Convergence of a mountain pass algorithm with projection

Ch. Troestler

Institut de Mathématique
Université de Mons-Hainaut
Mons, Belgium

Workshop on Theoretical and Computational Nonlinear Partial Differential Equations

Introduction

X a Hilbert space
$\mathscr{E} : X \to \mathbb{R}$ a $\mathscr{C}^1$ functional with the mountain-pass geometry

Compute MP type critical points for $\mathscr{E}$

- Choi & McKenna's
- Zhou's & al.

Ensure invariant solutions u are found, where by invariant it is meant

$$u \in K$$

where K is a closed convex cone (pointed at 0).

Introduction

X a Hilbert space
$\mathscr{E} : X \to \mathbb{R}$ a $\mathscr{C}^1$ functional with the mountain-pass geometry

Compute MP type critical points for $\mathscr{E}$

- Choi & McKenna's
- Zhou's & al.

Ensure invariant solutions u are found, where by invariant it is meant

$$u \in K$$

where K is a closed convex cone (pointed at 0).

Outline

1. Algorithm for invariant solutions
2. Examples
3. Open questions

Invariant solutions

K a closed convex cone (not necessarily salient).

1. $K = \{u \in H_0^1(\Omega) : u \geqslant 0\}$.

$$u \mapsto u^+$$

2. $K = \{u : \mathbb{R} \to \mathbb{R} : u$ is non-decreasing$\}$.

3. $K = \{u : \forall g \in G, \forall x \in \mathbb{R}^N, u(gx) = u(x)\}$ where G is a group acting on $\mathbb{R}^N$.

If $P : X \to K$ is a projector on K, $u \in K = \operatorname{Im} P$ iff

$$P(u) = u$$

Invariant solutions

K a closed convex cone (not necessarily salient).

1. $K = \{u \in H_0^1(\Omega) : u \geqslant 0\}$.

$$u \mapsto u^+$$

2. $K = \{u : \mathbb{R} \to \mathbb{R} : u \text{ is non-decreasing}\}$.

$$u \mapsto \int_0^{\cdot} |u'(t)| \, \mathrm{d}t$$

3. $K = \{u : \forall g \in G, \forall x \in \mathbb{R}^N, u(gx) = u(x)\}$ where G is a group acting on $\mathbb{R}^N$.

If $P : X \to K$ is a projector on K, $u \in K = \operatorname{Im} P$ iff

$$P(u) = u$$

Invariant solutions

K a closed convex cone (not necessarily salient).

1. $K = \{u \in H_0^1(\Omega) : u \geqslant 0\}$.

$$u \mapsto u^+$$

2. $K = \{u : \mathbb{R} \to \mathbb{R} : u \text{ is non-decreasing}\}$.

$$u \mapsto \int_0^{\cdot} |u'(t)| \, \mathrm{d}t$$

3. $K = \{u : \forall g \in G, \forall x \in \mathbb{R}^N, \; u(gx) = u(x)\}$ where G is a group acting on $\mathbb{R}^N$.

If $P : X \to K$ is a projector on K, $u \in K = \operatorname{Im} P$ iff

$$P(u) = u$$

Invariant solutions

K a closed convex cone (not necessarily salient).

1. $K = \{u \in H_0^1(\Omega) : u \geqslant 0\}$.

   $$u \mapsto u^+$$

2. $K = \{u : \mathbb{R} \to \mathbb{R} : u \text{ is non-decreasing}\}$.

   $$u \mapsto \int_0^{\cdot} |u'(t)| \, \mathrm{d}t$$

3. $K = \bigl\{u : \forall g \in G,\ \forall x \in \mathbb{R}^N,\ u(gx) = u(x)\bigr\}$ where G is a group acting on $\mathbb{R}^N$.

If $P : X \to K$ is a projector on K, $u \in K = \operatorname{Im} P$ iff

$$P(u) = u$$

Invariant solutions

K a closed convex cone (not necessarily salient).

1. $K = \{u \in H_0^1(\Omega) : u \geqslant 0\}$.
$$u \mapsto u^+$$
2. $K = \{u : \mathbb{R} \to \mathbb{R} : u \text{ is non-decreasing}\}$.
$$u \mapsto \int_0^{\cdot} |u'(t)| \, \mathrm{d}t$$
3. $K = \{u : \forall g \in G, \ \forall x \in \mathbb{R}^N, \ u(gx) = u(x)\}$ where G is a group acting on $\mathbb{R}^N$.

If $P : X \to K$ is a projector on K, $u \in K = \operatorname{Im} P$ iff
$$P(u) = u$$

Existence result

Theorem (Brezis & Nirenberg, '95)

Let X is a Banach space, $\mathscr{E} \in \mathscr{C}^1(X;\mathbb{R})$, $e \in X$ and $r > 0$ be s.t. $\|e\| > r$ and

$$b := \inf_{\|u\|=r} \mathscr{E}(u) > \mathscr{E}(0) \geqslant \mathscr{E}(e)$$

Let $P : X \to X$ be a continuous mapping s.t.

$$\forall u \in X,\ \mathscr{E}(Pu) \leqslant \mathscr{E}(u), \quad P(0) = 0 \text{ and } P(e) = e$$

Then there exists a sequence $(u_n) \subset X$ s.t.

$$\mathscr{E}(u_n) \to d, \quad \nabla\mathscr{E}(u_n) \to 0, \quad \operatorname{dist}(u_n, P(X)) \to 0$$

where

$$d := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathscr{E}(\gamma(t))$$

$$\Gamma := \{\gamma \in \mathscr{C}([0,1];X) : \gamma(0) = 0,\ \gamma(1) = e\}$$

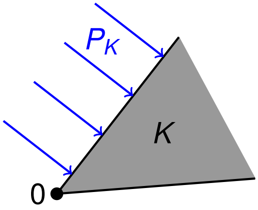
Projector

Definition

The *metric projector* on K, $P_K : X \to K$, is defined by: for all $u \in X$, $P_K(u)$ denotes the unique element of K satisfying

$$\|P_K(u) - u\| = \min_{v \in K} \|v - u\|$$

P_K is positively homegeneous and continuous.

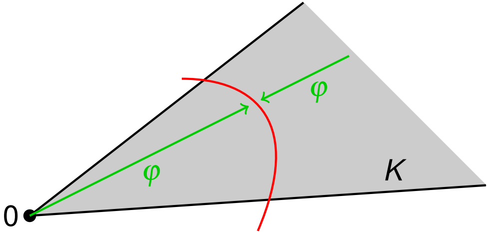


K-peak selection

Definition

A function $\varphi : K \setminus \{0\} \to K \setminus \{0\}$ is said to be a *K-peak selection* for $\mathscr{E}$ iff, for every $u \in K \setminus \{0\}$,

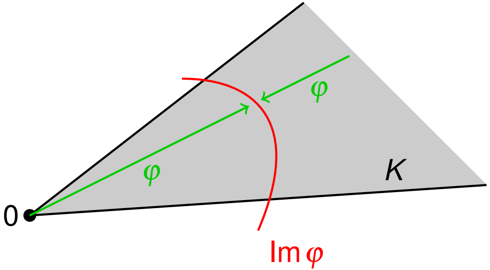
- $\varphi(u)$ is a local maximum point of $\mathscr{E}$ restricted to the half-line $\{tu : t \in]0, +\infty[\}$;
- $\forall \lambda > 0,\ \varphi(\lambda u) = \varphi(u)$.



Aim

Find $u \in \operatorname{Im}\varphi \subset K$ s.t.

$$\mathscr{E}(u) = \min_{\operatorname{Im}\varphi} \mathscr{E}$$



Algorithm (1/2)

MPAP algorithm

$$\begin{cases} \text{Choose } u_0 \in \operatorname{Im}\varphi, \\ \text{If } \nabla\mathscr{E}(u_n) = 0, \text{ then} \\ \quad \text{Stop: } u_n \text{ is a critical point} \\ \text{else} \\ \quad u_{n+1} := \varphi \circ P_K\Big(u_n - s_n \dfrac{\nabla\mathscr{E}(u_n)}{\|\nabla\mathscr{E}(u_n)\|}\Big), \quad \text{with } s_n \in S(u_n) \end{cases}$$

where $S(u_n)$ is the set of acceptable stepsizes at u_n.

Algorithm (2/2)

Definition (Stepsize)

Let $u_0 \in \operatorname{Im}\varphi$ and

$$S_{\downarrow}(u_0) := \Big\{ s > 0 : P_K(u_s) \neq 0 \text{ and } \mathscr{E}\big(\varphi \circ P_K(u_s)\big) - \mathscr{E}(u_0) < -\frac{s}{2}\|\nabla\mathscr{E}(u_0)\| \Big\}$$

where u_s is a shorthand for

$$u_s := u_0 - s\frac{\nabla\mathscr{E}(u_0)}{\|\nabla\mathscr{E}(u_0)\|}.$$

The *stepsize set* $S(u_0)$ at u_0 is $S_{\downarrow}(u_0) \cap \,]\frac{1}{2}\sup S_{\downarrow}(u_0), +\infty[$.

$\mathscr{E}$ bounded from below on $\operatorname{Im}\varphi \Rightarrow \sup S_{\downarrow}(u_0) < +\infty$

Geometry of $\mathscr{E}$

$\mathscr{E} : X \to \mathbb{R}$ has the appropriate "geometry" if

(E$_1$) $\forall u \in X,\ \mathscr{E}(P_K(u)) \leqslant \mathscr{E}(u)$;

(E$_2$) there exists a *continuous* K-peak selection $\varphi : K \setminus \{0\} \to K \setminus \{0\}$ for $\mathscr{E}$;

(E$_3$) $0 \notin \overline{\operatorname{Im} \varphi}$;

(E$_4$) $\inf\{\mathscr{E}(u) : u \in \operatorname{Im} \varphi\} > -\infty$;

(E$_5$) $\mathscr{E}$ satisfies the Palais-Smale condition i.e., any sequence $(u_n) \subset X$ such that $(\mathscr{E}(u_n))$ converges and $\nabla\mathscr{E}(u_n) \to 0$ possesses a convergent subsequence.

Geometry of $\mathscr{E}$

$\mathscr{E} : X \to \mathbb{R}$ has the appropriate "geometry" if

(E$_1$) $\forall u \in X,\ \mathscr{E}(P_K(u)) \leqslant \mathscr{E}(u)$;

(E$_2$) there exists a *continuous* K-peak selection $\varphi : K \setminus \{0\} \to K \setminus \{0\}$ for $\mathscr{E}$;

(E$_3$) $0 \notin \overline{\operatorname{Im} \varphi}$;

(E$_4$) $\inf\{\mathscr{E}(u) : u \in \operatorname{Im} \varphi\} > -\infty$;

(E$_5$) $\mathscr{E}$ satisfies the Palais-Smale condition i.e., any sequence $(u_n) \subset X$ such that $(\mathscr{E}(u_n))$ converges and $\nabla \mathscr{E}(u_n) \to 0$ possesses a convergent subsequence.

Geometry of $\mathscr{E}$

$\mathscr{E} : X \to \mathbb{R}$ has the appropriate "geometry" if

(E$_1$) $\forall u \in X,\ \mathscr{E}(P_K(u)) \leqslant \mathscr{E}(u)$;

(E$_2$) there exists a *continuous* K-peak selection $\varphi : K \setminus \{0\} \to K \setminus \{0\}$ for $\mathscr{E}$;

(E$_3$) $0 \notin \overline{\operatorname{Im} \varphi}$;

(E$_4$) $\inf\{\mathscr{E}(u) : u \in \operatorname{Im} \varphi\} > -\infty$;

(E$_5$) $\mathscr{E}$ satisfies the Palais-Smale condition i.e., any sequence $(u_n) \subset X$ such that $(\mathscr{E}(u_n))$ converges and $\nabla\mathscr{E}(u_n) \to 0$ possesses a convergent subsequence.

Geometry of $\mathscr{E}$

$\mathscr{E} : X \to \mathbb{R}$ has the appropriate "geometry" if

(E$_1$) $\forall u \in X,\ \mathscr{E}(P_K(u)) \leqslant \mathscr{E}(u)$;

(E$_2$) there exists a *continuous* K-peak selection $\varphi : K \setminus \{0\} \to K \setminus \{0\}$ for $\mathscr{E}$;

(E$_3$) $0 \notin \overline{\operatorname{Im} \varphi}$;

(E$_4$) $\inf\{\mathscr{E}(u) : u \in \operatorname{Im} \varphi\} > -\infty$;

(E$_5$) $\mathscr{E}$ satisfies the Palais-Smale condition i.e., any sequence $(u_n) \subset X$ such that $(\mathscr{E}(u_n))$ converges and $\nabla\mathscr{E}(u_n) \to 0$ possesses a convergent subsequence.

Geometry of $\mathscr{E}$

$\mathscr{E} : X \to \mathbb{R}$ has the appropriate "geometry" if

(E_1) $\forall u \in X,\ \mathscr{E}(P_K(u)) \leqslant \mathscr{E}(u)$;

(E_2) there exists a *continuous* K-peak selection $\varphi : K \setminus \{0\} \to K \setminus \{0\}$ for $\mathscr{E}$;

(E_3) $0 \notin \overline{\operatorname{Im} \varphi}$;

(E_4) $\inf\{\mathscr{E}(u) : u \in \operatorname{Im} \varphi\} > -\infty$;

(E_5) $\mathscr{E}$ satisfies the Palais-Smale condition i.e., any sequence $(u_n) \subset X$ such that $(\mathscr{E}(u_n))$ converges and $\nabla \mathscr{E}(u_n) \to 0$ possesses a convergent subsequence.

Does it work?

Theorem (Convergence of the MPAP)

Assume (E_1)–(E_5) hold. For any $u_0 \in \operatorname{Im} \varphi$, the sequence $(u_n)_{n \in \mathbb{N}}$ generated by the MPAP possesses a subsequence converging to a critical point of $\mathscr{E}$ in K. Moreover, the limit of any convergent subsequence of $(u_n)_{n \in \mathbb{N}}$ is a critical point of $\mathscr{E}$ in K.

If the critical point is a strict min on $\operatorname{Im} \varphi$, (u_n) converges.

Computational deformation lemma

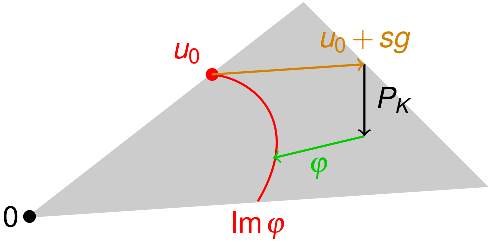
Lemma (Computational deformation lemma)

Assume (E_1) and that there exists a K-peak selection φ which is continuous at some $u_0 \in \operatorname{Im} \varphi$. If $\nabla \mathscr{E}(u_0) \neq 0$ then there exists some $s_0 > 0$ such that, for any $s \in]0, s_0[$,

$$\mathscr{E}(\varphi \circ P_K(u_s)) - \mathscr{E}(u_0) < -\tfrac{1}{2} s \|\nabla \mathscr{E}(u_0)\|$$

where $u_s = u_0 - s \frac{\nabla \mathscr{E}(u_0)}{\|\nabla \mathscr{E}(u_0)\|}$.

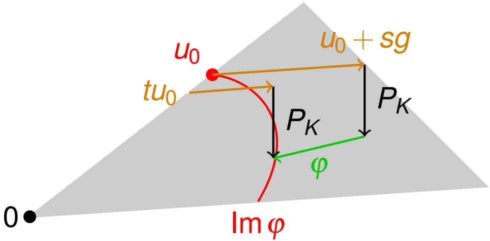
Proof of the computational deformation lemma



$$g = -\frac{\nabla\mathscr{E}(u_0)}{\|\nabla\mathscr{E}(u_0)\|}$$

$$\begin{aligned}
\mathscr{E}\big(\varphi P_K(u_0+sg)\big) &= \mathscr{E}\big(tP_K(u_0+sg)\big) \qquad \text{with } t \approx 1 \text{ for } s \approx 0\\
&= \mathscr{E}\big(P_K(tu_0+tsg)\big)\\
&\leqslant \mathscr{E}(tu_0+tsg)\\
&< \mathscr{E}(u_0) - \tfrac{1}{2}s\|\nabla\mathscr{E}(u_0)\|
\end{aligned}$$

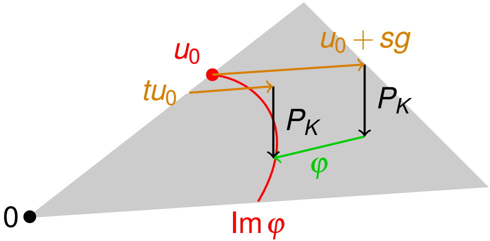
Proof of the computational deformation lemma



$$g = -\frac{\nabla \mathscr{E}(u_0)}{\|\nabla \mathscr{E}(u_0)\|}$$

$$\begin{aligned}
\mathscr{E}(\varphi P_K(u_0 + sg)) &= \mathscr{E}(tP_K(u_0 + sg)) \qquad \text{with } t \approx 1 \text{ for } s \approx 0\\
&= \mathscr{E}(P_K(tu_0 + tsg))\\
&\leqslant \mathscr{E}(tu_0 + tsg)\\
&< \mathscr{E}(u_0) - \tfrac{1}{2} s \|\nabla \mathscr{E}(u_0)\|
\end{aligned}$$

Proof of the computational deformation lemma



$$g = -\frac{\nabla\mathscr{E}(u_0)}{\|\nabla\mathscr{E}(u_0)\|}$$

$$\begin{aligned}
\mathscr{E}\big(\varphi P_K(u_0+sg)\big) &= \mathscr{E}\big(tP_K(u_0+sg)\big) \qquad \text{with } t \approx 1 \text{ for } s \approx 0 \\
&= \mathscr{E}\big(P_K(tu_0+tsg)\big) \\
&\leqslant \mathscr{E}(tu_0+tsg) \\
&< \mathscr{E}(u_0) - \tfrac{1}{2}s\|\nabla\mathscr{E}(u_0)\|
\end{aligned}$$

Local uniformity

The important consequence of the choice of the stepsize is the following.

Lemma

Let φ be a continuous K-peak selection such that P_K decreases $\mathscr{E}$. If $u_0 \in \operatorname{Im} \varphi$ is such that $\nabla \mathscr{E}(u_0) \neq 0$, then there exists an open neighborhood V of u_0 and a positive s_0 such that

$$S(u) \subset [s_0, +\infty[\quad \text{for all } u \in V \cap \operatorname{Im} \varphi.$$

Proof of convergence of the MPAP (1/8)

$$\begin{cases} \text{Choose } u_0 \in \operatorname{Im} \varphi, \\ \text{If } \nabla \mathscr{E}(u_n) = 0, \text{ then} \\ \quad \text{Stop: } u_n \text{ is a critical point} \\ \text{else} \\ \quad u_{n+1} := \varphi \circ P_K\Big(u_n - s_n \dfrac{\nabla \mathscr{E}(u_n)}{\|\nabla \mathscr{E}(u_n)\|}\Big), \quad \text{with } s_n \in S(u_n) \end{cases}$$

We want to show that $(u_n) \subset \operatorname{Im} \varphi$ converges up to a subsequence.

- If there exists a subsequence (u_{n_k}) s.t. $\nabla \mathscr{E}(u_{n_k}) \to 0$, we conclude by (PS).
- Otherwise, there exists $\delta > 0$, $\forall n$, $\|\nabla \mathscr{E}(u_n)\| \geqslant \delta$.

Proof of convergence of the MPAP (2/8)

$$\begin{cases} \text{Choose } u_0 \in \operatorname{Im}\varphi, \\ \text{If } \nabla\mathscr{E}(u_n) = 0, \text{ then} \\ \quad \text{Stop: } u_n \text{ is a critical point} \\ \text{else} \\ \quad u_{n+1} := \varphi \circ P_K\Big(u_n - s_n \dfrac{\nabla\mathscr{E}(u_n)}{\|\nabla\mathscr{E}(u_n)\|}\Big), \quad \text{with } s_n \in S(u_n) \end{cases}$$

We want to show that $(u_n) \subset \operatorname{Im}\varphi$ converges up to a subsequence.

- If there exists a subsequence (u_{n_k}) s.t. $\nabla\mathscr{E}(u_{n_k}) \to 0$, we conclude by (PS).
- Otherwise, there exists $\delta > 0$, $\forall n$, $\|\nabla\mathscr{E}(u_n)\| \geqslant \delta$.

Proof of convergence of the MPAP (3/8)

The computational deformation lemma implies

$$\mathscr{E}(u_{n+1}) - \mathscr{E}(u_n) \leqslant -\tfrac{1}{2} s_n \|\nabla \mathscr{E}(u_n)\| \leqslant -\tfrac{1}{2} s_n \delta$$

Adding up,

$$-\infty < \lim_{n\to\infty} \mathscr{E}(u_n) - \mathscr{E}(u_0) = \sum_{n=0}^{\infty} \big(\mathscr{E}(u_{n+1}) - \mathscr{E}(u_n)\big) \leqslant -\frac{\delta}{2} \sum_{n=0}^{\infty} s_n$$

Thus

$$\sum_{n=0}^{\infty} s_n < +\infty$$

which implies that (u_n) converges to a $u^* \in \operatorname{Im} \varphi$ s.t. $\|\nabla \mathscr{E}(u^*)\| \geqslant \delta$. By the local uniformity of the stepsize around u^*, $s_n \geqslant s^* > 0$ for n large contradicting $\sum_{n=0}^{\infty} s_n < +\infty$.

Proof of convergence of the MPAP (4/8)

The computational deformation lemma implies

$$\mathscr{E}(u_{n+1}) - \mathscr{E}(u_n) \leqslant -\tfrac{1}{2} s_n \|\nabla \mathscr{E}(u_n)\| \leqslant -\tfrac{1}{2} s_n \delta$$

Adding up,

$$-\infty < \lim_{n\to\infty} \mathscr{E}(u_n) - \mathscr{E}(u_0) = \sum_{n=0}^{\infty} \big(\mathscr{E}(u_{n+1}) - \mathscr{E}(u_n)\big) \leqslant -\frac{\delta}{2} \sum_{n=0}^{\infty} s_n$$

Thus

$$\sum_{n=0}^{\infty} s_n < +\infty$$

which implies that (u_n) converges to a $u^* \in \operatorname{Im}\varphi$ s.t. $\|\nabla \mathscr{E}(u^*)\| \geqslant \delta$. By the local uniformity of the stepsize around u^*, $s_n \geqslant s^* > 0$ for n large contradicting $\sum_{n=0}^{\infty} s_n < +\infty$.

Proof of convergence of the MPAP (5/8)

The computational deformation lemma implies

$$\mathscr{E}(u_{n+1}) - \mathscr{E}(u_n) \leqslant -\tfrac{1}{2} s_n \|\nabla \mathscr{E}(u_n)\| \leqslant -\tfrac{1}{2} s_n \delta$$

Adding up,

$$-\infty < \lim_{n\to\infty} \mathscr{E}(u_n) - \mathscr{E}(u_0) = \sum_{n=0}^{\infty} \big(\mathscr{E}(u_{n+1}) - \mathscr{E}(u_n)\big) \leqslant -\frac{\delta}{2} \sum_{n=0}^{\infty} s_n$$

Thus

$$\sum_{n=0}^{\infty} s_n < +\infty$$

which implies that (u_n) converges to a $u^* \in \operatorname{Im} \varphi$ s.t. $\|\nabla \mathscr{E}(u^*)\| \geqslant \delta$. By the local uniformity of the stepsize around u^*, $s_n \geqslant s^* > 0$ for n large contradicting $\sum_{n=0}^{\infty} s_n < +\infty$.

UMH

Proof of convergence of the MPAP (6/8)

$$\sum_{n=0}^{\infty} s_n < +\infty \;\Rightarrow\; (u_n) \text{ converges}$$

Let

$$v_n := \frac{u_n}{\|u_n\|} \quad (\text{thus } u_n = \varphi(v_n)), \qquad g_n := -\frac{\nabla\mathscr{E}(u_n)}{\|\nabla\mathscr{E}(u_n)\|}$$

It suffices to show that (v_n) converges.

$$\|v_{n+1} - v_n\| \leqslant \beta^{-1} \underbrace{\|P_K(u_n + s_n g_n) - u_n\|}_{P_K \text{ is the metric projector}} \leqslant 2 s_n \beta^{-1}$$

$$\left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \leqslant \beta^{-1}\|u - v\| \quad \text{if } \|u\|, \|v\| \geqslant \beta$$

So (v_n) is a Cauchy sequence.

Proof of convergence of the MPAP (7/8)

$$\sum_{n=0}^{\infty} s_n < +\infty \;\Rightarrow\; (u_n) \text{ converges}$$

Let

$$v_n := \frac{u_n}{\|u_n\|} \quad (\text{thus } u_n = \varphi(v_n)), \qquad g_n := -\frac{\nabla\mathscr{E}(u_n)}{\|\nabla\mathscr{E}(u_n)\|}$$

It suffices to show that (v_n) converges.

$$\|v_{n+1} - v_n\| \leqslant \beta^{-1} \underbrace{\|P_K(u_n + s_n g_n) - u_n\|}_{P_K \text{ is the metric projector}} \leqslant 2 s_n \beta^{-1}$$

$$\uparrow$$

$$\left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \leqslant \beta^{-1}\|u - v\| \quad \text{if } \|u\|, \|v\| \geqslant \beta$$

So (v_n) is a Cauchy sequence.

UMH

Proof of convergence of the MPAP (8/8)

$$\sum_{n=0}^{\infty} s_n < +\infty \;\Rightarrow\; (u_n) \text{ converges}$$

Let

$$v_n := \frac{u_n}{\|u_n\|} \quad (\text{thus } u_n = \varphi(v_n)), \qquad g_n := -\frac{\nabla\mathscr{E}(u_n)}{\|\nabla\mathscr{E}(u_n)\|}$$

It suffices to show that (v_n) converges.

$$\|v_{n+1} - v_n\| \leqslant \beta^{-1} \underbrace{\|P_K(u_n + s_n g_n) - u_n\|}_{P_K \text{ is the metric projector}} \leqslant 2 s_n \beta^{-1}$$

$$\uparrow$$

$$\left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \leqslant \beta^{-1} \|u - v\| \quad \text{if } \|u\|, \|v\| \geqslant \beta$$

So (v_n) is a Cauchy sequence.

1. Algorithm for invariant solutions
2. Examples
3. Open questions

Non-decreasing solutions: setting

Equation coming from solitary waves on lattices:

$$\begin{cases} u''(t) = V'\big(u(t+1)-u(t)\big) - V'\big(u(t)-u(t-1)\big), & t \in \mathbb{R} \\ u(0) = 0 \\ u \text{ non-decreasing} \end{cases}$$

This is equivalent to $\nabla\mathscr{E}(u) = 0$ with

$$\mathscr{E} : X \to \mathbb{R} : u \mapsto \frac{1}{2}\int_{\mathbb{R}} |u'(t)|^2 \,\mathrm{d}t - \int_{\mathbb{R}} V\big(u(t+1)-u(t)\big)\,\mathrm{d}t$$

where

$$X := \{u \in H^1_{\text{loc}}(\mathbb{R}) : u' \in L^2(\mathbb{R}) \text{ and } u(0) = 0\}$$

and

$$u \in K := \{u \in X : u \text{ is non-decreasing}\}$$

Non-decreasing solutions: assumptions

(V_1) $V \in \mathscr{C}^1(\mathbb{R};\mathbb{R})$, $V(0) = 0$,

$$V'(u) = o(|u|) \quad \text{as } u \to 0.$$

(V_2) There exists $\alpha > 2$ such that

$$\forall u \geqslant 0, \quad 0 \leqslant \alpha V(u) \leqslant V'(u)u$$

and there exists $u > 0$ such that $V(u) > 0$.

(V_3) $V'(u)/u$ is increasing w.r.t. $u \in]0, +\infty[$.

Non-decreasing solutions: projector on K

The metric projector $P_K : X \to X : u \mapsto P_K(u)$ on $K = \{u \in X : \ u \text{ is non-decreasing}\}$ can be written

$$P_K(u)(t) = \int_0^t (u')^+ \qquad \text{where } v^+ := \max\{v, 0\}.$$

It can be shown that $\mathscr{E}$ has the appropriate geometry and therefore the algorithm converges up to a subsequence and up to translations (where $\tau_a u(t) = u(t-a) - u(-a)$).

Non-decreasing solutions: projector on K

The metric projector $P_K : X \to X : u \mapsto P_K(u)$ on $K = \{u \in X : \ u \text{ is non-decreasing}\}$ can be written

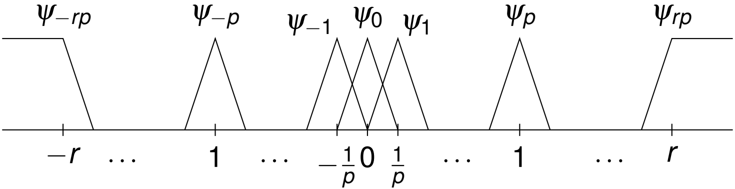
$$P_K(u)(t) = \int_0^t (u')^+ \qquad \text{where } v^+ := \max\{v, 0\}.$$

It can be shown that $\mathscr{E}$ has the appropriate geometry and therefore the algorithm converges up to a subsequence and up to translations (where $\tau_a u(t) = u(t-a) - u(-a)$).

Finite elements

$$X_{r,p} := \left\{ \sum_{i=-rp}^{rp} u_i \psi_i : u_0 = 0 \right\} \subset X$$

where the basis (ψ_i) is as follows:



Apply the algorithm to

$$\mathscr{E}\restriction_{X_{r,p}} : X_{r;p} \to \mathbb{R}$$

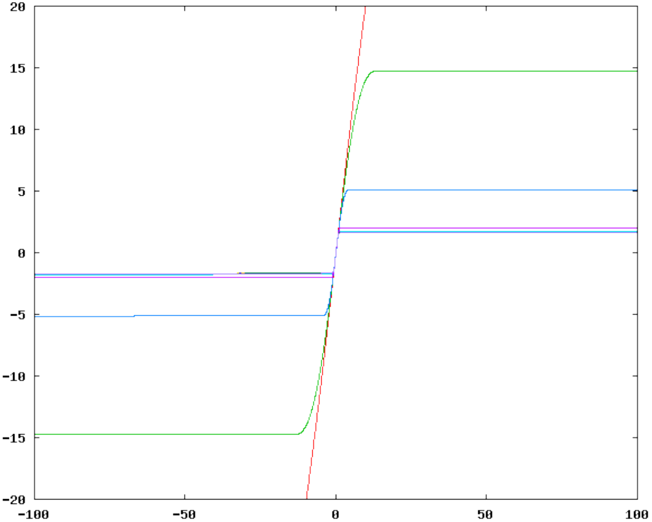
Computing the projector

Given $\mathbf{u} = \sum_{i=-rp}^{rp} u_i \psi_i$, its projection on the cone

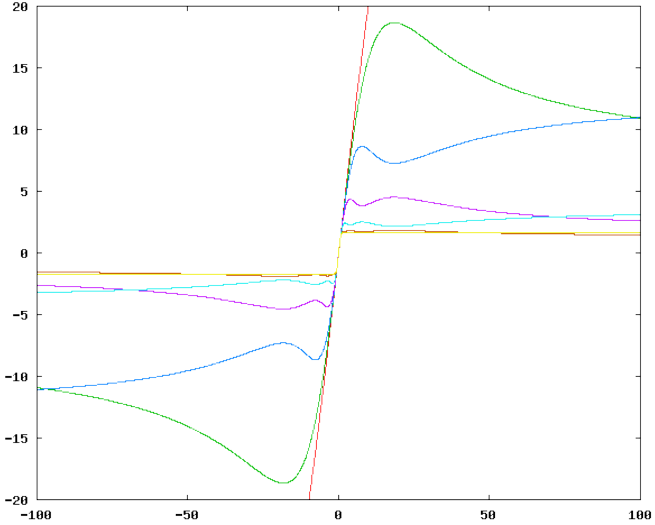
$P_K(\mathbf{u}) = \sum_{i=-rp}^{rp} v_i \psi_i$ is computed (exactly) by

$$
\begin{array}{|l}
v_0 = 0 \\
\textbf{for } i = 1, \ldots, rp \\
\quad \textbf{let } d = u_i - u_{i-1} \textbf{ in} \\
\quad v_i = (\textbf{if } d > 0 \textbf{ then } v_{i-1} + d \textbf{ else } v_{i-1}) \\
\textbf{for } i = -1, \ldots, -rp \\
\quad \textbf{let } d = u_{i+1} - u_i \textbf{ in} \\
\quad v_i = (\textbf{if } d > 0 \textbf{ then } v_{i+1} + d \textbf{ else } v_{i+1})
\end{array}
$$

Non-decreasing solutions: numerical results

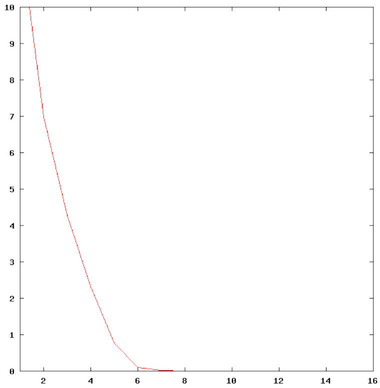


Non-decreasing solutions: numerical results

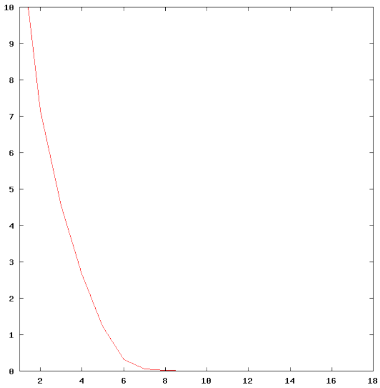


Non-decreasing solutions: numerical results

With P_K



Without P_K



Non-negative solutions: setting

Solutions of

$$\begin{cases} -\Delta u(x) = f(x, u(x)), & \text{for } x \in \Omega \subset \mathbb{R}^N \\ u = 0 & \text{on } \partial\Omega \\ u \geqslant 0 & \text{on } \Omega \end{cases}$$

are critical points of the functional

$$\mathscr{E} : H_0^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x - \int_\Omega F(x, u(x)) \,\mathrm{d}x$$

where $F(x,u) := \int_0^u f(x,v)\,\mathrm{d}v$, that belong to the cone

$$K := \{u \in H_0^1(\Omega) : u \geqslant 0 \text{ on } \Omega\}$$

Non-negative solutions: assumptions

(P1) For almost every $x \in \Omega$, $f(x,\xi)$ is continuous in ξ;

(P2) there exists two positives constants a_1, a_2 such that

$$|f(x,\xi)| \leqslant a_1 + a_2|\xi|^{s-1}$$

with $s \in [1, \frac{2N}{N-2}[$ if $N > 2$ and $s \in [1, +\infty[$ otherwise;

(P3) $f(x,\xi) = o(|\xi|)$ uniformly in x for $\xi \to 0$;

(P4) there exists two constants $\mu > 2$ and $r \geqslant 0$ such that

$$\forall |\xi| \geqslant r, \quad 0 < \mu F(x,\xi) \leqslant f(x,\xi)\xi$$

with $F(x,\xi) = \int_0^\xi f(x,t)\,\mathrm{d}t$;

(P5) finally, we will suppose that $\forall x \in]a,b[$, $f(x,\xi)/\xi$ is increasing and

$$\lim_{\xi\to\infty} \frac{f(x,\xi)}{\xi} = +\infty$$

Non-negative solutions: metric projector on K

$$\left.\begin{array}{l} \|P_K u\| \leqslant \|u\| \\ P_K u \geqslant \max\{u, 0\} \end{array}\right\} \Rightarrow \mathscr{E}_{\text{modif}}(P_K u) \leqslant \mathscr{E}_{\text{modif}}(u)$$

Characterisation of $P_K(u)$:

$$\forall v \geqslant 0, \quad (u - P_K u | v - P_K u) \leqslant 0$$

$$v = 0 \Rightarrow \quad (u - P_K u | -P_K u) \leqslant 0$$

$$\|P_K u\|^2 \leqslant (u | P_K u) \leqslant \|u\| \, \|P_K u\|$$

$$\forall v \geqslant 0, \quad (u - P_K u | v) \leqslant 0$$

$$\forall v \geqslant 0, \quad \int_\Omega -\Delta(u - P_K u) v \leqslant 0$$

$$-\Delta(u - P_K u) \leqslant 0$$

$$u - P_K u \leqslant 0$$

Non-negative solutions: metric projector on K

$$\left.\begin{aligned} \|P_K u\| &\leqslant \|u\| \\ P_K u &\geqslant \max\{u, 0\} \end{aligned}\right\} \Rightarrow \mathscr{E}_{\text{modif}}(P_K u) \leqslant \mathscr{E}_{\text{modif}}(u)$$

Characterisation of $P_K(u)$:

$$\forall v \geqslant 0, \quad (u - P_K u | v - P_K u) \leqslant 0$$

$$v = 0 \Rightarrow \quad (u - P_K u | -P_K u) \leqslant 0$$

$$\|P_K u\|^2 \leqslant (u | P_K u) \leqslant \|u\| \, \|P_K u\|$$

$$\forall v \geqslant 0, \quad (u - P_K u | v) \leqslant 0$$

$$\forall v \geqslant 0, \quad \int_\Omega -\Delta(u - P_K u) v \leqslant 0$$

$$-\Delta(u - P_K u) \leqslant 0$$

$$u - P_K u \leqslant 0$$

Non-negative solutions: metric projector on K

$$\left.\begin{array}{l} \|P_K u\| \leqslant \|u\| \\ P_K u \geqslant \max\{u, 0\} \end{array}\right\} \Rightarrow \mathscr{E}_{\text{modif}}(P_K u) \leqslant \mathscr{E}_{\text{modif}}(u)$$

Characterisation of $P_K(u)$:

$$\forall v \geqslant 0, \quad (u - P_K u | v - P_K u) \leqslant 0$$

$$v = 0 \Rightarrow \quad (u - P_K u | - P_K u) \leqslant 0$$

$$\|P_K u\|^2 \leqslant (u | P_K u) \leqslant \|u\| \, \|P_K u\|$$

$$\forall v \geqslant 0, \quad (u - P_K u | v) \leqslant 0$$

$$\forall v \geqslant 0, \quad \int_\Omega -\Delta(u - P_K u) v \leqslant 0$$

$$-\Delta(u - P_K u) \leqslant 0$$

$$u - P_K u \leqslant 0$$

Non-negative solutions: metric projector on K

$$\left.\begin{array}{l} \|P_K u\| \leqslant \|u\| \\ P_K u \geqslant \max\{u, 0\} \end{array}\right\} \Rightarrow \mathscr{E}_{\text{modif}}(P_K u) \leqslant \mathscr{E}_{\text{modif}}(u)$$

Characterisation of $P_K(u)$:

$$\forall v \geqslant 0, \quad (u - P_K u | v - P_K u) \leqslant 0$$

$$v = 0 \Rightarrow \quad (u - P_K u | - P_K u) \leqslant 0$$

$$\|P_K u\|^2 \leqslant (u | P_K u) \leqslant \|u\| \, \|P_K u\|$$

$$\forall v \geqslant 0, \quad (u - P_K u | v) \leqslant 0$$

$$\forall v \geqslant 0, \quad \int_\Omega -\Delta(u - P_K u) v \leqslant 0$$

$$-\Delta(u - P_K u) \leqslant 0$$

$$u - P_K u \leqslant 0$$

Non-negative solutions: projector in 1D

Theorem

The metric projector on K for the norm $\|u\| := \left(\int_{]a,b[} |u'|^2\right)^{1/2}$ is given by:

$$P_K(u) = u - \operatorname{conv} u$$

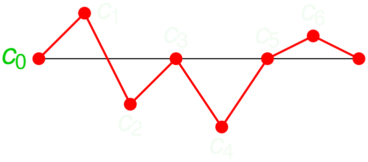
$\operatorname{conv} u$ is the convex hull hull of $u \in H^1_0(]a,b[)$ defined by

$$\operatorname{conv} u(x) := \sup\{\ell(x) : \ell \text{ is affine and } \forall y \in]a,b[,\ \ell(y) \leqslant u(y)\}$$

Non-negative solutions: algorithm for P_K

Let $\mathbf{u} := (u_i)_{i=0}^N$ be the discretization of u given by finite elements (with $u_0 = 0 = u_N$). One can compute $P_K\mathbf{u}$ with the following algorithm:

> Let $(c_i)_{i=0}^N$ be the *list* $(u_i)_{i=0}^N$
> **for** $i = 1, \dots, N$
>   **if** $\mathrm{slope}(c_{i-1}, c_i) \leqslant \mathrm{slope}(c_i, c_{i+1})$ **then**
>   Keep the node c_i
>   **else**
>   Remove c_i from the list
> The nodes c_i kept give the shape of $\operatorname{conv}\mathbf{u}$

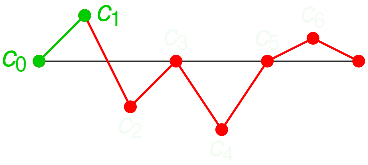


The cost of computing $\operatorname{conv}\mathbf{u}$ (hence $P_K\mathbf{u}$) is $O(N)$, thus comparable to the one for $\mathbf{u}^+$.

Non-negative solutions: algorithm for P_K

Let $\mathbf{u} := (u_i)_{i=0}^N$ be the discretization of u given by finite elements (with $u_0 = 0 = u_N$). One can compute $P_K\mathbf{u}$ with the following algorithm:

> Let $(c_i)_{i=0}^N$ be the *list* $(u_i)_{i=0}^N$
> **for** $i = 1, \dots, N$
>   **if** $\text{slope}(c_{i-1}, c_i) \leqslant \text{slope}(c_i, c_{i+1})$ **then**
>   Keep the node c_i
>   **else**
>   Remove c_i from the list
> The nodes c_i kept give the shape of $\operatorname{conv}\mathbf{u}$

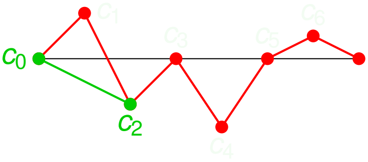


The cost of computing $\operatorname{conv}\mathbf{u}$ (hence $P_K\mathbf{u}$) is $O(N)$, thus comparable to the one for $\mathbf{u}^+$.

Non-negative solutions: algorithm for P_K

Let $\mathbf{u} := (u_i)_{i=0}^N$ be the discretization of u given by finite elements (with $u_0 = 0 = u_N$). One can compute $P_K\mathbf{u}$ with the following algorithm:

Let $(c_i)_{i=0}^N$ be the *list* $(u_i)_{i=0}^N$
for $i = 1, \dots, N$
  if slope$(c_{i-1}, c_i) \leqslant$ slope(c_i, c_{i+1}) **then**
    Keep the node c_i
  else
    Remove c_i from the list
The nodes c_i kept give the shape of conv $\mathbf{u}$

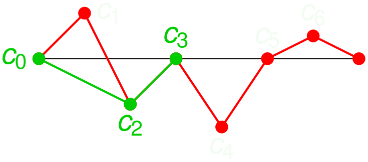


The cost of computing conv $\mathbf{u}$ (hence $P_K\mathbf{u}$) is $O(N)$, thus comparable to the one for $\mathbf{u}^+$.

Non-negative solutions: algorithm for P_K

Let $\mathbf{u} := (u_i)_{i=0}^N$ be the discretization of u given by finite elements (with $u_0 = 0 = u_N$). One can compute $P_K\mathbf{u}$ with the following algorithm:

> Let $(c_i)_{i=0}^N$ be the *list* $(u_i)_{i=0}^N$
> **for** $i = 1, \dots, N$
>   **if** $\text{slope}(c_{i-1}, c_i) \leqslant \text{slope}(c_i, c_{i+1})$ **then**
>   Keep the node c_i
>   **else**
>   Remove c_i from the list
> The nodes c_i kept give the shape of $\operatorname{conv}\mathbf{u}$

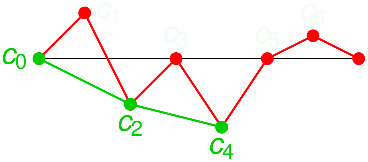


The cost of computing $\operatorname{conv}\mathbf{u}$ (hence $P_K\mathbf{u}$) is $O(N)$, thus comparable to the one for $\mathbf{u}^+$.

Non-negative solutions: algorithm for P_K

Let $\mathbf{u} := (u_i)_{i=0}^N$ be the discretization of u given by finite elements (with $u_0 = 0 = u_N$). One can compute $P_K\mathbf{u}$ with the following algorithm:

> Let $(c_i)_{i=0}^N$ be the *list* $(u_i)_{i=0}^N$
> **for** $i = 1, \dots, N$
>   **if** $\text{slope}(c_{i-1}, c_i) \leqslant \text{slope}(c_i, c_{i+1})$ **then**
>   Keep the node c_i
>   **else**
>   Remove c_i from the list
> The nodes c_i kept give the shape of $\operatorname{conv}\mathbf{u}$

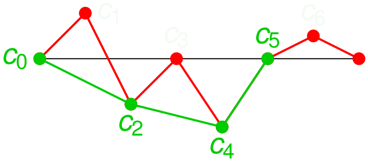


The cost of computing $\operatorname{conv}\mathbf{u}$ (hence $P_K\mathbf{u}$) is $O(N)$, thus comparable to the one for $\mathbf{u}^+$.

Non-negative solutions: algorithm for P_K

Let $\mathbf{u} := (u_i)_{i=0}^N$ be the discretization of u given by finite elements (with $u_0 = 0 = u_N$). One can compute $P_K\mathbf{u}$ with the following algorithm:

> Let $(c_i)_{i=0}^N$ be the *list* $(u_i)_{i=0}^N$
> **for** $i = 1, \dots, N$
>   **if** $\operatorname{slope}(c_{i-1}, c_i) \leqslant \operatorname{slope}(c_i, c_{i+1})$ **then**
>   Keep the node c_i
>   **else**
>   Remove c_i from the list
> The nodes c_i kept give the shape of $\operatorname{conv}\mathbf{u}$

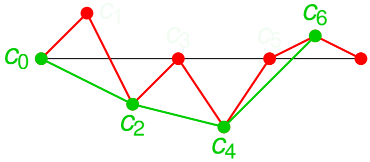


The cost of computing $\operatorname{conv}\mathbf{u}$ (hence $P_K\mathbf{u}$) is $O(N)$, thus comparable to the one for $\mathbf{u}^+$.

Non-negative solutions: algorithm for P_K

Let $\mathbf{u} := (u_i)_{i=0}^N$ be the discretization of u given by finite elements (with $u_0 = 0 = u_N$). One can compute $P_K\mathbf{u}$ with the following algorithm:

> Let $(c_i)_{i=0}^N$ be the *list* $(u_i)_{i=0}^N$
> **for** $i = 1, \dots, N$
>   **if** $\operatorname{slope}(c_{i-1}, c_i) \leqslant \operatorname{slope}(c_i, c_{i+1})$ **then**
>   Keep the node c_i
>   **else**
>   Remove c_i from the list
> The nodes c_i kept give the shape of $\operatorname{conv}\mathbf{u}$

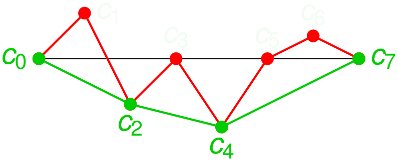


The cost of computing $\operatorname{conv}\mathbf{u}$ (hence $P_K\mathbf{u}$) is $O(N)$, thus comparable to the one for $\mathbf{u}^+$.

Non-negative solutions: algorithm for P_K

Let $\mathbf{u} := (u_i)_{i=0}^N$ be the discretization of u given by finite elements (with $u_0 = 0 = u_N$). One can compute $P_K\mathbf{u}$ with the following algorithm:

> Let $(c_i)_{i=0}^N$ be the *list* $(u_i)_{i=0}^N$
> **for** $i = 1, \dots, N$
>   **if** $\text{slope}(c_{i-1}, c_i) \leqslant \text{slope}(c_i, c_{i+1})$ **then**
>   Keep the node c_i
>   **else**
>   Remove c_i from the list
> The nodes c_i kept give the shape of $\operatorname{conv}\mathbf{u}$

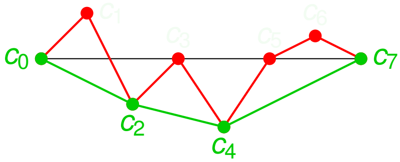


The cost of computing $\operatorname{conv}\mathbf{u}$ (hence $P_K\mathbf{u}$) is $O(N)$, thus comparable to the one for $\mathbf{u}^+$.

Non-negative solutions: algorithm for P_K

Let $\mathbf{u} := (u_i)_{i=0}^N$ be the discretization of u given by finite elements (with $u_0 = 0 = u_N$). One can compute $P_K\mathbf{u}$ with the following algorithm:

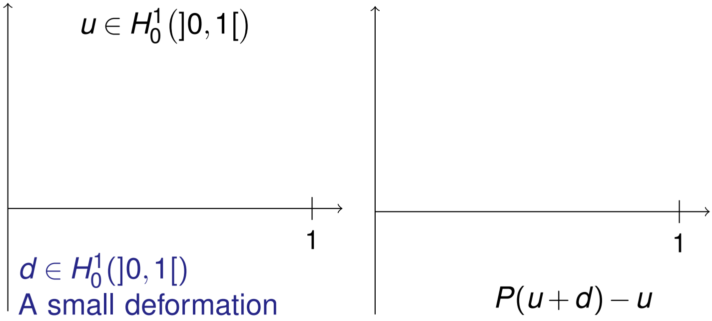
> Let $(c_i)_{i=0}^N$ be the *list* $(u_i)_{i=0}^N$
> **for** $i = 1, \dots, N$
>   **if** $\text{slope}(c_{i-1}, c_i) \leqslant \text{slope}(c_i, c_{i+1})$ **then**
>   Keep the node c_i
>   **else**
>   Remove c_i from the list
> The nodes c_i kept give the shape of $\text{conv}\,\mathbf{u}$



The cost of computing $\text{conv}\,\mathbf{u}$ (hence $P_K\mathbf{u}$) is $O(N)$, thus comparable to the one for $\mathbf{u}^+$.

Open questions & future work (1/2)

- Can we prove the convergence of the MPAP with the projector $u \mapsto u^+ := \max\{u, 0\}$ instead of P_K?
  Problem: $\|(u + sd)^+ - u^+\| \neq O(s)$.



Open questions & future work (2/2)

- Can we prove the convergence of a nodal algorithm?
  Problem: the natural projector is

$$u \mapsto \varphi(u^+) - \varphi(u^-)$$

  where $u^- := (-u)^+$.

- Can we reformulate the problems for invariant & nodal cases in order to use the ideas of Barutello & Terracini?

Open questions & future work (2/2)

- Can we prove the convergence of a nodal algorithm?
  Problem: the natural projector is
  $$u \mapsto \varphi(u^+) - \varphi(u^-)$$
  where $u^- := (-u)^+$.
- Can we reformulate the problems for invariant & nodal cases in order to use the ideas of Barutello & Terracini?

Thank you