

On PT-symmetric extensions of the Calogero and Sutherland models

Yves Brihaye*

Faculté des Sciences, Université de Mons-Hainaut,
B-7000 Mons, Belgium

Ancilla Nininahazwe †

Faculté des Sciences, Université du Burundi
P.O. Box 2700, Bujumbura, Burundi

February 1, 2008

Abstract

The original Calogero and Sutherland models describe N quantum particles on the line interacting pairwise through an inverse square and an inverse sinus-square potential. They are well known to be integrable and solvable. Here we extend the Calogero and Sutherland Hamiltonians by means of new interactions which are PT-symmetric but not self adjoint. Some of these new interactions lead to integrable PT-symmetric Hamiltonians; the algebraic properties further reveal that they are solvable as well. We also consider PT-symmetric interactions which lead to a new quasi-exactly solvable deformation of the Calogero and Sutherland Hamiltonians.

*yves.brihaye@umh.ac.be

†nininaha@yahoo.fr

1 Introduction

The Calogero and Sutherland models [1, 2] (CSM in the following) and various of their extensions have received a considerable new impetus of interest in the last years. We mention for example the recent preprint [3] where CSM are applied in the framework of conformal field theory. From the beginning, the CSM are known to be integrable both on the classical and quantum levels. In the framework of quantum systems, the distinction between integrable and exactly solvable models can further be made, according to the lines of Ref. [4] where it was shown that quantum CSM possess both properties: integrable and exactly solvable. Of course the notion of exactly solvable operators constitutes a particular case of Quasi-Exactly-Solvable (QES) operators [5] for which only a finite part of the spectrum can be computed algebraically. The possibility of having QES extensions of the Calogero and Sutherland Hamiltonians is a natural question which was addressed e.g. in [6, 7].

Recently again, it was recognized that the self-adjoint property is not necessary for an operator A to have a real spectrum. Instead the weaker condition of invariance of A under the combination of parity P and time-reversal T symmetries leads to a spectrum which is either real or composed of pairs of complex conjugate numbers [8]. It is therefore natural to consider non-self adjoint but PT-symmetric Hamiltonians describing N -particles quantum systems and to construct operators which are (i) integrable or (ii) exactly solvable or (iii) quasi-exactly solvable. Examples of such operators were studied in [9] and reconsidered more recently [10]

Of course the problem of classifying all PT-symmetric (quasi)- exactly integrable operators is vast, one possible way to approach it [9, 10] is to extend the well established CSM by suitably chosen non self-adjoint but PT-symmetric extra terms and to study which algebraic properties of the CSM are preserved or spoiled by the new terms. In this paper we consider more general types of PT-symmetric terms restricted by the following properties (i) translation invariant, (ii) first order in the derivatives, (iii) fully symmetric under the permutations of the N degrees of freedom. In the three cases we put the emphasis on the (quasi) solvability of the deformed model. We will exhibit two extensions which preserve the exact solvability and another one leading to a new type of QES N -body Hamiltonian.

2 PT-invariant Hamiltonians: rational case

We consider the quantum N -body Hamiltonians of the form

$$H = H_{cal} + H_{PT} \tag{1}$$

where H_{cal} denotes the rational Calogero[1] Hamiltonian :

$$H_{cal} = -\frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \frac{\omega^2}{2N} \left(\frac{1}{2} \tilde{r}_2 + \epsilon^2 \sigma_1^2 \right) + g \sum_{j < k} \frac{1}{(x_j - x_k)^2} \tag{2}$$

$$H_{PT} = \delta \sum_{j \neq k} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j} + \gamma \sum_{j \neq k} (x_j - x_k) \frac{\partial}{\partial x_j} \quad (3)$$

$$= \delta H_0 + \gamma H_1 \quad (4)$$

$$\tilde{\tau}_2 \equiv \sum_{j \neq k} (x_j - x_k)^2, \quad \sigma_1 \equiv \sum_{j=1}^N x_j, \quad (5)$$

For later convenience, we split the traditional harmonic part into the translation-invariant piece ($\tilde{\tau}_2$) and the "center of mass coordinate" σ_1 ;

$$\sum_{j=1}^N x_j^2 = \frac{1}{N} \left(\frac{1}{2} \tilde{\tau}_2 + \sigma_1^2 \right) \quad (6)$$

Setting $\epsilon = 0$ the above Hamiltonian is purely translation invariant, the standard case is recovered for $\epsilon = 1$. The part H_{PT} is not self-adjoint but symmetric under the PT reflexion $i \rightarrow -i$, $x_k \rightarrow -x_k$, $p_k \rightarrow p_k$ ($p_k = \frac{1}{i} \frac{\partial}{\partial x_k}$), it extends the interaction H_0 considered recently in [10]. In order to reveal the solvability property of H we first perform the standard (often called "gauge") transformation

$$\tilde{H} = \Psi_{gr}^{-1} H \Psi_{gr}, \quad \Psi_{gr} = \beta^\nu E \quad (7)$$

with

$$\beta = \prod_{j < k} (x_j - x_k), \quad E = \exp\left(-\frac{\alpha}{4} \tilde{\tau}_2 - \frac{\alpha'}{2} \sigma_1^2\right) \quad (8)$$

After some algebra, the new Hamiltonian \tilde{H} is obtained. The parameters ν, α, α' entering in Ψ_{gr} are fixed according to

$$g = \nu^2 - \nu(1 + 2\delta), \quad \omega^2 = \alpha^2 N + 2\alpha\gamma N, \quad \omega = \frac{\alpha' N}{\epsilon} \quad (9)$$

in such a way that the terms with the highest power in x_i as well as the most singular terms in x_i disappear in \tilde{H} . The condition on g, ν, δ is just identical as in the case of Ref.[9] and lead to the same restrictions on these constants for the eigenvectors to be nonsingular : (i) $\delta > -\frac{1}{2}$, $0 > g > -(\delta + \frac{1}{2})^2$ and $g > 0$ for arbitrary values of δ .

2.1 Exact solvability of H

The next step consists in writing \tilde{H} in terms of variables which encodes the full symmetry of the problem under permutations of the N degrees of freedom. We use here the variables introduced in [4] which we recall for completeness :

$$\sigma_1(x) = x_1 + x_2 + \dots + x_N \quad (10)$$

$$\sigma_2(x) = x_1 x_2 + x_1 x_3 + \dots + x_{N-1} x_N \quad (11)$$

$$\sigma_3(x) = \sum_{i < j < k} x_i x_j x_k, \quad \dots \quad (12)$$

$$\sigma_N(x) = x_1 x_2 x_3 \dots x_N \quad (13)$$

and for $k = 2, 3, \dots, N$ we further define the translation invariant variables

$$\tau_k(x) = \sigma_k(y) \quad y_j = x_j - \frac{1}{N}\sigma_1 \quad (14)$$

In particular, we have $\tilde{\tau}_2 = -4N\tau_2$. The Laplacian operator once expressed in terms of these new variables reads [4]

$$\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} \quad (15)$$

$$= N \frac{\partial^2}{\partial \sigma_1^2} + \sum_{j,k=2}^N A_{jk} \frac{\partial^2}{\partial \tau_j \partial \tau_k} + \sum_{j=2}^N B_j \frac{\partial}{\partial \tau_j} \quad (16)$$

$$A_{jk} = \frac{(N-j+1)(k-1)}{N} \tau_{j-1} \tau_{k-1} + \sum_{l \geq \max(1, k-j)}^N (k-j-2l) \tau_{j+l-1} \tau_{k-l-1} \quad (17)$$

$$-B_j = \frac{(N-j+2)(N-j+1)}{N} \tau_{j-2} \quad (18)$$

where the following conventions have to be understood

$$\tau_0 = 1 \quad \tau_1 = 0 \quad \tau_{-p} = 0 \quad \tau_{N+p} = 0 \quad \text{for } p > 0 \quad (19)$$

The PT invariant interactions H_0, H_1 defined in Eq.(3) can also be expressed in terms of the new variables by means of the following identities :

$$D = \sum_{j=1}^N x_j \frac{\partial}{\partial x_j} \quad (20)$$

$$= \sigma_1 \frac{\partial}{\partial \sigma_1} + \sum_{k=2}^N k \tau_k \frac{\partial}{\partial \tau_k} \quad (21)$$

$$\equiv \sigma_1 \frac{\partial}{\partial \sigma_1} + \tilde{D} \quad (22)$$

$$H_0 = -\frac{1}{2} \sum_{j=2}^N (N-j+2)(N-j+1) \tau_{j-2} \frac{\partial}{\partial \tau_j} \quad (23)$$

$$H_1 = N \tilde{D} \quad (24)$$

The final form of the operator \tilde{H} reads then

$$\begin{aligned} \tilde{H} &= \frac{-1}{2} \Delta + (\nu - \delta) H_0 \\ &+ \omega \sqrt{1 + \frac{\gamma^2 N^2}{\omega^2}} \tilde{D} + \omega \epsilon \sigma_1 \frac{\partial}{\partial \sigma_1} + E_0 \end{aligned} \quad (25)$$

$$E_0 = \alpha \frac{N(N-1)}{2} (\nu N + 1) + \frac{N\alpha'}{2} + \nu \gamma \frac{N^2(N-1)}{2} - \delta \alpha \frac{N^2(N-1)}{2} \quad (26)$$

The solvability of the operator \tilde{H} (and therefore of H in (1)) is revealed by the observation [4] that this operator (25) preserves the infinite flag of vector space

$$\mathcal{V}_n = \text{span}(\sigma_1^{n_1} \tau_2^{n_2} \dots \tau_N^{n_N}, \sum_{k=1}^N n_k \leq n), \quad n \in \mathbb{N} \quad (27)$$

The eigenvectors of \tilde{H} can therefore be constructed by considering the restriction of \tilde{H} to \mathcal{V}_n . As far as the construction of the eigenvalues is concerned, it can be realized that the restriction of \tilde{H} to \mathcal{V}_n leads effectively to a lower triangular matrix, the diagonal elements of which are issued from the matrix elements corresponding to the second line in Eq.(25). Of course the triangularity of this matrix is strongly related to the order adopted to enumerate the basis elements $\tau_2^{n_2} \dots \tau_N^{n_N}$. The relevant order can be set as follow

$$\begin{aligned} 0 &\leq n_N \leq n, \\ 0 &\leq n_{N-1} \leq n - n_N, \dots, \\ 0 &\leq n_2 \leq n - n_N - n_{N-1} \dots - n_3 \end{aligned} \quad (28)$$

The eigenvalues can therefore be determined in terms of the diagonal matrix elements by using the monomials defined in Eq.(27) as a basis. The spectrum reads

$$E_{n_1, n_2, \dots, n_N} = E_0 + \omega \epsilon n_1 + \omega \sqrt{1 + \frac{\gamma^2 N^2}{\omega^2}} (2n_2 + 3n_3 + \dots Nn_N) \quad (29)$$

The spectrum of the Calogero model is off course recovered with all its degeneracies in the limit $\gamma = 0$; the term H_0 does not affect these degeneracies. However for $\gamma \neq 0$ several degeneracies, but not all, are lifted by the H_1 interaction. The eigenstates which remain degenerate are those with $\sum_{k=2}^N kn_k = \sum_{k=2}^N kn'_k$.

2.2 Quasi-exactly Solvable case

In this section we consider the Hamiltonian

$$H = H_{cal} + \theta \sum_{i \neq j} (x_i - x_j)^3 \frac{\partial}{\partial x_i} + \tilde{\theta} \sum_{i \neq j} (x_i - x_j)^4 \quad (30)$$

where H_{cal} is supplemented by a PT-invariant term (with coupling constant θ) as well as a new term of the potential. This term is anharmonic, being of degree 4 in the position, and characterized by a coupling constant $\tilde{\theta}$. Performing the change of function by means of ψ_{gr} leads to

$$\begin{aligned} \tilde{H} &= -\frac{1}{2}\Delta + \nu H_0 + \omega D \\ &+ (\tilde{\theta} - \frac{\theta\alpha}{2}) \sum_{i \neq j} (x_i - x_j)^4 + 4N\tau_2 \left(\frac{\alpha^2 N}{4} - \frac{\omega^2}{4N} - \theta\nu - \frac{\theta N}{4} \tilde{D} \right) \end{aligned} \quad (31)$$

If the new coupling constant $\theta, \tilde{\theta}$ are chosen in such a way that

$$\tilde{\theta} = \frac{\theta\alpha}{2} \quad , \quad \frac{1}{\theta}(\alpha^2 - \frac{\omega^2}{N^2} - \frac{4\theta\nu}{N}) = m \in \mathbb{N} \quad , \quad (\leftrightarrow \quad \alpha^2 = \frac{\omega^2}{N^2} + 4\frac{\theta\nu}{N} + m\theta) \quad (32)$$

then the operator \tilde{H} preserves a finite dimensional vector space which we call $\tilde{\mathcal{V}}_m$. As so, it is quasi-exactly solvable [5]. The vector space $\tilde{\mathcal{V}}_m$ is characterized by the condition

$$\tau_2(m - \tilde{D})\tilde{\mathcal{V}}_m = \tau_2(m - \sum_{k=2}^N k\tau_k \frac{\partial}{\partial \tau_k})\tilde{\mathcal{V}}_m \subset \tilde{\mathcal{V}}_m \quad (33)$$

A little algebra shows that m can not be arbitrary. As an example, if $N = 3$ the vector space $\tilde{\mathcal{V}}_m$ has the form

$$\tilde{\mathcal{V}}_m = \mathcal{P}(\frac{m}{2}) \oplus \tau_3^2 \mathcal{P}(\frac{m}{2} - 3) \oplus \tau_3^4 \mathcal{P}(\frac{m}{2} - 6) \oplus \dots \quad (34)$$

if m is even ($m \geq 2$) and

$$\tilde{\mathcal{V}}_m = \tau_3 \mathcal{P}(\frac{m-3}{2}) \oplus \tau_3^3 \mathcal{P}(\frac{m-9}{2}) \oplus \tau_3^5 \mathcal{P}(\frac{m-15}{2}) \oplus \dots \quad (35)$$

if m is odd ($m \geq 5$). The sum of course runs with k as long as $(m - 6k)/2$ (resp. $(m - 3 - 6k)/2$) is positive for m even (resp. odd). Remark that we have the following inclusions

$$\tilde{\mathcal{V}}_m \subset \mathcal{V}_{m/2} \quad , \quad \tilde{\mathcal{V}}_m \subset \mathcal{V}_{(m-3)/2} \quad (36)$$

and that the vector spaces $\tilde{\mathcal{V}}_m$ are preserved separately by the operators Δ, H_0, D . In order to illustrate these results, we constructed the eigenvalues of the Hamiltonian in the case $N = 3, m = 6$, for which the vector space $\tilde{\mathcal{V}}_6$ has five dimensions. The coupling constant $\tilde{\theta}$ of the quartic interaction in (30) is then adjusted in terms of θ, ω, ν and m by Eq (32). Note that Changing the value of m results in another value for α and therefore another value for $\tilde{\theta}$. The eigenvalues E cannot be computed explicitly in terms of the coupling constants ν, ω, θ but, considering the PT-symmetric term as a perturbation of the Calogero Hamiltonian leads to the following result

$$\begin{aligned} E_{0,0} &= 0 + 27(\nu - 1)\frac{\theta}{\omega} + O(\theta^2) \\ E_{1,0} &= 2\omega + 9(\nu - 5)\frac{\theta}{\omega} + O(\theta^2) \\ E_{2,0} &= 4\omega - 9(\nu + 1)\frac{\theta}{\omega} + O(\theta^2) \\ E_{3,0} &= 6\omega - 27(\nu - 3)\frac{\theta}{\omega} + O(\theta^2) \\ E_{0,2} &= 6\omega \end{aligned} \quad (37)$$

Note that the only degenerate energy levels which occur in the diagonalisation of the restricted to \mathcal{V}_3 corresponding to $m_2 = 3, m_3 = 0$ and $m_2 = 0, m_3 = 2$. These vectors

are included in the subspace $\tilde{\mathcal{V}}_6$ and, again, this new term lift the degeneracy. The six extra vectors corresponding to \mathcal{V}_3 are not accessed algebraically in the presence of the new term. We also solve numerically the eigenvalue equation for generic values of ν, θ and found that the spectrum remain real.

3 PT-invariant Hamiltonians: trigonometric case

Here, we consider an extension of the Sutherland Hamiltonian H_{su} in the form

$$H = H_{su} + H_{PT} + V \quad (38)$$

$$H_{su} = -\frac{1}{2}\Delta + \frac{g}{4} \sum_{i<j} \frac{1}{\sin^2(\frac{x_i-x_j}{2})} \quad (39)$$

where the extra PT symmetric part is chosen of the form

$$H_{PT} = \delta H_0 + \gamma H_1 \quad (40)$$

$$\begin{aligned} &= \delta \sum_{i<j} \cot(\frac{x_i-x_j}{2}) \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \\ &+ \gamma \sum_{i<j} \sin(x_i-x_j) \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \end{aligned} \quad (41)$$

which provides a natural generalisation of Eq.(3) to the trigonometric case. The Sutherland inverse sine-square potential is also supplemented by the following terms

$$\begin{aligned} V &= \theta_1 \sum_{i \neq j \neq k} \cot(\frac{x_i-x_j}{2}) \cot(\frac{x_i-x_k}{2}) \\ &+ \theta_2 \sum_{i<j} \cos^2(\frac{x_i-x_j}{2}) \\ &+ \theta_3 \sum_{i \neq j \neq k} \cos(\frac{x_i-x_j}{2}) \cos(\frac{x_i+x_j-2x_k}{2}) \end{aligned} \quad (42)$$

where the θ_2 -term involves two body-interactions (like the original Sutherland potential) while the θ_1 and θ_3 -terms involve three body-interactions. Similarly the previous case, it is convenient to “gauge rotate” of H in (38) by using the ground state ψ_{gr} of the Sutherland model i.e

$$\psi_{gr} = \left(\prod_{j<k} \sin(\frac{x_j-x_k}{2}) \right)^\nu \quad \nu^2 + \nu = g \quad (43)$$

In this purpose, the following identities are usefull

$$\begin{aligned}\beta^{-1}H_0\beta &= H_0 + \sum_{i<j} \frac{1}{\sin^2(\frac{x_i-x_j}{2})} - \frac{N(N-1)}{2} \\ &+ \sum_{i\neq j\neq k} \cot(\frac{x_i-x_j}{2}) \cot(\frac{x_i-x_k}{2})\end{aligned}\quad (44)$$

$$\beta^{-1}H_1\beta = H_1 + \sum_{i<j} \cos^2(\frac{x_i-x_j}{2}) + \sum_{i\neq j\neq k} \cos(\frac{x_i-x_j}{2}) \cos(\frac{x_i+x_j-2x_k}{2})\quad (45)$$

The gauge rotation of the full Hamiltonian (38) by ψ_{gr} leads to the following equivalent operator

$$h = -2\beta^{-\nu}H\beta^\nu\quad (46)$$

$$= \Delta + (\nu - 2\delta)H_0 + (-2\gamma)H_1 + E_0\quad (47)$$

where $E_0 \equiv \nu^2 N(N^2 - 1)/12$ denotes the energy of the ground state. In order to obtain the above formula, the various coupling constants are choosen according to

$$g = \nu(\nu - 1) + \delta \quad , \quad \theta_1 = -\delta \quad , \quad \theta_2 = \theta_3 = -\gamma\quad (48)$$

in such a way that the singular terms and the non derivative parts of the operator h cancel.

3.1 Exact solvability

The next step to reveal the solvability of the Hamiltonian (46) is to use the following change of variables

$$\xi_N = e^{ix_1} + e^{ix_2} + \dots + e^{ix_N}\quad (49)$$

$$\eta_1 = e^{iy_1} + e^{iy_2} + \dots + e^{iy_N}\quad (50)$$

$$\eta_2 = e^{i(y_1+y_2)} + e^{i(y_1+y_3)} + \dots + e^{i(y_{N-1}+y_N)}\quad (51)$$

...

$$\eta_{N-1} = \eta_1^* \quad , \quad y_i = x_i - \frac{1}{N} \sum_{k=1}^N x_k\quad (52)$$

The different operators can then be expressed in terms of the new variables namely

$$\Delta = -N(\xi_N \frac{\partial}{\partial \xi_N})^2 - \sum_{j,k=1}^{N-1} A_{jk} \frac{\partial^2}{\partial \eta_j \partial \eta_k} - \frac{1}{N} \sum_{l=1}^{N-1} l(N-1) \eta_l \frac{\partial}{\partial \eta_l}\quad (53)$$

$$A_{jk} = \frac{k(N-j)}{N} \eta_j \eta_k + \sum_{l \geq \max(1, k-j)} (k-j-2l) \eta_{j+l} \eta_{k-l}\quad (54)$$

$$H_0 = - \sum_{l=1}^{N-1} l(N-l) \eta_l \frac{\partial}{\partial \eta_l}\quad (55)$$

Considering first the H_0 interaction only ($\gamma = \theta_2 = \theta_3 = 0$), we observe that this PT-symmetric term does not spoil the solvability of the Sutherland Hamiltonian [4] because both Δ , H_0 preserve the infinite flag of vector spaces $\mathcal{V}_n(\eta)$ (see (27) for the definition) as easily seen from Eq. (46). It is, however, worth to stress that the term H_0 leads to solvability only if the extra three-body potential corresponding to the θ_1 -term in Eq. (42) is supplemented to the original two-body Sutherland potential. This contrasts with the rational case where the addition of the term H_0 to the Calogero Hamiltonian leads to another solvable model without any extra pieces in the potential. The interaction H_1 is investigated in the next section.

3.2 Quasi Exact solvability

We could not evaluate H_1 in terms of η for generic cases of N but, for $N=2,3$, we find respectively

$$H_1 = \frac{1}{2}(\eta_1^2 - 4)\eta_1 \frac{\partial}{\partial \eta_1} \quad , \quad \eta_1 = 2 \cos \frac{x_1 - x_2}{2} \quad (56)$$

$$H_1 = \frac{1}{2}(\eta_1^2 \eta_2 - 2\eta_2^2 - 3\eta_1) \frac{\partial}{\partial \eta_1} + \frac{1}{2}(\eta_2^2 \eta_1 - 2\eta_1^2 - 3\eta_2) \frac{\partial}{\partial \eta_2} \quad (57)$$

$$\eta_1 = e^{iy_1} + e^{iy_2} + e^{iy_3} \quad , \quad \eta_2 = \eta_1^* \quad (58)$$

We see that, for $\gamma \neq 0$, the Hamiltonian h is likely nor solvable neither quasi-exactly solvable as suggested by the form of H_1 for $N=3$, (see Eq(57)). This operator obviously does not preserve any of the $\mathcal{V}_n(\eta)$ (see (27)).

For $N=2$, the operator (38) is nevertheless QES provided $\theta_2 = -2\gamma n, n \in \mathbb{N}$ in (42). Indeed, the ‘‘gauge rotated’’ operator reads in this case

$$h = \left(\frac{1}{2}\eta_1^2 - 2\right) \frac{\partial^2}{\partial \eta_1^2} + \left(\frac{1}{2} + \nu - \delta\right) \eta_1 \frac{\partial}{\partial \eta_1} + \frac{1}{2}\gamma(\eta_1^2 - 4)\eta_1 \frac{\partial}{\partial \eta_1} - \frac{1}{2}\gamma n \eta_1^2 \quad (59)$$

which can be easily checked to preserve the vector space

$$\mathcal{P}_n = \text{span}(\eta_1^n, \eta_1^{n-2}, \eta_1^{n-4}, \dots) \subset \mathcal{V}_n \quad (60)$$

Accordingly, $(n+2)/2$ (resp. $(n+1)/2$) eigenvectors can be constructed algebraically for n even (resp. odd).

Once more we notice that, in spite of the fact that (40) constitutes the natural counterpart of Eq. (3) for the trigonometric case, the solvability is preserved only for the interaction H_0 ; in contrast to the rational case, the Hamiltonian H_1 spoils the solvability of the model for $N > 2$.

4 Conclusion

We have considered several extensions of the Calogero and Sutherland models by means of PT-symmetric terms which are translation invariant, completely symmetric

and involve first derivatives. The algebraic structure of these extended Hamiltonians is revealed by (i) the change of function involving a common ground state function ψ_{gr} , (ii) the change to appropriate symmetric coordinates and (iii) the construction of some vector spaces of polynomials preserved by these operators. The completely solvable character of the Calogero Hamiltonians is preserved for the terms called H_0 and H_1 . Another choice of interaction, involving a PT-symmetric term supplemented by an extra anharmonic potential, leads to a new Quasi-Exactly-Solvable extension of the Calogero model. Note that this QES extension differs from the one obtained first in [6] where a potential depending of the variable τ_2 was added.

Applying the same kind of methods to the trigonometric (or Sutherland) model reveals that some PT-invariant terms lead to completely solvable extended models only provided a suitable extra three-body interaction is added to the potential. For another natural PT-symmetric term, the solvability is spoiled even at the price of adding new terms to the Sutherland original potential. Only for $N = 2$ the corresponding operator turns out to be quasi-exactly solvable.

Acknowledgements

A. N. is supported by a grant of the C.U.D..

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