Hairy Black Holes & Boson Stars: From shift-symmetry to spontaneous scalarization

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2. Model
3. Equations of motion
   - Equations of motion
   - Ansatz
   - Boundary conditions
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   - Black holes
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     - Spontaneous scalarization
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     - Domain of existence
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Introduction: Why should we modify general relativity?
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Despite consequential success . . .

- Offer a geometrical explanation of gravitational process [elegant]
- Allow to explain many phenomenons:
  1. Mercury perihelion problem
  2. Existence and shape of gravitational waves: GW150914 (2016)

[many experimental checks]
Introduction: *Why should we modify general relativity?*

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  [many experimental checks]

. . . there are some unexplained phenomena within General Relativity (GR):

- Origin and value of the cosmological constant
- Low intensity of gravitational interaction
- Existence of singularities within space-time
- Origin and composition of dark matter and dark energy
- Accelerated expansion of the universe

*Not all of them* are related to quantum correction problems!
Introduction: *How should we modify general relativity?*

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One of them is to consider that the unrated phenomena are due to unknown degrees of freedom (that can be interpreted as new particles or as a new component in the description of gravity).

In GR, all the degrees of freedom are encoded in the metric $g_{\mu\nu}$. But, formally, the equivalence principle does not rule out the possible existence of other kind of fields in the model.
Introduction : *How should we modify general relativity?*

The most simple candidate for these degrees of freedom is a scalar field.
The most simple candidate for these degrees of freedom is a scalar field.

- Simplest covariant object
- Important element of many models:
  - Cosmology
  - Standard model of particle physics
  - Kaluza-Klein reduction
  - Effective theory
  - ...
- Also experimentally motivated since the Brout-Englert-Higgs boson’s discovery (CERN 2012)
Introduction: Why not considering the simplest case?

Why not just using $\mathcal{L}_{EKG} = \kappa (R - 2\Lambda) - \nabla_\mu \phi \nabla^{\mu} \phi - V(\phi)$?
Introduction : Why not considering the simplest case?

No Hair Theorem (*Schematically*)

*Consider an asymptotically flat black hole spacetime*

**Hypothesis 1 :** (Symmetries of spacetime)

**Hypothesis 2 :** (Coupling condition)

**Hypothesis 3 :** (Symmetries of the scalar field)

**Hypothesis 4 :** (“Energetic” condition)

Then, the scalar field must be trivial: \( \phi (x^\mu) = c^te, \forall x^\mu \).
Introduction: Why not considering the simplest case?

No Hair Theorem (Schematically)

Consider an asymptotically flat black hole spacetime

Hypothesis 1: (Symmetries of spacetime)
Hypothesis 2: (Coupling condition)
Hypothesis 3: (Symmetries of the scalar field)
Hypothesis 4: ("Energetic" condition)

Then, the scalar field must be trivial: \( \phi (x^\mu) = c^{te}, \forall x^\mu \).

Note: Generically, the proof makes no use of the Einstein's equations. It just uses the scalar field equation defined thanks to hypothesis 2.
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Model: Curvature induced scalarization

We are interested in scalar tensor theory where a complex scalar field $\phi$ is non-minimally coupled to gravity by means of a curvature invariant $I(g)$:
We are interested in salar tensor theory where a complex scalar field $\phi$ is non-minimally coupled to gravity by means of a curvature invariant $\mathcal{I}(g)$:

$$S = \int \left[ \frac{1}{16\pi G} R - \nabla_\mu \phi^* \nabla^\mu \phi - V(\phi) + f(\phi) \mathcal{I}(g) \right] \sqrt{-g} d^4x.$$
Model : Curvature induced scalarization

We are interested in salar tensor theory where a complex scalar field \( \phi \) is non-minimally coupled to gravity by means of a curvature invariant \( \mathcal{I}(g) \):

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\]

If we assume that both \( V(\phi) \) and \( f(\phi) \) are functions of \( |\phi| = \sqrt{\phi^* \phi} \), the model possess a global \( U(1) \) symmetry : \( \phi \rightarrow e^{i\alpha} \phi \).
Model: Curvature induced scalarization

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We will focus on a coupling to the Gauss-Bonnet invariant:

\[ \mathcal{I}(g) = \mathcal{L}_{GB} \equiv R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}. \]
Model : Curvature induced scalarization

\[ S = \int \left[ \frac{1}{16\pi G} \left( R - \nabla_\mu \phi^* \nabla^\mu \phi - V(\phi) + f(\phi) \mathcal{I}(g) \right) \right] \sqrt{-g} d^4 x. \]

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The functions \( V \) and \( f \) are choosen as:

\[ V(\phi) = m^2 |\phi|^2 + \lambda_4 |\phi|^4 + \lambda_6 |\phi|^6, \]

\[ f(\phi) = \gamma_1 |\phi| + \gamma_2 |\phi|^2. \]
Equations of motion

For the metric function:

\[ G_{\mu\nu} = 8\pi G (T(\phi)_{\mu\nu} + T(I)_{\mu\nu}) , \]

where

\[ T(\phi)_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi^* - (\nabla_\alpha \phi \nabla_\alpha \phi^* + V(\phi)) g_{\mu\nu} , \]

and

\[ T(I)_{\mu\nu} = - (g_{\mu\rho} g_{\nu\sigma} + g_{\nu\rho} g_{\mu\sigma}) \varepsilon^{\rho\alpha\gamma\delta} \varepsilon^{\beta\sigma\lambda\tau} R_{\gamma\delta\lambda\tau} \nabla_\alpha \nabla_\beta f(\phi) , \]

with \( \varepsilon^{\rho\alpha\gamma\delta} \) the Levi-Civita tensor.

For the scalar field:

\[ -\Box \phi = - \frac{\partial V}{\partial \phi} \phi^* + \frac{\partial f}{\partial \phi} \phi^* I(g) , \]

with \( \Box = \nabla_\mu \nabla_\mu \).
Equations of motion

For the metric function :

\[ G_{\mu\nu} = 8\pi G \left( T^{(\phi)}_{\mu\nu} + T^{(I)}_{\mu\nu} \right), \]

where

\[ T^{(\phi)}_{\mu\nu} = \nabla_{(\mu} \phi \nabla_{\nu)} \phi^* - (\nabla_{\alpha} \phi^* \nabla^{\alpha} \phi + V(\phi)) g_{\mu\nu}, \]

and

\[ T^{(I)}_{\mu\nu} = - (g_{\mu\rho} g_{\nu\sigma} + g_{\nu\rho} g_{\mu\sigma}) \varepsilon^{\rho\alpha\gamma\delta} \varepsilon^{\beta\sigma\lambda\tau} R_{\gamma\delta\lambda\tau} \nabla_{\alpha} \nabla_{\beta} f(\phi), \]

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with \( \Box = \nabla^\mu \nabla_\mu \).
Ansatz

For the metric function:

\[ d\sigma^2 = -N(r) \, dt^2 + \frac{1}{N(r)} \, dr^2 + g(r) \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right), \]

where we fix the gauge freedom in the definition of the \( r \) coordinate by setting \( g(r) = r^2 \).

For the scalar field:

\[ \phi(x^\mu) = e^{-i\omega t} \phi(r), \]

where \( \omega \) is a constant real parameter.
Ansatz

For the metric function:
We will focus on a spherically symmetric space-time.
On an appropriate coordinate system \((t, r, \theta, \varphi)\), the metric read

\[
\text{d}s^2 = -N(r)\sigma^2(r)\text{d}t^2 + \frac{1}{N(r)}\text{d}r^2 + g(r)(\text{d}\theta^2 + \sin^2 \theta \text{d}\varphi^2),
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g(r) = r^2.
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where we fix the gauge freedom in the definition of the \(r\) coordinate by setting

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\]

For the scalar field:

In the same coordinate system, we choose a scalar field of the form

\[
\phi(x^\mu) = e^{-i\omega t} \phi(r),
\]

where \(\omega\) is a constant real parameter.
Reduced equations

Within this ansatz, the field equations can be rewritten in the form

\[ N' = F_1(N, \sigma, \phi, \phi'; \omega), \]
\[ \sigma' = F_2(N, \sigma, \phi, \phi'; \omega), \]
\[ \phi'' = F_3(N, \sigma, \phi, \phi'; \omega), \]

where the functions \( F_1, F_2 \) and \( F_3 \) are involved algebraic functions of the fields \( N, \sigma, \phi \) and \( \phi' \).
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where the functions \( F_1, F_2 \) and \( F_3 \) are involved algebraic functions of the fields \( N, \sigma, \phi \) and \( \phi' \).

Note that we can reduce ourself to a real scalar field via a \( \omega \to 0 \) limit.
Boundary conditions for Black Holes

We impose an horizon at radius $r = r_h$:

$$N(r_h) = 0.$$ 

We further demand regularity of the solution at the horizon. This constrains the first derivative of the scalar field $\phi'$ at $r_h$:

$$\phi'(r_h) = -\frac{r_h^2}{\sqrt{\Delta}} \pm \sqrt{\Delta},$$

where

$$\Delta = r_h^4 - 96\gamma_2^2 - 384(\gamma_1^2 + 2\gamma_2\phi(r_h))^2 + \gamma_1\gamma_2\phi(r_h).$$

Finally, we require asymptotic flatness:

$$\sigma(r \to \infty) = 1, \quad \phi(r \to \infty) = 0.$$
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  \[ \phi'(r_h) = \frac{-r_h^2 \pm \sqrt{\Delta}}{8r_h(\gamma_1 + 2\gamma_2\phi(r_h))}, \]
  where
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- Finally, we require asymptotic flatness:

$$\sigma(r \to \infty) = 1, \quad \phi(r \to \infty) = 0.$$
Boundary conditions
for Boson Stars

The regularity of the solutions at the origin impose $N(0) = 1$, $\phi'(0) = 0$.

The asymptotic flatness is ensured by setting $\sigma(r \to \infty) = 1$, $\phi(r \to \infty) = 0$. 
Boundary conditions
for Boson Stars

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**Boundary conditions**

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- The regularity of the solutions at the origin impose

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5 Conclusion
The scalar field is real, i.e. \( \omega = 0 \) in \( \varphi(x) = e^{i\omega t} \varphi(r) \).

The potential contain no self-interaction so \( \lambda_4 = 0 \) in \( V(\varphi) = m_2 |\varphi|^2 + \lambda_4 |\varphi|^4 + \lambda_6 |\varphi|^6 \).

Unless explicitly stated, we will also assume the scalar field to be massless: \( m = 0 \).

The behaviour of the solutions is only due to the coupling function \( f(\varphi) = \gamma_1 \varphi + \gamma_2 \varphi^2 \).\[\text{Ludovic Ducobu}\]
\[\text{RTG Workshop}\]
\[\text{October 2019}\]
The scalar field is real, i.e. $\omega = 0$ in $\phi(x^\mu) = e^{i\omega t}\phi(r)$. 
Hypothesis

- The scalar field is real, i.e. $\omega = 0$ in $\phi(x^\mu) = e^{i\omega t} \phi(r)$.
- The potential contain no self-interaction so $\lambda_4 = 0 = \lambda_6$ in $V(\phi) = m^2 |\phi|^2 + \lambda_4 |\phi|^4 + \lambda_6 |\phi|^6$.
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The behaviour of the solutions is only due to the coupling function

$$f(\phi) = \gamma_1 \phi + \gamma_2 \phi^2.$$
Shift-symmetry \((\gamma_1 \neq 0, \gamma_2 = 0)\)

The equation of motion for \(\phi\) read

\[
\Box \phi = -\gamma_1 \mathcal{I}(g).
\]

The condition of regularity at the horizon reduces to

\[
\phi'(r_h) = \frac{-r_h^2 \pm \sqrt{\Delta}}{8r_h \gamma_1}, \quad \Delta = r_h^4 - 96\gamma_1^2.
\]
Shift-symmetry \( (\gamma_1 \neq 0, \gamma_2 = 0) \)

The equation of motion for \( \phi \) read

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\]

Consequently, the condition of positivity of the discriminant \( \Delta \) constraint the accessible values of \( \gamma_1 \):

\[
\Delta \geq 0 \iff \gamma_1 \leq r_h^2 \sqrt{1/96} \approx r_h^2 \times 0.1021.
\]
Shift-symmetry \((\gamma_1 \neq 0, \gamma_2 = 0)\)

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\phi'(r_h) = \frac{-r_h^2 \pm \sqrt{\Delta}}{8r_h \gamma_1}, \quad \Delta = r_h^4 - 96\gamma_1^2.
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$$\phi'(r_h) = \frac{-r_h^2 \pm \sqrt{\Delta}}{8 r_h \gamma_1}, \quad \Delta = r_h^4 - 96 \gamma_1^2.$$ 

- In the following, we will focus on solutions corresponding to the “+” sign.
  - Solution corresponding to “+” sign $\leftrightarrow$ approach regularly Schwarschild solution in the $\gamma_1 \to 0$ limit.
  - Solution corresponding to “−” sign $\leftrightarrow$ no regular limit for $\gamma_1 \to 0$. 

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Shift-symmetry ($\gamma_1 \neq 0, \gamma_2 = 0$)

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  - Solution corresponding to “+” sign ↔ approach regularly Schwarschild solution in the $\gamma_1 \to 0$ limit.
  - Solution corresponding to “−” sign ↔ no regular limit for $\gamma_1 \to 0$.

- On this branch, solutions exists for $\gamma_1 \in \left[0, r_h^2\sqrt{1/96}\right]$. 
Shift-symmetry \((\gamma_1 \neq 0, \gamma_2 = 0)\)

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\phi'(r_h) = \frac{-r_h^2 \pm \sqrt{\Delta}}{8r_h \gamma_1}, \quad \Delta = r_h^4 - 96\gamma_1^2.
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  - Solution corresponding to “+” sign \(\leftrightarrow\) approach regularly Schwarschild solution in the \(\gamma_1 \rightarrow 0\) limit.
  - Solution corresponding to “−” sign \(\leftrightarrow\) no regular limit for \(\gamma_1 \rightarrow 0\).

- On this branch, solutions exists for \(\gamma_1 \in \left[0, r_h^2 \sqrt{1/96}\right]\).

- Since \(\phi'(r_h)\) does only depend on \(r_h\) and \(\gamma_1\), for a fixed \(r_h\), there is only one possible solution for each value of \(\gamma_1\). (no excited solutions)
Spontaneous scalarization \((\gamma_1 = 0, \gamma_2 \neq 0)\)

The equation of motion for \(\phi\) read

\[
\Box \phi = -2\gamma_2 \phi \mathcal{I}(g) \iff \hat{D} \phi = \gamma_2 \phi.
\]

The condition of regularity at the horizon reduces to

\[
\phi'(r_h) = \frac{-r_h^2 \pm \sqrt{\Delta}}{16r_h \gamma_2 \phi(r_h)}, \quad \Delta = r_h^4 - 384\gamma_2^2 \phi(r_h)^2.
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\]

In this case the pattern of solutions is very different:

- Solutions exists only for \(\gamma_2 \in [\gamma_{2,c}, \gamma_{2,\text{max}}]\), whith \(\gamma_{2,c} > 0\).
- Excited solutions exists.
Spontaneous scalarization \( (\gamma_1 = 0, \gamma_2 \neq 0) \)

Origin of the critical values

The existence of regular solutions require 3 conditions:

\[ \Delta \geq 0, \; \gamma_2 \neq 0 \; \text{and} \; \phi(r_h) \neq 0 \]

\[ \rightarrow \gamma_{2,c} : \text{Correspond to} \; \Delta \rightarrow 0. \]

\[ \rightarrow \gamma_{2,max} : \text{Correspond to} \; \phi(r_h) \rightarrow 0. \]
Spontaneous scalarization \((\gamma_1 = 0, \gamma_2 \neq 0)\)

Origin of the critical values

The existence of regular solutions require 3 conditions :

\[ \Delta \geq 0, \ \gamma_2 \neq 0 \text{ and } \phi(r_h) \neq 0 \]

\[ \rightarrow \gamma_2,c : \text{Correspond to } \Delta \to 0. \]

\[ \rightarrow \gamma_2,\text{max} : \text{Correspond to } \phi(r_h) \to 0. \]

This pattern can be understood when examining the case of a test field :
On a fixed Schwarzschild background the equation for \(\phi\) can be written as

\[
\frac{r^4}{48M} \frac{d}{dr} \left[ r^2 \left( 1 - \frac{2M}{r} \right) \frac{d}{dr} \phi \right] = \gamma_2 \phi \iff \hat{D}_{Sch} \phi = \gamma_2 \phi.
\]

\[ \Rightarrow \gamma_2 \text{ must be an eigen value of the differential operator } \hat{D}_{Sch} \leftrightarrow \gamma_{2,\text{max}}. \]
Spontaneous scalarization \((\gamma_1 = 0, \gamma_2 \neq 0)\)

unexcited solutions

\[
\Delta \phi(r_h)
\]
New results \((\gamma_1 \neq 0, \gamma_2 \neq 0)\)

unexcited solutions
New results \( (\gamma_1 \neq 0, \gamma_2 \neq 0) \)

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New results \((\gamma_1 \neq 0, \gamma_2 \neq 0)\)

influence of a mass term

![Graph showing the influence of a mass term with different mass values]
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5 Conclusion
Hypothesis
Hypothesis

- The scalar field is complex, of the form $\phi(x^\mu) = e^{i\omega t} \phi(r)$ with $\omega \neq 0$. Accordingly, the Lagrangian possesses a global $U(1)$ symmetry. The associated Noether charge will be denoted $Q$. 

$V(\phi) = m^2 |\phi|^2 + \lambda_4 |\phi|^4 + \lambda_6 |\phi|^6$ and should contain at least a mass term, so $m > 0$.

More precisely, we will concentrate our study to two cases:

• no self-interaction: $m \neq 0$, $\lambda_4 = 0$, $\lambda_6 = 0$,

• self-interaction: $m \neq 0$, $\lambda_4 = -2m^2 \phi^2$, $\lambda_6 = m^2 \phi^4$.

In this case, the potential is $V(\phi) = m^2 \phi^2 (1 - \phi^2 \phi^2)^2$. It possesses three degenerate minima located at $\phi = 0, \pm \phi_c$.
Hypothesis

- The scalar field is complex, of the form \( \phi(x^\mu) = e^{i\omega t} \phi(r) \) with \( \omega \neq 0 \). Accordingly, the Lagrangian possesses a global \( U(1) \) symmetry. The associated Noether charge will be denoted \( Q \).

- The potential is of the form \( V(\phi) = m^2|\phi|^2 + \lambda_4|\phi|^4 + \lambda_6|\phi|^6 \) and should contain at least a mass term, so \( m > 0 \). More precisely, we will concentrate our study to two cases:
  - no self-interaction: \( m \neq 0, \lambda_4 = 0, \lambda_6 = 0 \),
  - self-interaction: \( m \neq 0, \lambda_4 = -2 \frac{m^2}{\phi_c^2}, \lambda_6 = \frac{m^2}{\phi_c^4} \). In this case, the potential is \( V(\phi) = m^2\phi^2 \left( 1 - \frac{\phi^2}{\phi_c^2} \right)^2 \). It possesses three degenerate minima located at \( \phi = 0, \pm \phi_c \).
Hypothesis

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  - no self-interaction: $m \neq 0$, $\lambda_4 = 0$, $\lambda_6 = 0$,
  - self-interaction: $m \neq 0$, $\lambda_4 = -2 \frac{m^2}{\phi_c^2}$, $\lambda_6 = \frac{m^2}{\phi_c^4}$. In this case, the potential is $V(\phi) = m^2 \phi^2 \left(1 - \frac{\phi^2}{\phi_c^2}\right)^2$. It possesses three degenerate minima located at $\phi = 0, \pm \phi_c$.

- The linear coupling to the Gauss-Bonnet term will be set to zero, so $\gamma_1 = 0$. In other words: $f(\phi) = \gamma_2 |\phi|^2$. 
Solutions without self-interaction

\[ \omega \]

\[ M \]

- \( \gamma_2 = 0 \)
- \( \gamma_2 = 0.02 \)
- \( \gamma_2 = 0.05 \)
- \( \gamma_2 = 0.1 \)
- \( \gamma_2 = 0.2 \)
Solutions with self-interaction
The Noether charge associated to the global $U(1)$ symmetry, i.e. $Q$, will be interpreted as a number of particles. More precisely, $Q$ will be seen as the number of bosons of mass $m$ forming the star (of mass $M$). Within this interpretation, it is natural to perform a comparison between $M$ and $mQ$:

- If $M < mQ$, we will say that the boson star is classically stable, since the total mass of the star $M$ is lower than the "sum of its constituents" $mQ$.
- If $M > mQ$, following the same lines, we will say that the boson star is classically unstable.

In the following, we will report our results in terms of the quantity $\frac{M}{mQ}$:

$$\frac{M}{mQ} > \frac{1}{2} \Leftrightarrow \text{(un)stable}$$

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RTG Workshop
October 2019
Classical stability

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Plan

1 Introduction
2 Model
3 Equations of motion
   - Equations of motion
   - Ansatz
   - Boundary conditions
4 Results
   - Black holes
     - Shift-symmetry
     - Spontaneous scalarization
     - New results
   - Boson stars
     - Domain of existence
     - Classical stability
5 Conclusion
Conclusions & outlooks

\[ S = \int \left[ \frac{1}{16\pi G} R - \nabla_\mu \phi^* \nabla^\mu \phi - V(\phi) + f(\phi) \mathcal{I}(g) \right] \sqrt{-g} d^4 x. \]

\[ f(\phi) = \gamma_1 |\phi| + \gamma_2 |\phi|^2, \quad \mathcal{I}(g) = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \]
Conclusions & outlooks

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Conclusions :

- We have illustrated how \( f(\phi) \) can lead to very different patterns when coupled to the Gauss-Bonnet invariant,
- In the case of black holes: our results link shift-symmetric theory to spontaneous scalarization,
- In the case of boson stars: we shown how a coupling function \( \gamma_2 |\phi|^2 \) could enlarge the domain and improve the stability of the solutions.
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Conclusions:
- We have illustrated how \( f(\phi) \) can lead to very different patterns when coupled to the Gauss-Bonnet invariant,
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Outlooks:
- Charged scalar field: \( \nabla_\mu \phi \rightarrow D_\mu \phi = (\partial_\mu + i e A_\mu) \phi \),
- Other types of coupling: \( \nabla_\mu \phi^* g^{\mu\nu} \nabla_\nu \phi \rightarrow \nabla_\mu \phi^* (\alpha g^{\mu\nu} + \eta G^{\mu\nu}) \nabla_\nu \phi \),
- Influence on matter: \( T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu} \).
Thank you for your attention!
Y. Brihaye & L. Ducobu,
“Hairy black holes, boson stars and non-minimal coupling to curvature invariants,”
[arXiv :1812.07438 [gr-qc]].