Symmetry of least energy nodal solutions for the Lane-Emden equation

Ch. Grumiau

Institut de Mathématique
Université de Mons-Hainaut
Mons, Belgium

May 2007, Northern Arizona University,
Flagstaff, USA
From what will we speak?

Lane-Emden problem

Let $\Omega$ open bounded of $\mathbb{R}^N$ and $2 < p < 2^*$, we consider the super-linear elliptic problem

$$(P) \begin{cases} \Delta u(x) + |u(x)|^{p-2}u(x) = 0, & \text{if } x \in \Omega \\ u(x) = 0, & \text{if } x \in \partial \Omega. \end{cases}$$

Question

What are the symmetries of least energy nodal solution?
The problem (P) is a variational problem. The solutions are **critical points** of the functional $J_p$, called **functional energy**, 

$$J_p : H_0^1(\Omega) \rightarrow \mathbb{R} : u \mapsto \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{p} \|u\|_{L^p}$$

$$\partial J_p(u) : H_0^1(\Omega) \rightarrow \mathbb{R} : v \mapsto \int_{\Omega} \nabla u \nabla v - \int_{\Omega} |u|^{p-2} uv$$
Solutions (1)

Theorem (A. Castro-J. Cossio-J. M. Neuberger, ’97)

A super-linear elliptic problem has got at least 3 non zero solutions: a positive, a negative and a nodal solution with two nodes.

For our problem (P), let us the Nehari variety
\[ \mathcal{N}_p := \{ u \in H^1_0(\Omega) \setminus \{0\} : (\partial J_p(u))(u) = 0 \}. \]

- Solution ⇒ \( \mathcal{N} \)
- Positive solutions ⇒ \( \mathcal{N}_p^+ := \{ u > 0 : u \in \mathcal{N}_p \} \)
- Negative solutions ⇒ \( \mathcal{N}_p^- := \{ u < 0 : u \in \mathcal{N}_p \} \)
- Nodale solutions ⇒ \( \mathcal{N}_p^1 := \{ u : u^{\pm} \in \mathcal{N}_p \} \) (not a differential variety)
Solutions (2)

- \( \exists u_1 \in \mathcal{N}_p^+ : J_p(u_1) = \inf_{\mathcal{N}_p^+} J_p \) and
  \( \exists u_2 \in \mathcal{N}_p^- : J_p(u_2) = \inf_{\mathcal{N}_p^-} J_p \)

- \( u_1 \) and \( u_2 \) are critical points (ground states)

- \( \exists u_3 \in \mathcal{N}_p^1 : J_p(u_3) = \inf_{\mathcal{N}_p^1} J_p \)

- \( u_3 \) is critical point (least energy nodal solution)
Symmetries of least energy nodal solutions?

**Theorem (T. Bartsch, T. Weth, M. Willem, '05)**

*On a radial domain, a nodal solution of least energy respects a Schwarz foliated symmetry. So, it’s pair in $N - 1$ independent directions.*

........problem......les dessins ne s’affichent pas.................
 Questions?

- What happens for other domains?
- On a disk, is it odd in one direction?
- Does the nodal line intersect (from an orthogonal manner) the boundary?
Theorem (G., V. Bouchez, J. Van Schaftingen)

If \( e_2 \) is simple, a least energy nodal solution respects all the symmetries and antisymmetries of \( e_2 \) for \( p \) close to two. So, if \( \Omega \) has got \( N \) independant symmetry directions, it’s at least pair in \( N - 1 \) directions.

where \( e_2 \) is the second eigenfunction of the Laplacian with Dirichlet boundary conditions.

Remark

In the case where \( e_2 \) isn’t simple, we give assumptions such that least energy nodal solutions respects symmetries of one second eigenfunction (but we doesn’t know the symmetries of this function).
Example: rectangle
Theorem (G., C. Troestler)

On a disk of dimension 2 or 3, a nodal solution of least energy is odd in one direction for $p$ close to two.
Equivalent problem

Studying symmetries of solutions of the problem (P) is the same that studying symmetries of solutions of the problem

\[
(P2) \begin{cases}
\Delta u(x) + \lambda_2 |u(x)|^{p-2} u(x) = 0, & \text{if } x \in \Omega \\
u(x) = 0, & \text{if } x \in \partial \Omega'
\end{cases}
\]

where \( \lambda_2 \) is the second eigenvalue of the Laplacian with Dirichlet boundary conditions.

In fact, \( u_p \) is a least nodal solution of the problem (P) if and only if \( u_{p,2} := \lambda_2^{\frac{-1}{p-2}} u_p \) is a nodal solution of least energy of \( (P2) \).
Let $E_2$ the second \textbf{eigenspace} of the Laplacian with Dirichlet boundary conditions and $u_{p,2}$ a family of least energy nodal solutions of (P2).

- $u_{p,2}$ is upper bounded
- $u_{p,2}$ isn’t close to zero
- The accumulation points of $u_{p,2}$ are second eigenfunctions
- Characterizing this points
- Using IFT (implicit function theorem) to obtain a single curve of solutions in start of the accumulation points
- Concluding the symmetries obtained
Upper bounded

Lemma

Let $u_{p,2}$ a solution of least energy of (P2). The family $\{u_{p,2}\}_{p>2}$ is upper bounded.

- Let $v \in E_2$, we obtain

$$\left(\frac{1}{2} - \frac{1}{p}\right)\|u_{p,2}\|_{H_0^1} \leq \left(\frac{1}{2} - \frac{1}{p}\right)\left\{t_p^+\|v^+\|_{H_0^1}^2 + t_p^-2\|v^-\|_{H_0^1}^2\right\},$$

- We’ve got control about $t_p^+$ and $t_p^-$ when $p \to 2$. 
Not close to two(1)

Lemma

The family \( \{u_{p,2}\}_{p>2} \) isn't close to two.

- For all nodal function \( u \in H^1_0 \), there exists a function \( \nu \) in \( \langle u^+, u^- \rangle \) belongs to variety of Nehari and orthogonal at \( e_1 \) in \( L^2 \), where \( e_1 \) is the first eigenfunction of the Laplacian.
- If \( u \) belongs to Nehari variety, \( \nu \) is smaller on norm than \( u \).
- Then, we proof that the family \( (\nu_{p,2})_{p>2} \) constructs with \( (u_{p,2})_{p>2} \) can't be close to zero.
**Proposition**

The family \( \{u_{p,2}\}_{p>2} \) converges to a non zero function in \( E_2 \).

- By upper boundary, we obtain the weakly convergence in \( H^1_0 \).
- In fact, we’ve got a strongly convergence.
- Then, we prove that it’s solution of the second problem of Dirichlet.
- By the lower boundary, we obtain our thesis.
What about this eigenfunction which is chosen?

**Proposition**

An accumulation point of \((u_{p,2})_{p>2}\) is second eigenfunction \(u\) of the Laplacian such that, for all \(v \in \mathcal{E}_2\),

\[
\int_{\Omega} \ln(|u|) uv = 0.
\]

W.l.o.g, we consider \(u_{p,2} \to u\) when \(p \to 2\).

- For all \(v \in \mathcal{E}_2\) and \(p > 2\), \(\int_{\Omega} (|u_{p,2}|^{p-2}u_{p,2} - u_{p,2}) v = 0\)
- \(\lim_{p\to 2} \int_{\Omega} \frac{(|u_{p,2}|^{p-2}u_{p,2} - u_{p,2})}{p-2} v = \int_{\Omega} \ln|u| uv\)

**Remark**

If the second eigenfunction is simple, at rescaling by -1, this function is unique.
For all second eigenfunction respecting the last property, there exists only one curve of solutions which converges to this point?
IFT : equivalent equations

The following problems are equivalent :

\[ \Delta u(x) + \lambda_2 |u(x)|^{p-2}u(x) = 0, \]

\[ -\Delta u - \lambda_2 u = \lambda_2 (|u|^{p-2}u - u), \]

\[
\begin{cases}
(-\Delta - \lambda_2 \text{id})u = P \left( \lambda_2 (|u|^{p-2}u - u) \right) \\
\int_{\Omega} \frac{(|u|^{p-2}u-u)}{p-2} v = 0, \text{ for all } v \in E_2
\end{cases},
\]

where \( P \) (resp. \( Q := \text{id} - P \)) is the orthogonal projection from \( L^2 \) to \( E_2^\perp \) (resp. \( E_2 \)).

The limit problem when \( p \to 2 \) is

\[
(LP) \begin{cases}
(-\Delta - \lambda_2 \text{id})u = 0 \\
\int_{\Omega} \ln|u| uv = 0, \text{ for all } v \in E_2
\end{cases}.
\]
We consider the function

$$\varphi : [2, 2^*] \times E_2^\perp \times E_2 \to E_2^\perp \times E_2 :$$

$$(p, Pu, Qu) \mapsto \begin{cases} 
(Pu - T(P(\lambda_2(|u|^{p-2} - u))), Q\left(\frac{\lambda_2(|u|^{p-2} - u)}{p-2}\right)) \\
(Pu, Q(ln|u|u)), \text{ if } p = 2,
\end{cases}$$

where, for $v \in E_2^\perp$, $T(v)$ is the unique solution in $E_2^\perp$ of the problem $(-\Delta - \lambda_2 id)(u) = v$. The roots of $\varphi$ are the solutions of \((P2)\) and \((LP)\). For $(2, 0, u)$ where $u$ is an accumulation point of $u_p,2$ (and so solution of the limit problem \((LP)\)), we’ve got

$$\partial_{E_2^\perp \times E_2} \varphi(2, 0, u)(Ph, Qh) = (Ph, Q(h + ln|u|h)).$$
Using IFT

Proposition

If \( \partial E_2^\perp \times E_2 \) \( \phi(2,0,u) \) is injective, there exists, for \( p \) close to 2, one and only one continuous curve \( p \mapsto (P_{u,p,2}, Q_{u,p,2}) \) in the start of \((2,0,u)\) such that \( u_{p,2} \) solves the problem \((P2)\).

Corollary

If the second eigenfunction is simple, there exists, for \( p \) close to 2, one and only one continuous curve \( p \mapsto (P_{u,p,2}, Q_{u,p,2}) \) in the start of \((2,0,u)\) such that \( u_{p,2} \) solves the problem \((P2)\).

- As \( Ph = 0 \), we obtain \( h = \alpha e_2 \)
- As \( Q(h + \ln|u|h) = 0 \), we've got \( \int_\Omega \alpha e_2^2 + \int_\Omega \ln|u|\alpha e_2^2 = 0 \)
- As \( \int_\Omega \ln|u|\alpha e_2^2 = 0 \), we've got \( h = 0 \).
Respect all symmetry of a second eigenfunction

**Theorem**

*If we can use the IFT (b.e. for the simple second eigenfunction), a least energy nodal solution respects the symmetry and antisymmetry of one second eigenfunction (of \(e_2\) if it’s simple) for \(p\) close to 2.*

**Sketch when it’s simple:** Else,

- we’ve got a sequence \(u_{p,n,2}\) which doesn’t respect the symmetries of \(e_2\) by respect \(H\)
- by symmetry reflect following \(H\), we obtain a second curve of solutions of (P2) who converges to unique solution (by rescaling -1) of (LP).

**Remark**

*When \(e_2\) is simple, \(e_2\) is symmetric or antisymmetric in all symmetric directions of \(\Omega\).*
Proposition

In fact, the accumulation points of $u_{p,2}$ minimize the norm in $H_0^1$ of the second eigenfunction such that, for all $v \in E_2$,

$$\int_{\Omega} \ln(|u|) uv = 0.$$ 

Remark

It’s probably possible using this proposition to characterize least energy nodal solutions on a square.
The preceding reasoning does not go any more

Theorem (A. Aftalion F. Pacella, ’04)

On a radial domain, a nodal solution of least energy can’t be radial.

So, we’ve got an orbit of least energy nodal solutions. We can’t use the implicit function theorem. Nevertheless, on a disk, $u_{p,2}$ converge to a non-zero second eigenfunction of the Laplacian.

Idea

Using the fact that a least energy nodal solution respects a Schwarz foliated symmetry. W.l.o.g, we can assume that for all $p > 2$, $u_{p,2}$ respects a Schwarz foliated symmetry in the direction $H$. Then, we use a similar argument by working in $H^1_0$ restricted to functions pairs in orthogonal directions of $H$. The accumulations points of $u_{p,2}$ are pair in all orthogonal directions of $H$. 
Unique second eigenfunction with good symmetry for $N = 2$

**Proposition**

For $N = 2$, there exists one and only one (by rescaling $-1$) second eigenfunction of the Laplacian with Dirichlet conditions which is pair in all orthogonale directions of $H$. More, this function is **odd** in the direction $H$.

- Eigenfunctions are
  \[
  u_{n,m} = J_n \left( \frac{\alpha_{n,m}}{r} \right) (A_{n,m} \cos(n\theta) + B_{n,m} \sin(n\theta)).
  \]
- The second eigenvalue is $\alpha_{1,1}^2$ and eigenfunctions are
  \[
  J_1(\alpha(1,1)r)(C\sin(\theta + \varphi)).
  \]
- There exists one and only one function pair (by rescaling $-1$) in the good direction. More, it’s odd in the orthogonal direction.
Bessel functions for $N = 2$
Unique second eigenfunction with good symmetry for $N = 3$

**Proposition**

For $N = 3$, there exists one and only one (by rescaling $-1$) second eigenfunction of the Laplacian with Dirichlet conditions which is pair in all orthogonale directions of $H$. More, this function is **odd** in the direction $H$.

- Eigenfunctions are $u_{n,m,l} = r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\alpha_{n,l} r) Y_{n,m}(\varphi, \theta)$.
- $\left(\alpha_{1,1}\right)^2$ is the second eigenvalue and the second eigenfunction are

$$r^{-\frac{1}{2}} J_{\frac{3}{2}}\left(\alpha_{1,1}\right)^2(A\cos \varphi + B\cos \theta \sin \varphi + C\sin \theta \sin \varphi).$$

- There exist one and only one function pair (by rescaling $-1$) in the good directions. More, it’s odd in the direction $H$. 

Ch. Grumiau

Least energy nodal solutions
Bessel functions for $N = 3$
Let us $G := \mathbb{Z}_2$ (resp. $\mathbb{Z}_3$) and the action $\{T_k\}_{k \in G}$ on $H^1_0$ such that $\text{Fix}(G)$ is $H^1_0$ restricted to functions pair in orthogonal directions of $H$. Let the function

$$\varphi : [2, 2^*] \times \text{Fix}(G) \times \mathbb{R}$$

$$(p, u, \lambda) \mapsto (-(-\Delta)^{-1}(\lambda|u|^{p-2}u) + u, \|u\|^2 - 1).$$

Let $e_2$ the unique second eigenfunction with the good symmetry. We obtain by using IFT that there exist one and only one curve $p \mapsto (u_p, \lambda_p)$ such that $u_p$ is normed solution of

$$\Delta u + \lambda_p |u|^{p-2}u = 0.$$  

At a rescaling, it’s the solutions of the problem (P2).

**Theorem**

*For $p$ close to 2 and $N = 3$ or 3, a least energy nodal solution on a disk respects all symmetry of $e_2$. So, it’s odd in one direction.*
In the future

- On a disk, studying spherical harmonic to prove that least energy nodal solutions are odd in one direction for all dimension.
Open problems

- What happens for $p$ close to $2^*$?
- What happens for $p$ big?
- What happens for a super-linear elliptic equation in general?
- Nodal line intersects the boundary from an orthogonal manner?