Higher-spin gravities with bifundamental fields and dynamical 2-form

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SISSA, 29 July 2014

Based on works – to appear soon – done in collaboration with E. Sezgin and P. Sundell

Plan

1. Introduction to unfolding and Vasiliev's equations
2. Extension with bifundamental fields
3. Underlying Frobenius algebra ; Action
4. Conclusions

1. INTRO TO UNFOLDING ; VASILIEV'S EQUNS

- Weyl's Gauge Principle : HS theories contain gravity. ∞-dim gauge algebra ;
- Vasiliev's unfolding : A geometric, Cartan-like approach to field theory ;
- AdS/CFT dualities for Vasiliev's theory : AdS_4/CFT_3 [Sezgin-Sundell, Klebanov-Polyakov] and AdS_3/CFT_2 [Gaberdiel-Gopakumar]. Relations with statistical physics, non-commutative field theory, strings.

The gauge principle [H. Weyl, 1929]

M. A. Vasiliev : *fully nonlinear field equations* for higher-spin gauge fields in 4D [Vasiliev, 1990 – 1992] and in D space-time dimensions [hep-th/0304049]. Salient features :

- Manifest diffeomorphism invariance, no explicit d^2s ;
- Cartan integrability $\Rightarrow$ *gauge invariance* under $\mathfrak{hs}_D$;
- Two ∞-dim $\mathfrak{so}(2, D-1)$ modules : *adjoint* and *twisted-adjoint* representations $\rightsquigarrow$ master 1-form and master zero-form. Uses unfolding in terms of FDA.

A free (graded commutative, associative) differential algebra $\mathfrak{R}$ is set $\{X^\alpha\}$ of *a priori* independent variables, locally-defined differential forms obeying first-order equations of motion

$$\mathscr{R}^\alpha = \mathrm{d}X^\alpha + Q^\alpha(X) = 0 \,, \quad Q^\alpha(X) = \sum_n f^\alpha_{\beta_1 \ldots \beta_n} X^{\beta_1} \cdots X^{\beta_n} \,.$$

Nilpotency of d and integrability condition $\mathrm{d}\mathscr{R}^\alpha = 0$ require

$$Q^\beta \, \frac{\partial Q^\alpha}{\partial X^\beta} \equiv 0 \quad .$$

For $X^\alpha_{[p_\alpha]}$ with $p_\alpha > 0$, gauge transformation preserving $\mathscr{R}^\alpha \approx 0$:

$$\delta_\epsilon X^\alpha = \mathrm{d}\epsilon^\alpha - \epsilon^\beta \frac{\partial^L}{\partial X^\beta} Q^\alpha \quad .$$

The principle of unfolding [Vasiliev, 1988 –]

- The concepts of spacetime, dynamics and observables are *derived* from infinite-dimensional FDA's.
- Unfolded dynamics is an inclusion of local d.o.f. into field theories described *on-shell* by flatness conditions on generalized curvatures.
- The local, perturbative d.o.f. are contained in the zero-forms ;
- Lorentz-covariant derivative, minimal coupling.

- A master 1-form $A = \sum_{s=2,4,\dots} A_{(s)}$ where

$$A_{(s)} = -i \sum_{t=0}^{s-1} dx^\mu \, A_\mu^{a(s-1),b(t)}(x) M^{a_1 b_1} \dots M^{a_t b_t} P^{a_{t+1}} \dots P^{a_s - 1} \; ;$$

- A master zero-form $\Phi = \sum_{s=0,2,4,\dots} \Phi_{(s)}$ where

$$\Phi_{(s)} = \sum_{k=0}^{\infty} \tfrac{1}{k!} \, \Phi^{a(s+k),b(s)}(x) M^{a_1 b_1} \dots M^{a_s b_s} P^{a_{s+1}} \dots P^{a_s + k} \; ;$$

Vasiliev's eqns : $F + \sum_{n=1}^{\infty} J_{(n)}(A, A; \Phi, \dots, \Phi) = 0 \,,$

$$D_x \Phi + \sum_{n=2}^{\infty} P_{(n)}(A; \Phi, \dots, \Phi) = 0 \,,$$

$$F := \mathrm{d}_x A + A \star A \,, \; D_x \Phi := \mathrm{d}_x \Phi + [A, \Phi]_\pi \,, \; \pi(P, M) = (-P, M) \,,$$

Master fields of the *minimal bosonic model* :

$$\star \ \underline{\text{adjoint}} \quad A \ = \ A_x + A_z \ ,$$

$$A_x \ = \ \mathrm{d}x^M A_M(x,Z;Y) \ , \qquad A_z \ = \ \mathrm{d}Z^{\underline{\alpha}} A_{\underline{\alpha}}(x,Z;Y) \ ,$$

and a

$$\star \ \underline{\text{twisted-adjoint zero-form}} \ \Phi \ = \ \Phi(x,Z;Y) \ ,$$

where the x^M's are commuting coordinates, while

$$(Y^{\underline{\alpha}}, Z^{\underline{\alpha}}) = (y^{\alpha}, \bar{y}^{\dot{\alpha}}; z^{\alpha}, -\bar{z}^{\dot{\alpha}})$$ are non-commutative.

Minimal bosonic higher-spin gravity :

$$F + \Phi \star J \;=\; 0 \;, \qquad D\,\Phi \;=\; 0 \;, \qquad \mathrm{d}J = 0 \;,$$

$$F \;:=\; \mathrm{d}A + A \star A \;, \qquad D\Phi \;:=\; \mathrm{d}\Phi + [A, \Phi]_\pi \;,$$

$$\tau(A, \Phi) \;=\; (-A, \pi(\Phi)) \;, \qquad (A, \Phi)^\dagger \;=\; (-A, \pi(\Phi)) \;,$$

$$\hookrightarrow [A, J]_\pi = 0 = [\Phi, J]_\pi \;.$$

[The integrability of $F + \Phi \star J = 0$ implies that $D\Phi \star J = 0$, that is, $D\Phi = 0$, where the twisted-adjoint covariant derivative $D\Phi = d\Phi + A \star \Phi - \Phi \star \pi(A)$.

This constraints is integrable since

$D^2\Phi \;=\; F \star \Phi - \Phi \star \pi(F) \;=\; -\Phi \star J \star \Phi + \Phi \star \pi(\Phi) \star J$ gives zero, using the constraint on F and $0 = [\Phi, J]_\pi$ with $\pi(J) = J$.]

↪ Integrability implies invariance under Cartan gauge transformations

$$\delta_\epsilon A = D\epsilon \,, \qquad \delta_\epsilon \Phi = -[\epsilon, \Phi]^\pi_\star \,,$$

for zero-form gauge parameters $\epsilon(x, Z; Y)$ obeying the same kinematic constraints as the master one-form, *i.e.* $\tau(\epsilon) = -\epsilon$ and $(\epsilon)^\dagger = -\epsilon$.

↪ The closure of the gauge transformations reads

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{\epsilon_{12}} \,, \qquad \epsilon_{12} = [\epsilon_1, \epsilon_2]_\star \,,$$

defining the algebra $\mathfrak{hs}(4)$.

2. EXTENSION WITH BIFUNDAMENTAL FIELDS

Vasiliev's equations can be written as

$$F_{[2]} + B_{[0]} \star J_{[2]} = 0 \ , \qquad DB_{[0]} = 0 \ ,$$

where

$$F_{[2]} := \mathrm{d}A_{[1]} + A_{[1]} \star A_{[1]} \ , \ DB_{[0]} := \mathrm{d}B_{[0]} + A_{[1]} \star B_{[0]} - B_{[0]} \star A_{[1]} \ ,$$

$$J_{[2]} \ := \ -\tfrac{i}{4} \left(dz^{\alpha} dz_{\alpha} \, k \, \kappa + d\bar{z}^{\dot{\alpha}} d\bar{z}_{\dot{\alpha}} \, \bar{k} \, \bar{\kappa} \right) \ ,$$

satisfying

$$\mathrm{d}J_{[2]} = 0 \ , \qquad [A_{[1]}, J_{[2]}]_{\star} = 0 \ , \qquad [B_{[0]}, J_{[2]}]_{\star} = 0 \ .$$

- Extension : on $\mathscr{M}_8 = \mathscr{X}_4 \times \mathscr{Z}_4$,

$$A = A_{[1]} + A_{[3]} + A_{[5]} + A_{[7]} \ ,$$
$$B = B_{[0]} + B_{[2]} + B_{[4]} + B_{[6]} + B_{[8]} \ .$$

Does not change the perturbative spectrum around AdS_4 and facilitates deformations of the action [N.B.-P.Sundell (2011)] by HS invariants $Tr[(\Phi\kappa)^n]$, $Tr[(\Phi\kappa)^n\kappa\bar{\kappa}]$ --→ nontrivial on-shell action yielding correct higher spin amplitudes [Colombo-Sundell, Didenko-Skvortsov (2012)] reproducing $\langle J_{s_1}(\vec{x}_1)\dots J_{s_n}(\vec{x}_n)\rangle$ for free $O(N)$ model.

- The deformations of the action : ∞-many in the original model. Arbitrariness in the relative coefficients of the n-point functions ;
- New extension presented here : enlarges the symmetries $\Rightarrow$ restricts the space of higher spin invariants $\Rightarrow$ eliminating all degrees of arbitrariness implied for the tree amplitudes in the original formulation.
- Left to verify : Remaining invariants gives rise to all holographic n-point functions.

BIFUNDAMENTAL EXTENSION ; DYNAMICAL 2-FORM.

Closed and central 2-form $J_{[2]}$, in an otherwise *dynamical* set of fields, suggests $J_{[2]}$ vacuum expectation value of a *dynamical* 2-form $\widetilde{B}_{[2]}$, subject to some differential constraint

$$\widetilde{D}\widetilde{B}_{[2]} = 0 \ ,$$

such that

$$F_{[2]} + B_{[0]} \star \widetilde{B}_{[2]} = 0$$

remains Cartan-integrable.

Cartan-integrability can be achieved by adding a new $\widetilde{A}_{[1]}$.

Finally, Cartan-integrable system with dynamical 2-form :

$$F_{[2]} + B_{[0]} \star \widetilde{B}_{[2]} = 0 \ , \quad \widetilde{F}_{[2]} + \widetilde{B}_{[2]} \star B_{[0]} = 0 \ ,$$

$$DB_{[0]} := \mathrm{d}B_{[0]} + A_{[1]} \star B_{[0]} - B_{[0]} \star \widetilde{A}_{[1]} = 0 \ ,$$

$$\widetilde{D}\widetilde{B}_{[2]} := \mathrm{d}\widetilde{B}_{[2]} + \widetilde{A}_{[1]} \star \widetilde{B}_{[2]} - \widetilde{B}_{[2]} \star A_{[1]} = 0 \ ,$$

$$F_{[2]} := \mathrm{d}A_{[1]} + A_{[1]} \star A_{[1]} \ , \quad \widetilde{F}_{[2]} := \mathrm{d}\widetilde{A}_{[1]} + \widetilde{A}_{[1]} \star \widetilde{A}_{[1]} \ .$$

Without adding new degrees of freedom, extend the system universally. In particular, on $\mathscr{M}_8 = \mathscr{X}_4 \times \mathscr{Z}_4$

$$A = A_{[1]} + A_{[3]} + A_{[5]} + A_{[7]} \,, \quad \widetilde{A} = \widetilde{A}_{[1]} + \widetilde{A}_{[3]} + \widetilde{A}_{[5]} + \widetilde{A}_{[7]} \,,$$

$$B = B_{[0]} + B_{[2]} + B_{[4]} + B_{[6]} + B_{[8]} \,, \quad \widetilde{B} = \widetilde{B}_{[2]} + \widetilde{B}_{[4]} + \widetilde{B}_{[6]} + \widetilde{B}_{[8]}$$

obeying the duality extended version of the BiF system :

$$\mathrm{d}A + A \star A + B \star \widetilde{B} = 0 \,,$$

$$\mathrm{d}\widetilde{A} + \widetilde{A} \star \widetilde{A} + \widetilde{B} \star B = 0 \,,$$

$$\mathrm{d}B + A \star B - B \star \widetilde{A} = 0 \,,$$

$$\mathrm{d}\widetilde{B} + \widetilde{A} \star \widetilde{B} - \widetilde{B} \star A = 0 \,.$$

System invariant under

$$
\begin{aligned}
\delta A &= d\epsilon + [A, \epsilon]_\star - \eta \star \widetilde{B} - B \star \widetilde{\eta} \ , \\
\delta \widetilde{A} &= d\widetilde{\epsilon} + [\widetilde{A}, \widetilde{\epsilon}]_\star - \widetilde{\eta} \star B - \widetilde{B} \star \eta \ , \\
\delta B &= d\eta + A \star \eta + \eta \star \widetilde{A} - \epsilon \star B + B \star \widetilde{\epsilon} \ , \\
\delta \widetilde{B} &= d\widetilde{\eta} + \widetilde{A} \star \widetilde{\eta} + \widetilde{\eta} \star A - \widetilde{\epsilon} \star \widetilde{B} + \widetilde{B} \star \epsilon \ .
\end{aligned}
$$

The resulting field content can be extended into a bulk manifold $\mathscr{B}_9$ with boundary $\mathscr{M}_8 = \partial \mathscr{B}_9$ after which a bulk action of generalized Hamiltonian type can be constructed by introducing dual Lagrange multipliers. Before going to action, relation to Vasiliev's model.

- Defining $W = \frac{1}{2}(A + \widetilde{A})$, $V = \frac{1}{2}(A - \widetilde{A})$, the field equations

$$\begin{aligned}\hookrightarrow \quad \mathrm{d}W + W \star W + V \star V + \tfrac{1}{2}\{B, \widetilde{B}\} &= 0 \,, \\ \mathrm{d}V + \{W, V\}_\star - \tfrac{1}{2}[B, \widetilde{B}]_\star &= 0 \,, \\ \mathrm{d}B + [W, B]_\star - \{V, B\} &= 0 \,, \\ \mathrm{d}\widetilde{B} + [W, \widetilde{B}]_\star + \{V, \widetilde{B}\}_\star &= 0 \,;\end{aligned}$$

- Corresponding gauge transformations

$$\begin{aligned}\delta W &= D\varepsilon + \ldots \,, \qquad & \delta V &= D\beta + \ldots \\ \delta B &= D\eta + \ldots \qquad & \delta\widetilde{B} &= D\widetilde{\eta} + \ldots \,.\end{aligned}$$

- Assume existence of globally-defined, closed 2-form $J_{[2]}$ on base manifold ;
- Set VEV : $(W^{(0)}, V^{(0)}, B^{(0)}, \widetilde{B}^{(0)}) = (\Omega, 0, 0, J)$;
- Expand $\widetilde{B} = J + C$, $W = \sum_{n\geqslant 0} W^{(n)}$, $(B, V, C) = \sum_{n\geqslant 1}(B^{(n)}, V^{(n)}, C^{(n)})$.
- At the first order, $D_0 V^{(1)} = 0$, $D_0 C^{(1)} + 2\, V^{(1)} \star J_{[2]} = 0$;
- gauge transformations $V^{(1)} = D_0 \beta^{(1)}$, $\delta C^{(1)} = D_0 \widetilde{\eta}^{(1)} + 2\beta^{(1)} \star J_{[2]}$;

- $\hookrightarrow$ Set $C^{(1)} = 0 = V^{(1)}$. Continuing, the full $(V, C) \to (0,0)$;
- Original system $\longrightarrow$ Vasiliev's equations

  $\mathrm{d}W_{[1]} + W_{[1]}^2 + B_{[0]} \star J_{[2]} = 0$, $dB_{[0]} + [W_{[1]}, B_{[0]}] = 0$, setting to zero the higher forms in W and B .
- The gauge invariance of the original system has an important consequence : Not any HS invariant of the Vasiliev system descends from an invariant of the original extended system upon gauge fixing ;
- invariants are highly restricted ! given on-shell by

$$\mathscr{I}_{(1)} = \int_{\mathscr{Z}_4} \mathrm{Tr}_{\mathfrak{h}} \left(B_{[0]} \star J_{[2]} \star B_{[0]} \star J_{[2]} \right) ,$$

$$\left[\text{Also} \quad \mathscr{I}_{(2)} = \int_{\mathscr{Z}_4} \mathrm{Tr}_{\mathfrak{h}} \left(B_{[0]} \widetilde{B}_{[4]} + B_{[2]} \widetilde{B}_{[2]} \right) \right]$$

3. UNDERLYING FROBENIUS ALGEBRA ; ACTION

Introduce

$$e_{ij}e_{kl} = \delta_{jk}e_{il} \ , \qquad i = 1, 2 \ ,$$

and combine

$$\mathscr{A} \ := \ A\, e_{11} + \widetilde{A}\, e_{22} \ , \quad \mathscr{B} \ := \ B\, e_{12} - \widetilde{B}\, e_{21} \ ,$$

$$\Lambda \ := \ \epsilon\, e_{11} + \widetilde{\epsilon}\, e_{22} \ , \qquad \Sigma \ := \ \eta\, e_{12} - \widetilde{\eta}\, e_{21} \ .$$

Then, the extended system :

$$\mathrm{d}\mathscr{A} + \mathscr{A}^2 - \mathscr{B}^2 = 0 \ , \qquad \mathrm{d}\mathscr{B} + \mathscr{A}\mathscr{B} - \mathscr{B}\mathscr{A} = 0 \ ,$$

$$\delta\mathscr{A} = \mathrm{d}\Lambda + [\mathscr{A}, \Lambda] + \{\Sigma, \mathscr{B}\} \ , \qquad \delta\mathscr{B} = \mathrm{d}\Sigma + \{\mathscr{A}, \Sigma\} - [\Lambda, \mathscr{B}] \ .$$

Assembling the fields into a single master field

$$\Psi := \mathscr{A} + \mathscr{B} \,, \qquad \theta := \Lambda + \Sigma \,,$$

the equations of motion, gauge transformations, generalized Bianchi identities and curvature gauge covariance :

$$R := \mathrm{d}\Psi + \bar{\Psi}\Psi = 0 \,, \qquad \delta_\theta \Psi = \mathrm{d}\theta + \Psi\theta - \bar{\theta}\Psi \,,$$

$$\mathrm{d}R + \Psi R - \bar{R}\Psi \equiv 0 \,, \qquad \delta R_\theta = [R, \theta] \,,$$

where $\bar{\Psi} := \mathscr{A} - \mathscr{B}$, $\bar{\theta} := \Lambda - \Sigma$ and $\bar{R} := \mathrm{d}\bar{\Psi} + \Psi\bar{\Psi}$.

Introducing extra Kleinian h with

$$h^2 = 1 \;, \qquad [h, e_{11}] = 0 = [h, e_{22}] \;, \qquad \{h, e_{12}\} = 0 = \{h, e_{21}\} \;,$$

and defining

$$\boldsymbol{\Psi} := h\Psi \;, \qquad \boldsymbol{\theta} := \theta \;, \qquad \boldsymbol{q} := h\mathrm{d}$$

it follows from $\bar{\Psi} = h\Psi h$ that

$$\boldsymbol{R} := \boldsymbol{q}\boldsymbol{\Psi} + \boldsymbol{\Psi}^2 = 0 \;, \qquad \delta_{\boldsymbol{\theta}}\boldsymbol{\Psi} = \boldsymbol{q}\boldsymbol{\theta} + [\boldsymbol{\Psi}, \boldsymbol{\theta}] \;,$$
$$\boldsymbol{q}\boldsymbol{R} + [\boldsymbol{\Psi}, \boldsymbol{R}] \equiv 0 \;, \quad \delta_{\boldsymbol{\theta}}\boldsymbol{R} = [\boldsymbol{R}, \boldsymbol{\theta}] \;.$$

- A one-parameter family of cubic bulk actions $S^{\rm cubic}_{\rm bulk} = \int_{\mathscr{M}} L^{\rm cubic}_{\rm bulk}$ with non-trivial Poisson three-vector :

$$L^{\rm cubic}_{\rm bulk} = Tr_{\mathscr{A}} \left(\begin{array}{l} \widetilde{U} D B + V[F + B\widetilde{B} + \alpha(U\widetilde{U} - \frac{1}{3}V^2)] \\ + U\widetilde{D}\widetilde{B} + \widetilde{V}[\widetilde{F} + \widetilde{B}B + \alpha(\widetilde{U}U - \frac{1}{3}\widetilde{V}^2)] \end{array} \right) ;$$

- $U = U_{[2]} + \ldots + U_{[8]}$ conjugated to B, $V = V_{[1]} + \ldots + V_{[7]}$ conjugated to A, *idem* $(\widetilde{U}, \widetilde{B})$ and $(\widetilde{V}, \widetilde{A})$;
- Cartan transformations send the Lagrangian to a total derivative, hence $(U, V, \widetilde{V}, \widetilde{U}; \eta^U, \eta^V, \eta^{\tilde{V}}, \eta^{\tilde{U}})|_{\partial\mathscr{M}} = (0,0,0,0;0,0,0,0)$, while $(B, A, \widetilde{A}, \widetilde{B}; \epsilon^B, \epsilon^A, \epsilon^{\tilde{A}}, \epsilon^{\tilde{B}})$ can be defined locally and left free to fluctuate on $\partial\mathscr{M}$.

- Upon introducing the momenta-like variables inside $\boldsymbol{\Psi}$,

$$\boldsymbol{A} := \sum_I A^I e_I \ , \qquad \boldsymbol{B} := \sum_P B^P f_P \ ,$$

  the action takes the form

$$\begin{aligned} S^{\rm cubic}_{\rm bulk}[Z] &= \int_{\mathscr{M}} Tr_{\mathscr{A}} {\rm Tr}_{\mathscr{F}} \left[\tfrac{1}{2}\, Z{\rm q}Z + \tfrac{g}{3}\, Z^3\right] \\ & \quad -\tfrac{1}{4} \oint_{\partial\mathscr{M}} Tr_{\mathscr{A}} {\rm STr}_{\mathscr{F}} \left[\pi_h(Z)Z\right] \ , \end{aligned}$$

- $\mathscr{F}$ is an internal Frobenius algebra [an unital associative algebra with a non-degenerate invariant bilinear form].

4. CONCLUSIONS

I) By imposing boundary conditions by hand, Vasiliev's higher-spin gravity is embedded into a larger theory without altering the perturbative spectrum ;

II) What used to be interaction ambiguities are now 2-form moduli ;

III) Dramatic reduction in the number of higher-spin invariants, as the new model has more gauge symmetries inside the bulk ;

IV) New master theory free of parameters. Direct contact to Frobenius algebras, 2D TFT ... and OSFT ?