Games where you can play optimally with finite memory

Patricia Bouyer\textsuperscript{1}  Stéphane Le Roux\textsuperscript{1}  Youssouf Oualhadj\textsuperscript{2}  Mickael Randour\textsuperscript{3}  Pierre Vandenhove\textsuperscript{3}

\textsuperscript{1}LSV – CNRS & ENS Paris-Saclay  \textsuperscript{2}LACL – UPEC  \textsuperscript{3}F.R.S.-FNRS & UMONS – Université de Mons

October 10, 2019

GT ALGA annual meeting 2019
Games where you can play optimally with finite memory

Patricia Bouyer\textsuperscript{1}  Stéphane Le Roux\textsuperscript{1}  Youssouf Oualhadj\textsuperscript{2}  Mickael Randour\textsuperscript{3}  Pierre Vandenhove\textsuperscript{3}

\textsuperscript{1}LSV – CNRS & ENS Paris-Saclay  \textsuperscript{2}LACL – UPEC  \textsuperscript{3}F.R.S.-FNRS & UMONS – Université de Mons

October 10, 2019

GT ALGA annual meeting 2019
Games where you can play optimally with finite memory

A sequel to the critically acclaimed blockbuster by Gimbert & Zielonka

Patricia Bouyer\textsuperscript{1}  Stéphane Le Roux\textsuperscript{1}  Youssouf Oualhadj\textsuperscript{2}  Mickael Randour\textsuperscript{3}  Pierre Vandenhove\textsuperscript{3}

\textsuperscript{1}LSV – CNRS & ENS Paris-Saclay  \textsuperscript{2}LACL – UPEC  \textsuperscript{3}F.R.S.-FNRS & UMONS – Université de Mons

October 10, 2019

GT ALGA annual meeting 2019

Games where you can play optimally without any memory *

Hugo Gimbert and Wieslaw Zielonka

Université Paris 7 and CNRS, LIAFA, case 7014
2, place Jussieu
75251 Paris Cedex 05, France
\{hugo,zielonka\}@liafa.jussieu.fr

Abstract. Reactive systems are often modelled as two person antagonistic games where one player represents the system while his adversary represents the environment. Undoubtedly, the most popular games in this setting are parity games and their cousins (Rabin, Streett and Muller games). We also games with other types of payments, like safety games. In this paper we investigate strategies for parity games with infinite memory. This questions has previously used only in economic theory, which is our main source of motivation.
The talk in one slide

Strategy synthesis for two-player turn-based games

Finding good controllers for systems interacting with an antagonistic environment.
The talk in one slide

Strategy synthesis for two-player turn-based games

Finding good controllers for systems interacting with an antagonistic environment.

▷ Good? Performance evaluated through objectives / payoffs.
The talk in one slide

Strategy synthesis for two-player turn-based games
Finding **good** controllers for systems interacting with an **antagonistic** environment.

▶ Good? Performance evaluated through *objectives* / *payoffs*.

**Question**
When are *simple* strategies sufficient to play optimally?
The talk in one slide

Strategy synthesis for two-player turn-based games
Finding good controllers for systems interacting with an antagonistic environment.

▷ Good? Performance evaluated through objectives / payoffs.

Question
When are simple strategies sufficient to play optimally?

Two directions for finite-memory determinacy:
The talk in one slide

Strategy synthesis for two-player turn-based games

Finding good controllers for systems interacting with an antagonistic environment.

▷ Good? Performance evaluated through objectives / payoffs.

Question

When are simple strategies sufficient to play optimally?

Two directions for finite-memory determinacy:

1. lifting under objective combination (with S. Le Roux and A. Pauly, in FSTTCS’18 [LPR18]),
The talk in one slide

**Strategy synthesis for two-player turn-based games**

Finding **good** controllers for systems interacting with an **antagonistic** environment.

▶ Good? Performance evaluated through objectives / payoffs.

**Question**

When are **simple** strategies sufficient to play optimally?

Two directions for **finite-memory determinacy**:

1. lifting under **objective combination** (with S. Le Roux and A. Pauly, in FSTTCS’18 [LPR18]),
2. complete characterization and **lifting from one-player games** (ongoing work).
1. Memoryless determinacy
2. Finite-memory determinacy and Boolean combinations
3. Characterization and lifting corollary
4. Conclusion
<table>
<thead>
<tr>
<th></th>
<th>Memoryless determinacy</th>
<th>Finite-memory determinacy and Boolean combinations</th>
<th>Characterization and lifting corollary</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Memoryless determinacy</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Finite-memory determinacy and Boolean combinations</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Characterization and lifting corollary</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Conclusion</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Games where you can play optimally with finite memory
Two-player turn-based zero-sum games on graphs
Two-player turn-based zero-sum games on graphs

We consider finite arenas with vertex colors in $C$. Two players: circle ($P_1$) and square ($P_2$). Strategies $C^* \times V_i \rightarrow V$.

A winning condition is a set $W \subseteq C^\omega$. 
Two-player turn-based zero-sum games on graphs

We consider *finite* arenas with vertex *colors* in $C$. Two players: circle ($P_1$) and square ($P_2$). Strategies $C^* \times V_i \to V$.

- A **winning condition** is a set $W \subseteq C^\omega$.

From where can $P_1$ ensure to reach $v_6$?

**How complex is his strategy?**
Two-player turn-based zero-sum games on graphs

We consider finite arenas with vertex colors in $C$. Two players: circle ($\mathcal{P}_1$) and square ($\mathcal{P}_2$). Strategies $C^* \times V_i \to V$.

- A winning condition is a set $W \subseteq C^\omega$.

From where can $\mathcal{P}_1$ ensure to reach $v_6$?
How complex is his strategy?

Memoryless strategies ($V_i \to V$) always suffice for reachability (for both players).
When are memoryless strategies sufficient to play optimally?
When are memoryless strategies sufficient to play optimally?

Virtually always for simple winning conditions!

Examples: reachability, safety, Büchi, parity, mean-payoff, energy, total-payoff, average-energy, etc.
When are memoryless strategies sufficient to play optimally?

Virtually always for *simple* winning conditions!

Examples: reachability, safety, Büchi, parity, mean-payoff, energy, total-payoff, average-energy, etc.

Can we characterize when they are?
When are memoryless strategies sufficient to play optimally?

Virtually always for simple winning conditions!

Examples: reachability, safety, Büchi, parity, mean-payoff, energy, total-payoff, average-energy, etc.

Can we characterize when they are?

Yes, thanks to Gimbert and Zielonka [GZ05].
Gimbert and Zielonka’s characterization

Memoryless strategies suffice for a preference relation $\leq$ (and the induced winning conditions) if and only if

1. it is monotone,

2. it is selective.
Gimbert and Zielonka’s characterization

Memoryless strategies suffice for a *preference relation* \( \sqsubseteq \) (and the induced winning conditions) if and only if

1. it is **monotone**,
   - Intuitively, stable under prefix addition.

2. it is **selective**.
Gimbert and Zielonka’s characterization

Memoryless strategies suffice for a preference relation $\sqsubseteq$ (and the induced winning conditions) if and only if

1. it is monotone,\[\Rightarrow\]Intuitively, stable under prefix addition.

2. it is selective.\[\Rightarrow\]Intuitively, stable under cycle mixing.
Gimbert and Zielonka’s characterization

Memoryless strategies suffice for a *preference relation* \( \sqsubseteq \) (and the induced winning conditions) **if and only if**

1. it is **monotone**,
   - Intuitively, stable under prefix addition.

2. it is **selective**.
   - Intuitively, stable under cycle mixing.

Example: reachability.
Gimbert and Zielonka’s corollary

If $\square$ is such that
Gimbert and Zielonka’s corollary

If $\subseteq$ is such that

- in all $\mathcal{P}_1$-arenas, $\mathcal{P}_1$ has an optimal memoryless strategy,
Gimbert and Zielonka’s corollary

If \( \sqsubseteq \) is such that

- in all \( \mathcal{P}_1 \)-arenas, \( \mathcal{P}_1 \) has an optimal memoryless strategy,
- in all \( \mathcal{P}_2 \)-arenas, \( \mathcal{P}_2 \) has an optimal memoryless strategy (i.e., for \( \sqsubseteq^{-1} \)),

Games where you can play optimally with finite memory
Gimbert and Zielonka’s corollary

If $\sqsubseteq$ is such that

- in all $P_1$-arenas, $P_1$ has an optimal memoryless strategy,
- in all $P_2$-arenas, $P_2$ has an optimal memoryless strategy (i.e., for $\sqsubseteq^{-1}$),

then both players have optimal memoryless strategies in all two-player arenas.

*Extremely useful in practice!*
Going further: finite memory

Memoryless strategies do not always suffice!
Going further: finite memory

Memoryless strategies do not always suffice!

Examples:

- Büchi for $v_1$ and $v_3 \rightarrow$ finite (1 bit) memory.
Going further: finite memory

Memoryless strategies do not always suffice!

Examples:
- Büchi for $v_1$ and $v_3 \rightarrow$ finite (1 bit) memory.
- Mean-payoff (average weight per transition) $\geq 0$ on all dimensions $\rightarrow$ infinite memory!
Going further: finite memory

Memoryless strategies do not always suffice!

Examples:

- Büchi for $v_1$ and $v_3 \rightarrow$ finite (1 bit) memory.
- Mean-payoff (average weight per transition) $\geq 0$ on all dimensions $\rightarrow$ infinite memory!

Two directions:

1. single-objective $\sim$ multi-objective [LPR18],
2. GZ-like characterization and one-player $\sim$ two-player.
1 Memoryless determinacy

2 Finite-memory determinacy and Boolean combinations

3 Characterization and lifting corollary

4 Conclusion

Games where you can play optimally with finite memory

Mickael Randour
Combining winning conditions

Our goal

We want a *general* and *abstract* theorem guaranteeing the sufficiency of *finite-memory strategies*\(^a\) in games with *Boolean combinations of objectives* provided that the underlying *simple objectives* fulfill some criteria.

\(^a\)Implementable via a finite-state machine.
Combining winning conditions

Our goal

We want a *general* and *abstract* theorem guaranteeing the sufficiency of *finite-memory strategies*\(^a\) in games with *Boolean combinations of objectives* provided that the underlying *simple objectives* fulfill some criteria.

\(^a\)Implementable via a finite-state machine.

**Advantages:**

▷ study of core features ensuring finite-memory determinacy,
▷ works for almost all existing settings and many more to come.
Combining winning conditions

Our goal

We want a *general* and *abstract* theorem guaranteeing the sufficiency of finite-memory strategies\(^a\) in games with *Boolean combinations of objectives* provided that the underlying *simple objectives* fulfill some criteria.

\(^a\)Implementable via a finite-state machine.

Advantages:

- study of core features ensuring finite-memory determinacy,
- works for almost all existing settings and many more to come.

Drawbacks:

- concrete memory bounds are huge (as they depend on the most general upper bound).
- sufficient criterion, not full characterization.
The building blocks

The full approach is technically involved but can be sketched intuitively.
The building blocks

The full approach is technically involved but can be sketched intuitively.

Criterion outline

Any *well-behaved* winning condition combined with conditions traceable by finite-state machines (i.e., *safety-like* conditions) preserves finite-memory determinacy.
The building blocks

The full approach is technically involved but can be sketched intuitively.

Criterion outline

Any well-behaved winning condition combined with conditions traceable by finite-state machines (i.e., safety-like conditions) preserves finite-memory determinacy.

To state this theorem formally, we need three ingredients:
The building blocks

The full approach is technically involved but can be sketched intuitively.

Criterion outline

Any well-behaved winning condition combined with conditions traceable by finite-state machines (i.e., safety-like conditions) preserves finite-memory determinacy.

To state this theorem formally, we need three ingredients:

1. regularly-predictable winning conditions,
The building blocks

The full approach is technically involved but can be sketched intuitively.

Criterion outline

Any well-behaved winning condition combined with conditions traceable by finite-state machines (i.e., safety-like conditions) preserves finite-memory determinacy.

To state this theorem formally, we need three ingredients:

1. regularly-predictable winning conditions,
2. regular languages,
The building blocks

The full approach is technically involved but can be sketched intuitively.

Criterion outline

Any *well-behaved* winning condition combined with conditions traceable by finite-state machines (i.e., *safety-like* conditions) preserves finite-memory determinacy.

To state this theorem formally, we need three ingredients:

1. *regularly-predictable* winning conditions,
2. *regular* languages,
3. *hypothetical* subgame-perfect equilibria (hSPE).
The building blocks

The full approach is technically involved but can be sketched intuitively.

Criterion outline

Any *well-behaved* winning condition combined with conditions traceable by finite-state machines (i.e., *safety-like* conditions) preserves finite-memory determinacy.

To state this theorem formally, we need three ingredients:

1. *regularly-predictable* winning conditions,
2. *regular* languages,
3. *hypothetical* subgame-perfect equilibria (hSPE).

We match the FM-determinacy frontier almost exactly!
The building blocks

The full approach is technically involved but can be sketched intuitively.

Criterion outline

Any well-behaved winning condition combined with conditions traceable by finite-state machines (i.e., safety-like conditions) preserves finite-memory determinacy.

To state this theorem formally, we need three ingredients:

1. regularly-predictable winning conditions,
2. regular languages,
3. hypothetical subgame-perfect equilibria (hSPE).

We match the FM-determinacy frontier almost exactly!

⇒ Only one exception AFAWK (hSPE vs. opt. strategies).
Comments

- Combining similar simple objectives leads to contrasting behaviors: difficult to extract the core features leading to FM determinacy.
Comments

- Combining similar simple objectives leads to contrasting behaviors: difficult to extract the core features leading to FM determinacy.
- Our main result is a sufficient criterion, not a full characterization.
Comments

- Combining similar simple objectives leads to contrasting behaviors: difficult to extract the core features leading to FM determinacy.
- Our main result is a **sufficient criterion**, not a full characterization.
  - In practice, it does cover everything except *average-energy with a lower-bounded energy condition* – a very strange corner case.
Comments

■ Combining similar simple objectives leads to contrasting behaviors: difficult to extract the core features leading to FM determinacy.

■ Our main result is a sufficient criterion, not a full characterization.
  ▶ In practice, it does cover everything except average-energy with a lower-bounded energy condition – a very strange corner case.
  ▶ Any weakening of our hypotheses almost immediately leads to falsification.
Comments

- Combining similar simple objectives leads to contrasting behaviors: difficult to extract the core features leading to FM determinacy.
- Our main result is a sufficient criterion, not a full characterization.
  - In practice, it does cover everything except average-energy with a lower-bounded energy condition — a very strange corner case.
  - Any weakening of our hypotheses almost immediately leads to falsification.
  - We also have several more precise results (e.g., much lower bounds) for specific combinations and/or restrictive hypotheses.
Comments

- Combining similar simple objectives leads to contrasting behaviors: difficult to extract the core features leading to FM determinacy.
- Our main result is a **sufficient criterion**, not a full characterization.
  - In practice, it does cover everything except *average-energy with a lower-bounded energy condition* — a very strange corner case.
  - Any weakening of our hypotheses almost immediately leads to falsification.
  - We also have several **more precise results** (e.g., much lower bounds) for specific combinations and/or restrictive hypotheses.

Almost complete picture of the frontiers of FM determinacy for *combinations of objectives* but still **not a complete characterization à la Gimbert and Zielonka.**
1. Memoryless determinacy

2. Finite-memory determinacy and Boolean combinations

3. Characterization and lifting corollary

4. Conclusion

Games where you can play optimally with finite memory

With P. Bouyer, S. Le Roux, Y. Oualhaadj & P. Vandenbrouck.

*ongoing work*
Reminder: memoryless determinacy
Reminder: memoryless determinacy

1. **Complete characterization** using
   - monotony,
   - selectivity.
Reminder: memoryless determinacy

1. **Complete characterization** using
   - monotony,
   - selectivity.

2. **Lifting corollary**: extremely useful in practice!
Reminder: memoryless determinacy

1. **Complete characterization** using
   - monotony,
   - selectivity.

2. **Lifting corollary**: extremely useful in practice!

**Our dream**: exact equivalent in the finite-memory case.
A partial counter-example (lifting corollary)

Let $C \subseteq \mathbb{Z}$ and the winning condition for $\mathcal{P}_1$ be

$$\overline{TP}(\pi) = \infty \lor \exists \infty i \in \mathbb{N}, \sum_{i=0}^{n} c_i = 0$$
A partial counter-example (lifting corollary)

Let $C \subseteq \mathbb{Z}$ and the winning condition for $P_1$ be

$$\overline{TP}(\pi) = \infty \lor \exists \infty i \in \mathbb{N}, \sum_{i=0}^{n} c_i = 0$$

Both 1-player variants are finite-memory determined.
A partial counter-example (lifting corollary)

Let $C \subseteq \mathbb{Z}$ and the winning condition for $\mathcal{P}_1$ be

$$\overline{TP}(\pi) = \infty \lor \exists \infty i \in \mathbb{N}, \sum_{i=0}^{n} c_i = 0$$

Both 1-player variants are finite-memory determined.

But the two-player one is not!

$\Rightarrow$ $\mathcal{P}_1$ needs infinite memory to win.
A partial counter-example (lifting corollary)

Let $C \subseteq \mathbb{Z}$ and the winning condition for $P_1$ be

$$TP(\pi) = \infty \lor \exists i \in \mathbb{N}, \sum_{i=0}^{n} c_i = 0$$

Both 1-player variants are finite-memory determined.

But the two-player one is not!

$\implies P_1$ needs infinite memory to win.

*Hint*: non-monotony is a bigger threat in two-player games. In one-player games, *finite* memory may help.
A new hope

Our goal

GZ-like characterization for finite-memory strategies.

Two tricks:
A new hope

Our goal

GZ-like characterization for finite-memory strategies.

Two tricks:

1. **Monotony as hypothesis** (cf. counter-example).
A new hope

Our goal

GZ-like characterization for finite-memory strategies.

Two tricks:

1. **Monotony as hypothesis** (cf. counter-example).
2. From selectivity to *S*-selectivity and cyclic covers for arenas.
   \[\implies\text{Intuitively, selectivity } modulo \text{ a memory skeleton.}\]
A new hope

Our goal

GZ-like characterization for finite-memory strategies.

Two tricks:

1. **Monotony as hypothesis** (cf. counter-example).
2. From selectivity to *S*-selectivity and cyclic covers for arenas.

\[ \Rightarrow \quad \text{Intuitively, selectivity } \text{modulo a memory skeleton.} \]

We obtain a natural GZ-equivalent for FM determinacy, including the lifting corollary (1-p. to 2-p.)!

*Still some elements to flesh out.*

\[ \Rightarrow \quad \text{Preprint writing in progress.} \]
1 Memoryless determinacy

2 Finite-memory determinacy and Boolean combinations

3 Characterization and lifting corollary

4 Conclusion
Conclusion

Our goal

Understand and characterize the frontiers of FM-determinacy.
Conclusion

Our goal

Understand and characterize the frontiers of FM-determinacy.

Two directions

Combinations of objectives

- Matches our current knowledge almost-exactly.
- Useful when the underlying obj. are well-understood.
Conclusion

Our goal

Understand and characterize the frontiers of FM-determinacy.

Two directions

Combinations of objectives

➤ Matches our current knowledge almost-exactly.
➤ Useful when the underlying obj. are well-understood.
➤ With Le Roux and Pauly [LPR18] (on arXiv).

GZ-like criterion

➤ No exact equivalent.
➤ Natural criterion and useful lifting corollary.
➤ With Bouyer, Le Roux, Oualhadj and Vandenhove, ongoing work.
Thank you! Any question?
References I

Infinite runs in weighted timed automata with energy constraints.

Patricia Bouyer, Piotr Hofman, Nicolas Markey, Mickael Randour, and Martin Zimmermann.
Bounding average-energy games.

Véronique Bruyère, Quentin Hautem, and Mickael Randour.
Window parity games: an alternative approach toward parity games with time bounds.

Véronique Bruyère, Quentin Hautem, and Jean-François Raskin.
On the complexity of heterogeneous multidimensional games.

Average-energy games.

Games where you can play optimally with finite memory

Mickael Randour
References II

Krishnendu Chatterjee and Laurent Doyen.
Energy parity games.

Krishnendu Chatterjee, Laurent Doyen, Mickael Randour, and Jean-François Raskin.
Looking at mean-payoff and total-payoff through windows.

Krishnendu Chatterjee, Thomas A. Henzinger, and Marcin Jurdzinski.
Mean-payoff parity games.

Krishnendu Chatterjee, Thomas A. Henzinger, and Nir Piterman.
Generalized parity games.

Krishnendu Chatterjee, Mickael Randour, and Jean-François Raskin.
Strategy synthesis for multi-dimensional quantitative objectives.
Hugo Gimbert and Wieslaw Zielonka.  
Games where you can play optimally without any memory.  

Marcin Jurdzinski, Ranko Lazic, and Sylvain Schmitz.  
Fixed-dimensional energy games are in pseudo-polynomial time.  

Stéphane Le Roux, Arno Pauly, and Mickael Randour.  
Extending finite-memory determinacy by Boolean combination of winning conditions.  

Yaron Velner, Krishnendu Chatterjee, Laurent Doyen, Thomas A. Henzinger, Alexander Moshe Rabinovich, and Jean-François Raskin.  
The complexity of multi-mean-payoff and multi-energy games.  
References IV

Yaron Velner.
Robust multidimensional mean-payoff games are undecidable.