an R -module. Then, $L.T^{-1}$ is naturally endowed with a structure of $K^{\alpha^{-1}}$ -vector space. Indeed, let $k^{\alpha^{-1}} \in K^{\alpha^{-1}}$, and $n \in L.T^{-1}$, we define $n.k^{\alpha^{-1}} := n.t.k.t^{-1}$. Let us check that for an element $n \in L$ and $k^{\alpha^{-1}} \in K$, this action coincides with the action of K on L . If $k^{\alpha^{-1}} \in K$, then there exists $\ell \in K$ such that $\ell^{\alpha} = k$, so $n.t.k.t^{-1} = n.t.\ell^{\alpha}.t^{-1}$ and using the commutation rule in the skew polynomial ring R we get that $n.t.\ell^{\alpha}.t^{-1} = n.\ell.t.t^{-1} = n.\ell = n.k^{\alpha^{-1}}$ (note that this is well-defined). Now, $L.T^{-1}$ is again T -torsion-free by Proposition 9.12 in [8]. \dashv

NOTATION 3.3. Let L belong to \mathcal{R} . We will denote by $L.K^{\alpha^{-1}}$ the $K^{\alpha^{-1}}$ -vector space of $L.T^{-1}$ generated by L . So it is the set of all finite sums of the form

$$\sum_i n_i.t.k_i.t^{-1} = \sum_i n_i.k_i^{\alpha^{-1}}.$$

Note that we have that $\sum_i (n_i.k).k_i^{\alpha^{-1}} = \sum_i n_i.(k_i^{\alpha^{-1}}.k)$, for any $k \in K$. Indeed, $n.t.k^{\alpha}.t^{-1} = n.k$. So we have a map from $L \otimes_K K^{\alpha^{-1}}$ to $L.K^{\alpha^{-1}}$.

LEMMA 3.5. Suppose that $N \in \mathcal{R}$. Then, $M \subseteq_t N$ implies that for any $n_i \in N$, $k_i \in K - \{0\}$, the element $\sum_{i=1}^l n_i.k_i^{\alpha^{-1}}$ of $N.K^{\alpha^{-1}}$ has the following property: if $\sum_{i=1}^l n_i.k_i^{\alpha^{-1}}.t \in M$ then either all the $n_i \in M$ or $\sum_{i=1}^l n_i.k_i^{\alpha^{-1}} = 0$.

PROOF. Suppose that $\sum_{i=1}^l n_i.k_i^{\alpha^{-1}}.t \in M$ with $n_i \in N$ and $k_i \in K - \{0\}$.

If $l = 1$, then $n.k^{\alpha^{-1}}.t \in M$ implies that $n.t.k \in M$ and so $n.t \in M$ which implies that $n \in M$ (since $M \subseteq_t N$).

Suppose that not all $n_i \in M$, since we have the property that (if $n \notin M$, then $n.t \notin M$), we may extract a non empty maximal subset from $\{n_1.t, \dots, n_l.t\}$ of K -linearly independent elements over M . W.l.o.g., assume that this subset is equal to $\{n_1.t, \dots, n_s.t\}$ with $s \geq 1$.

If $s = l$, then $\sum_{i=1}^l n_i.k_i^{\alpha^{-1}}.t = \sum_{i=1}^l n_i.t.k_i \in M$ and so $n_i.t \in M$ which entails that $n_i \in M$.

If $1 \leq s < l$, we express for $l \geq j > s$ $n_j.t \equiv_M \sum_{i=1}^s n_i.t.k_{ij}$ with $k_{ij} \in K - \{0\}$. So we get $\sum_{i=1}^l n_i.t.k_i \equiv_M \sum_{i=1}^s n_i.t.(\sum_{j=s+1}^l k_i + k_{ij}.k_j) \equiv_M 0$. This implies that $\bigwedge_{i=1}^s \sum_{j=s+1}^l k_i + k_{ij}.k_j = 0$ and so $\sum_{i=1}^l n_i.t.k_i = 0$. Rewriting $\sum_{i=1}^l n_i.t.k_i$ as $\sum_{i=1}^l n_i.k_i^{\alpha^{-1}}.t$ and applying the fact that $L.T^{-1} \in \mathcal{R}$, we get that

$$\sum_{i=1}^l n_i.k_i^{\alpha^{-1}} = 0. \tag{†}$$

In the following definition we have $M \subseteq N$ belonging to \mathcal{R} and in what follows, we will consider N embedded in $N.T^{-1}$ and so its extension $N.K^{\alpha^{-1}}$ endowed with a structure of $K^{\alpha^{-1}}$ -vector space.

DEFINITION 3.3. N is linearly disjoint from $K^{\alpha^{-1}}$ over M if any set of elements of N which are K -linearly independent over M remains in the extension $N.K^{\alpha^{-1}}$, $K^{\alpha^{-1}}$ -linearly independent over M , namely

$$\forall n_i \in N \left[(\exists k_i^{\alpha^{-1}} \in K^{\alpha^{-1}} - \{0\} \sum_i n_i.k_i^{\alpha^{-1}} \equiv_M 0) \implies (\exists k_i' \in K - \{0\} \sum_i n_i.k_i' \equiv_M 0) \right].$$

LEMMA 3.6. Let $M \subseteq_t N$ belong to \mathcal{R} , then

N is linearly disjoint from $K^{\alpha^{-1}}$ over M

iff the following property holds:

for all $(e_i) \subset K^{\alpha^{-1}} - \{0\}$ such that that $\exists (l_i) \subset N - M$ with $\sum_i l_i.e_i \equiv_M 0$, there exist $(k_i) \in K$ with $k_i \neq 0$ for some i , such that $\sum_i k_i.e_i = 0$.

PROOF.

(\rightarrow) Let $(e_i) \subset K^{\alpha^{-1}} - \{0\}$ and $(l_i) \subset N - M$ and suppose that $\sum_i l_i.e_i \equiv_M 0$. Among the elements l_i , select a K -linearly independent subset modulo M , say l_1, \dots, l_k and express the others as K -linear combinations modulo M , namely $l_j \equiv_M \sum_{i=1}^k l_i.k_{ij}$. So, we get that $(\sum_{i=1}^k l_i.(e_i + \sum_{j>k} k_{ij}.e_j)) \equiv_M 0$ and so by hypothesis, we have that $\bigwedge_{i=1}^k (e_i + \sum_{j>k} k_{ij}.e_j = 0)$.

(\leftarrow) Suppose that $\sum_i l_i.e_i \equiv_M 0$ with (e_i) belonging to $K^{\alpha^{-1}} - \{0\}$. Let e_1, \dots, e_k be a maximal subset of the (e_i) consisting of K -linearly independent elements. So, for each $j > k$, there exist $(k_{ij}) \in K$ such that $e_j = \sum_{i=1}^k e_i.k_{ij} = \sum_{i=1}^k k_{ij}.e_i$ since $K^{\alpha^{-1}}$ is a commutative field. We get that $\sum_{i=1}^k (l_i + \sum_{j>k} l_j.k_{ij}).e_i \equiv_M 0$. This implies that $\bigwedge_{i=1}^k (l_i + \sum_{j>k} l_j.k_{ij} \equiv_M 0)$. -1

PROPOSITION 3.7. *Let $M \subseteq_t N$ belong to \mathcal{R} . Then,*

N and $K^{\alpha^{-1}}$ are linearly disjoint over M

iff

there is an α -basis of K which is α -free in N over M

iff

every α -basis of K is α -free in N over M .

PROOF.

(1. \rightarrow 3.) Let (c_i) be an α -basis of K . Suppose that $\sum_i l_i.t.c_i \equiv_M 0$, where $l_i \in N$. Then, in $N.T^{-1}$ we have $\sum_i l_i.c_i^{\alpha^{-1}}.t \equiv_M 0$ (definition of the action of $K^{\alpha^{-1}}$ on N). So, by Lemma 3.5, either all the l_i belong to M and so we are done, or $\sum_i l_i.c_i^{\alpha^{-1}} = 0$. In the second case using the preceding lemma, we get that there exist $k_i \in K$ such that $\sum_i k_i.c_i^{\alpha^{-1}} = 0$. So, $\sum_i k_i^{\alpha}.c_i = 0$, which contradicts the fact that the c_i 's are α -free in K .

(1. \leftarrow 2.) Assume that (c_i) is an α -basis of K which is α -free in N over M . Suppose that $\sum l_i.e_i \equiv_M 0$ with $l_i \in N$, $e_i \in K^{\alpha^{-1}} - \{0\}$. Applying t , we get $\sum_i l_i.t.e_i^{\alpha} \equiv_M 0$. Express $e_i^{\alpha} = \sum_j k_{ij}^{\alpha}.c_j$ with $k_{ij} \in K - \{0\}$. Then, $\sum_i l_i.t.(\sum_j k_{ij}^{\alpha}.c_j) \equiv_M 0$, so $\sum_j (\sum_i l_i.k_{ij}).t.c_j \equiv_M 0$. By hypothesis, we get that $\bigwedge_j (\sum_i l_i.k_{ij} \equiv_M 0)$. (Note that in this direction we did not use the assumption that $M \subseteq_t N$.) -1

REMARK 3. When K is viewed with its full field structure, a well-known construction in order to add solutions to one variable difference equations is the construction of the Picard-Vessiot extensions (see [12]).

Let us recall this construction below. First, we note the following: let $p(t) \in X$ of the form: $t^n + t^{n-1}.a_{n-1} + \dots + a_0$, with $a_0 \neq 0$.

To the formula:

$$v.(t^n + t^{n-1}.a_{n-1} + \dots + a_0) = 0,$$

we will associate the equation:

$$V.t = V.A,$$

where $A \in GL_n(K)$ and V is the tuple $(v \ v.t \ \dots \ v.t^{n-1})$

$$\begin{pmatrix} v.t & v.t^2 & \dots & v.t^n \end{pmatrix} = \begin{pmatrix} v & v.t & \dots & v.t^{n-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \dots & -a_0 \\ 1 & 0 & 0 & \dots & -a_1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & -a_{n-1} \end{pmatrix}$$

In the case where α is an automorphism, to the equation of the form $m.p(t) = 0$, with $p(t) \in X$, corresponds a Picard-Vessiot extension of K (see [12] Chapter 1), namely a simple difference K -algebra containing a solution of that equation and in the case the field of constants is algebraically closed, minimal with this property. (Recall that a difference ring is simply a ring with an endomorphism. A simple difference ring is one without non-trivial difference ideals and a difference ideal is a ring ideal with the additional property that if $a \in I \implies a.t \in I$, so it is simply an R -submodule. Note also that in a simple difference ring the action of t is injective (the kernel of t is a difference ideal).)

Now if α is an endomorphism and not necessarily an automorphism, we can perform the same construction to obtain a simple difference K -algebra.

Indeed, one forms the K -algebra $K[x_{ij}, 1/det]$, where det is the determinant of (x_{ij}) and we define the action of t by setting $(x_{ij}).t = (x_{ij}).A$ and extend it by linearity. We note that the radical of a difference ideal is again a difference ideal. So, the quotient of $K[x_{ij}, 1/det]$ by a maximal difference ideal is a simple difference ring without nilpotent elements.

Now we are ready to consider extensions of K (viewed as an R -module) where one increases the torsion submodules. We place ourselves in the general setting of models of T_α .

PROPOSITION 3.8. *Let M be a model of T_α . Then M can be embedded in a model of $T(R)$.*

PROOF. First, note that if $L \in \mathcal{R}$ is an α -separable extension of M , then we may expand L to an \mathcal{L} -structure and this expansion becomes a model of T_α . Indeed, by Lemma 3.1 (2), $M \subseteq_t L$ and so we may apply Proposition 3.3. Denote by X_p the set of prime separable elements of X .

Set $M = L_0$, we will construct a chain of α -separable extensions L_{i+1} of L_i in \mathcal{R} such that (*) for every $n \in L_i$, for every $q(t) \in X_p$, there exists $m \in L_{i+1}$ such that $m.q(t) = n$.

Set $L := \bigcup_{i \in \omega} L_i$. Then, L is an α -separable extension of M as a union of such, and so it can be expanded to an \mathcal{L} -structure. Let us check that it satisfies axiom schemes 4 and 5. Let $n \in L$, so $n \in L_i$ for some i ; let $q(t) \in X$ and let $q(t) = q_1(t) \cdot \dots \cdot q_s(t)$ be a decomposition of $q(t)$ in prime factors $q_i(t) \in X_p$. Set $n := m_0$, by construction, there exists $m_j \in L_{i+j}$ such that $m_{j-1} = m_j.q_{s-j+1}(t)$, $1 \leq j \leq s$. We obtain that $n = m_0 = m_s.q_1(t) \cdot \dots \cdot q_s(t)$.

So, either L_i satisfies property (*), in this case we set $L_{i+1} = L_i$, or we proceed as follows.

We will first explain how to construct an α -separable extension of L_i belonging to \mathcal{R} and containing an element e_0 with $e_0.q(t) = m \in L_i$, where m was not already divisible by $q(t) \in X_p$ in L_i .

More generally, we will assume that we already constructed an α -separable extension L'_i of L_i in \mathcal{R} and that $m \in L_i$ is not divisible in L'_i by $q(t) \in X_p$.

Let $q(t) = \sum_{i=0}^d t^i.k_i$ with $k_i \in K$ and $k_0, k_d \neq 0$. Consider the following extension of L'_i : the sum of L'_i with a K -vector space of dimension d , namely $\tilde{L}'_i := L'_i + (e_0.K \oplus e_1.K \oplus \dots \oplus e_{d-1}.K)$. We want to endow \tilde{L}'_i with an R -module structure. We first define the action of t on the e_i 's and then extend it on \tilde{L}'_i as follows: $(l + \sum_i e_i.k_i).t := l.t + \sum_i e_i.t.k_i^\alpha$, where $l \in L'_i$. We then define the action of $r(t) = \sum_i t^i.k'_i \in R$ with $k'_i \in K$ on each e_j by $\sum_i e_j.t^i.k'_i$; we extend it the same way as before on \tilde{L}'_i . For $0 \leq i \leq d-2$, set $e_i.t := e_{i+1}$ and $e_{d-1}.t := n_0 - \sum_{i=0}^{d-1} e_i.k_i$. So, $e_0.q(t) = n_0$ and \tilde{L}'_i is generated as an R -module by L'_i and e_0 . Since $q(t)$ is prime, \tilde{L}'_i is a direct sum of L'_i with the K -vector space generated by $\{e_0.t^i; 0 \leq i \leq d-1\}$. Indeed, suppose that $e_0.p(t) \in L'_i$ and take such element $p(t) \in R$ of minimal degree strictly less than d . Applying the Euclidean algorithm, we get that $q(t) = p(t).r_1(t) + r_2(t)$ with $r_1(t), r_2(t) \in R$ and $\deg(r_2(t)) < \deg(p(t)) < d$. Since $e_0.q(t) \in L_i$ and $e_0.p(t) \in L'_i$ by the minimality assumption (on the degree of $p(t)$) we get that $r_2(t) = 0$. But $q(t)$ is prime, so we reach a contradiction.

Let us check that \tilde{L}'_i is α -separable over L'_i (and so will be over L_i). It suffices to prove that each $e_i = e_0.t^i, 0 \leq i \leq (d-1)$, is α -separable over L'_i . Indeed, by the fact that X is a right denominator set, a sum of α -separable elements over L'_i is α -separable over L'_i (see Lemma 3.1 (3)). Now, since $t^i.q^{\alpha^i}(t) = q(t).t^i$, we have that $e_i.q^{\alpha^i}(t) = n_0.t^i \in L_i$.

Finally, let us show that $\tilde{L}'_i \in \mathcal{R}$. Suppose that $a + e_0.r(t) \in \tilde{L}'_i$, with $a \in L'_i$ and $\deg(r(t)) < d$, be such that $(a + e_0.r(t)).t \in L'_i$, so $(e_0.r(t)).t \in L'_i$. Using the Euclidean division algorithm, since the degree of $r(t).t$ is less than or equal to d , we get that $q(t) = r(t).t.r_1(t) + r_2(t)$ with degree of $r_2(t)$ strictly less than d . Since $q(t) \in X$, if $r_2(t) \neq 0$, then $r_2(t) \in X$. So, $e_0.q(t) = e_0.r(t).t.r_1(t) + e_0.r_2(t) \in L'_i$, but we have shown above that this implies that $r_2(t) = 0$. So, $q(t) = r(t).t.r_1(t)$, which contradicts the fact that $q(t) \in X_p$.

Let $\{q_\gamma(t) : \gamma < \delta\}$ be an enumeration of X_p and let $\{m_\beta : \beta < \lambda\}$ be an enumeration of the elements of L_i . We take the first element in the enumeration of L_i which is not divisible by an element of X_p and choose the first such element of X_p , say m_β is not divisible by q_γ . We showed how to construct an α -separable extension in \mathcal{R} of L_i of the form $L_i + \langle e_{\beta,\gamma} \rangle$, where $e_{\beta,\gamma}$ has the property that $e_{\beta,\gamma}.q_\gamma(t) = m_\beta$. We denote such extension by $L_{i,\beta,\gamma}$.

Then fixing β , we will look for the next element in the enumeration of X_p such that m_β is not divisible by it in $L_{i,\beta,\gamma}$. Then, we move to the next element of L_i . More precisely, we proceed as follows.

By induction on the ordinals, we define the following extensions of L_i .

Set $L_{i,0} := L_i$ and for $0 < \beta < \lambda$, define $L_{i,\beta+1} := \bigcup_{\gamma < \delta} L_{i,\beta,\gamma}$, where for $0 \leq \beta < \lambda$ we have defined $L_{i,\beta,\gamma}$ as follows:

$$\begin{aligned} &\text{if } \gamma = 0, \\ &L_{i,\beta,0} := L_{i,\beta} + \langle e_{\beta,0} \rangle, \text{ if } m_\beta \text{ is not divisible in } L_{i,\beta} \text{ by } q_0(t) \text{ and} \\ &L_{i,\beta,0} := L_{i,\beta} \text{ otherwise,} \end{aligned}$$

if γ is successor,

$L_{i,\beta,\gamma} := L_{i,\beta,\gamma-1} + \langle e_{\beta,\gamma} \rangle$, if m_β is not divisible in $L_{i,\beta,\gamma-1}$ by $q_\gamma(t)$ and
 $L_{i,\beta,\gamma} := L_{i,\beta,\gamma-1}$ otherwise,

if γ is limit, define

$L_{i,\beta,\gamma} := \bigcup_{v < \gamma} L_{i,\beta,v} + \langle e_{\beta,\gamma} \rangle$, if m_β is not divisible in $L_{i,\beta,v}$ for some $v < \gamma$
 by $q_\gamma(t)$ and
 $L_{i,\beta,\gamma} := \bigcup_{v < \gamma} L_{i,\beta,v}$ otherwise.

Note that each $L_{i,\beta,\gamma}$ belongs to \mathcal{R} and that it is α -separable, and so each $L_{i,\beta}$ has those two properties.

Finally, for β a limit ordinal, we define $L_{i,\beta} := \bigcup_{\rho < \beta} L_{i,\rho}$.

Set $L_{i+1} := L_{i,\lambda}$, and note that it is an α -separable extension of L_i in \mathcal{R} . \dashv

COROLLARY 3.9. $T(R)$ is consistent.

PROOF. The additive structure $(K, +, 0, \alpha)$ of the field K with its endomorphism α is a model of T_α and by the preceding proposition can be embedded in a model of $T(R)$. \dashv

COROLLARY 3.10. Suppose that $\text{Fix}(\alpha)$ is infinite. Then, the theory $T(R)$ is the model completion of T_α .

PROOF. Apply Proposition 3.8 and Corollary 2.11. \dashv

In case $\text{Fix}(\alpha)$ is finite, one may obtain a similar statement for the different completions of $T(R)$, using Corollary 2.12.

§4. Ultraproducts of separably closed fields. Let K_n be a separably closed field of characteristic p_n and imperfection degree e , $p_n \in \mathcal{P}$, $e \in \omega \cup \{\infty\}$. Let U be a non principal ultrafilter on ω and consider the ultraproduct $K := \prod_U K_n$. Either, $(p_n)_{n \in \omega}$ is a strictly increasing sequence and K is an algebraically closed field of characteristic zero. Or, K is a separably closed field of characteristic p , of fixed finite imperfection degree e . In each K_n , we may consider the non standard Frobenius maps either $x \rightarrow x^{p_n}$ or $x \rightarrow x^{p^n}$.

Let α be the non-standard Frobenius map sending $[x_n]_U$ to either $[x_n^{p_n}]_U$ or to $[x_n^{p^n}]_U$. In this case, we obtain as the subfield fixed by α , a pseudo-finite field either $\prod_U \mathbb{F}_{p_n}$ or $\prod_U \mathbb{F}_{p^n}$.

Let us first consider the characteristic zero case. The characteristic p case is rather similar, replacing p_n by p^n . In each K_n , we have a p_n -basis $\mathcal{B}_n := \{b_1^n, \dots, b_e^n\}$. Denote by \mathcal{M}_{p_n} the corresponding set of p_n -monomials. Let \mathcal{E} be a basis of K over $\alpha(K)$ with $\alpha = [\alpha_n]$. Let R be the skew polynomial ring $K[t; \alpha]$, with the usual commutation rule $k.t = t.k^\alpha$, $k \in K$. Let R_n be the skew polynomial ring $K_n[t; \alpha_n]$ with $n \in \omega - \{0\}$.

Corresponding to this particular ring R , we may write as before a theory $T(R)$.

LEMMA 4.1. Let $M_n \models T(R_n)$, then any non-principal ultraproduct $M := \prod_U M_n$, where U is a non-principal ultrafilter, is a model of the theory $T(R)$. In particular, $K := \prod_U K_n \models T(R)$.

PROOF. Let $\{e_{j,n} : j \in p_n^e\} = \mathcal{M}_{p_n}$. The action of R on $M := \prod_U M_n$ is defined by $[m_n].[r_n] := [m_n.r_n]$, where $[m_n] \in M$ and $[r_n] \in R$. Now if $m = \sum_{c_i \in \bar{c}} m_i.t.c_i$ where \bar{c} is an ℓ -tuple included in \mathcal{E} , we define $\lambda_i^{\bar{c}}(m) = m_i$. Let us check it is

well-defined (axiom scheme 3) i.e., suppose that

$$(*) \quad \sum_{c_i \in \bar{c}} [m_{n,i}] \cdot t \cdot c_i = 0.$$

Each $c_i \in \prod_U K_n$, so $c_i = [c_{i,n}]$, where $c_{i,n} \in K_n - K_n^{\alpha_n}$. Write $c_{i,n} = \sum_{j \in p_n^e} k_{ji}^{\alpha_n} \cdot e_{j,n}$ with the $k_{ji} \in K_n - \{0\}$. In matrix notation, we have:

$$\begin{pmatrix} c_{1,n} \\ c_{2,n} \\ \dots \\ c_{\ell,n} \end{pmatrix} = \begin{pmatrix} \dots & k_{j1}^{\alpha_n} & \dots \\ & & \\ & & \\ \dots & k_{j\ell}^{\alpha_n} & \dots \end{pmatrix} \begin{pmatrix} e_{1,n} \\ \dots \\ e_{p_n^e,n} \end{pmatrix}.$$

Set

$$A^{\alpha_n} := \begin{pmatrix} \dots & k_{j1}^{\alpha_n} & \dots \\ & & \\ & & \\ \dots & k_{j\ell}^{\alpha_n} & \dots \end{pmatrix}.$$

Since, for almost all n , the $c_{i,n}$, $1 \leq i \leq \ell$ are linearly independent over $K_n^{\alpha_n}$, there exist a permutation matrix P and an invertible matrix Q such that $Q^{\alpha_n} \cdot A^{\alpha_n} \cdot P$ is equal to an upper triangular matrix with only non zero elements on its diagonal. So, let us rewrite $(*)$ as $\sum_{i \leq \ell} [m_{n,i}] \cdot t \cdot (\sum_{j \in p_n^e} k_{ji}^{\alpha_n} \cdot e_{j,n}) = 0$. By interchanging the two sums, we get $\sum_{j \in p_n^e} (\sum_{i \leq \ell} m_{n,i} \cdot k_{ji}) \cdot t \cdot e_{j,n} = 0$, for almost all n . The equality $(*)$ holds iff $N := \{n \in \omega : \sum_{j \in p_n^e} (\sum_{i \leq \ell} m_{n,i} \cdot k_{ji}) \cdot t \cdot e_{j,n} = 0\}$ belongs to U . For $n \in N$, since the $\{e_{j,n}\}_{j \in p_n^e}$ form a basis, we get that $\bigwedge_{j \in p_n^e} \sum_{i \leq \ell} m_{n,i} \cdot k_{ji} = 0$, which in matrix form gives:

$$\begin{pmatrix} m_{1,n} & \dots & m_{\ell,n} \end{pmatrix} \cdot \begin{pmatrix} \dots & k_{j1} & \dots \\ & & \\ & & \\ \dots & k_{j\ell} & \dots \end{pmatrix} = 0.$$

Note that $Q \cdot A \cdot P$ is again a upper triangular matrix with only non zero elements on its diagonal, say $(B, 0)$ where B is upper triangular with only non zero elements on its diagonal.

Now, $(m_{1,n}, \dots, m_{\ell,n}) \cdot Q^{-1} \cdot B = 0$, for any $n \in N$. So, $[m_{1,n}], \dots, [m_{\ell,n}]$ are equal to zero.

Let $q(t) \in X$ of degree d , then $q(t) = [q_n(t)]$, where $q_n(t)$ belongs to R_n . Then $\text{ann}(q(t))$ is non zero, since it is non zero in each M_n . To check axiom scheme 5, we do it in each M_n using the fact that each one is a model of $T(R_n)$.

To prove that $K \models T(R)$, by the first part it suffices to prove that $K_n \models T(R_n)$. Note that in addition, we have that the annihilator of a polynomial $q_n(t)$ of degree d is a vector space of dimension d over the finite field \mathbb{F}_{p_n} (respectively \mathbb{F}_{p^n}) and this is equivalent to the set of solutions of a polynomial of degree p_n^d (respectively $(p^n)^d$) is a \mathbb{F}_{p^n} -vector-space (respectively a \mathbb{F}_{p^n} -vector-space) of dimension d . \dashv

COROLLARY 4.2. *$T(R)$ admits quantifier elimination, is consistent, complete and any model of $T(R)$ is elementarily equivalent to a non-principal ultraproduct of models of $T(R_n)$.*

PROOF. The field $Fix(\alpha)$ is a pseudo-finite field, isomorphic either to $\prod_U \mathbb{F}_{p^n}$ or to $\prod_U \mathbb{F}_{p^n}$ and so infinite. So we may apply Corollary 2.10. The last assertions follow from the preceding Lemma. \dashv

§5. Decidability. In our context, it seems more adequate to use for the notion of decidability, instead of the classical one, the notion developed by L. Blum, M. Shub and S. Smale ([2], chapters 2 and 3). This last notion seems to be more natural in the case where the ring R is uncountable since the cardinality of the ring does not seem here to play any model-theoretic role. We will assume as in Corollary 2.11, that $Fix(\alpha)$ is infinite. We replace a Turing machine by a BSS-machine where the set of constants is the ring R and the basic operations are the ones given by the language \mathcal{L} . It remains to check that $T(R)$ and its complement in R^ω are halting sets over R .

First, given an \mathcal{L} -formula, one constructs an equivalent \mathcal{L}_R -formula, replacing the functions symbols $\lambda_i^{\bar{c}}$ by their \mathcal{L}_R -definitions. Then, we use the quantifier elimination result in language \mathcal{L} and its effectiveness (see Proposition 7.2 in [5]) and so using the p.p. elimination in theories of modules and noting that in $T(R)$, if the index of two p.p. formulas is strictly bigger than 1, then it is infinite (see Corollary 2.11), it remains to check the validity of sentences of the form:

$$\exists x \bigvee_{\ell} \left[\bigwedge_k t_k(x) = 0 \wedge s_\ell(x) \neq 0 \right],$$

where $t_k(x), s_\ell(x)$ are \mathcal{L} -terms. Each \mathcal{L} -term t_k (respectively s_ℓ) is equivalent to a term of the form $\sum_{\bar{r}_i} \lambda_{\bar{r}_i}^{\bar{c}}(x) \cdot r_i$ (respectively $\sum_{\bar{s}_j} \lambda_{\bar{s}_j}^{\bar{c}}(x) \cdot s_j$), where $r_i, s_j \in R - \{0\}$ (see Lemma 4.1 in [5]). Notice that we may assume that the λ functions have the same superscript $\bar{c} \in \mathcal{E}$.

Each disjunct can be put in the form $\bar{x}.A = 0 \wedge \bar{x}.\bar{c} \neq 0 \wedge x = \sum_{i \in p^{en}} x_i.t.m_i$, where $A \in M_{p^{en} \times p^{en}}(R)$, $\bar{c} \in R$. Then we reduce the matrix A to a lower triangular matrix in order to get the equivalent formula

$$\exists \bar{z} [\bar{z} = \bar{x}.P^{-1} \wedge \bar{z}.\tilde{A} = 0 \wedge \bar{z}.P.\bar{c} \neq 0 \wedge x = \sum_{i \in p^{en}} x_i.t.m_i],$$

where \tilde{A} is a lower triangular matrix and P a permutation matrix; it is an effective procedure (see Proposition 6.1 in [5]), it involves in particular repeated applications of the Euclidean algorithm in R . Considering now the formula $\bar{z}.P.\bar{c} \neq 0$, it remains to apply Gauss elimination and so whether

$$\exists x (\bar{x}.A = 0 \wedge \bar{x}.\bar{c} \neq 0 \wedge x = \sum_{i \in p^{en}} x_i.t.m_i)$$

holds, is equivalent to check whether a coefficient of z_j is non zero, where j is bigger than the number of non zero columns of \tilde{A} .

Acknowledgments. We would like to thank Pilar Dellunde and Françoise Delon with whom we have written two previous articles, on which this last one is based.

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