Kaluza-Klein towers for spinors in flat space

Fernand Grard[1], Jean Nuyts[2]

Abstract

Considering a massive or massless free spinor field propagating in a flat five dimensional space with its fifth dimension compactified either on a strip or on a circle, we analyse the procedure of generation of the four dimensional Kaluza-Klein spinor mass towers. Requiring the five dimensional Dirac operator to be symmetric, the set of all the allowed boundary conditions is obtained. In the determination of the boundary conditions and in the Kaluza-Klein reduction equations, the SO(3,1) and parity invariances in the space-time subspace are carefully taken into account. The equations determining the mass towers are written in full generality. A few numerical examples are given.

[1] *Fernand.Grard@umh.ac.be, Physique Générale et Physique des Particules Elémentaires, Université de Mons-Hainaut, 20 Place du Parc, 7000 Mons, Belgium*

[2] *Jean.Nuyts@umh.ac.be, Physique Théorique et Mathématique, Université de Mons-Hainaut, 20 Place du Parc, 7000 Mons, Belgium*

1 Introduction

In recent articles, we have reanalysed in a mathematically complete and fully consistent way, the generation of Kaluza-Klein mass towers [1] in five dimensional theories with a compactified fifth dimension. This study was carried out for a scalar field supposed to propagate in the bulk, at first in a flat space [2] and then in a warped space without metric singularities [3] and in a warped space with metric singularities [4]. The mathematical approach relies heavily on a precise study of the hermiticity properties of the Kaluza-Klein reduction equations which are of second order in derivatives.

This has allowed us to classify all the sets of allowed boundary conditions. These sets include, as particular cases, the usual box and periodic or antiperiodic boundary conditions which are currently invoked. We found, as a main result, that the Kaluza-Klein mass states may form non-regular towers which depend on the specific set of boundary conditions considered for the various possible metric configurations.

As the future high energy colliders will look for the possible appearance of Kaluza-Klein mass towers as evidence for the existence of fields propagating in higher dimensions and that these towers, if they exist, may well be composed of spinor states, we were led to extend our work to the Dirac fields. At first sight, this problem appeared simpler as the Dirac equation is of first order only in derivatives. However, the presence of multicomponents spinors offers new perspectives and hence increases the complexity of the solutions.

In this article, we restrict ourselves, as a first step in a more general approach, to a five-dimensional flat space. This leads to a convenient toy model where the degrees of freedom of a free Dirac field propagating in the bulk already show up and play a major role.

The article is organized as follows. In Section (2), we recall a few properties of the Dirac equation in five dimensions, putting forward the specific characteristics that are needed to construct and classify the Kaluza-Klein towers. In Section (3), taking into account the underlying invariances and symmetries, in particular covariance and parity invariance in the four dimensional space time, the Kaluza-Klein reduction equations together with the set of all allowed boundary conditions are established. The resulting mass equations, from which the Kaluza-Klein mass towers are built, are given in Section (4). In Section (5), some numerical examples are presented and discussed.

Our approach will be extended in a forthcoming article to a five dimensional warped space, which is known to provide a natural and elegant solution to the hierarchy problem [6] as an alternative to the solution based on large extra dimensions and gravity considerations [5]. Its peculiar characteristics will allow, in particular, the generation of Kaluza-Klein towers with an expected more realistic physical content.

2 Dirac equation in a five-dimensional flat space

We consider a free spinor field with mass M satisfying the five dimensional Dirac equation

$$(i\gamma^a\partial_a - M)\,\Psi = 0\ . \tag{1}$$

The field is supposed to propagate in the bulk, a flat five dimensional space with coordinates $x^a,\ a \equiv \{\mu, 5\} \equiv \{\mu, s\},\ \mu = 0, 1, 2, 3$ and a metric

$$\mathrm{diag}(\eta_{_{ab}}) = \{+1, -1, -1, -1, -1\} \tag{2}$$

giving rise to an invariance group SO(4,1) (related for the spinor representation to the symplectic group Sp(4)). The fifth dimension s is compactified either on a strip or on a circle ($0 \leq s \leq 2\pi R$).

The five Dirac matrices in this space satisfy

$$\left[\gamma^a, \gamma^b\right]_+ = 2\eta^{ab}\ . \tag{3}$$

There are two inequivalent sets of γ^a matrices which can be built from the four usual 4×4 Dirac matrices γ^μ and

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 \quad , \quad \left(\gamma^5\right)^2 = -1\ . \tag{4}$$

They are

$$\gamma^{[I]a} \equiv \{\gamma^\mu, \gamma^5\} \tag{5}$$

or

$$\gamma^{[II]a} \equiv \{-\gamma^\mu, -\gamma^5\}\ . \tag{6}$$

Contrary to the four dimensional case, there is no transformation mapping one set to the other

$$\left\{\ \nexists\ P\ |\ \ \gamma^{[II]a} = P\gamma^{[I]a}P^{-1}\right\}\ . \tag{7}$$

If there is another set of 4×4 γ^a matrices satisfying (3), this set is equivalent through a change of basis, either to the set $\gamma^{[I]a}$ or to the set $\gamma^{[II]a}$. In particular, the sets $(\gamma^a)^+$ and $(\gamma^a)^t$ satisfy (3) and are equivalent to the set $\gamma^{[I]a}$

$$(\gamma^a)^+ = A\gamma^a A^{-1} \quad , \quad A = \gamma^0 \tag{8}$$
$$(\gamma^a)^t = D\gamma^a D^{-1} \quad , \quad D = -D^t = C\gamma^5 \tag{9}$$

where C is the usual four dimensional charge conjugation matrix satisfying $C\gamma^\mu C^{-1} = -(\gamma^\mu)^t$ and where the antisymmetric matrix D is related to the symplectic metric of Sp(4).

It should be remarked, taking into account (7), that M and $-M$ correspond to distinct fields. The covariance, under the covering group of SO(4,1), of the Dirac equation in a five dimensional space follows exactly the same pattern as the four dimensional one. In particular, the infinitesimal generators of the spinor transformations $\psi'(x') = S\psi(x)$ are $\sigma^{ab} = \frac{i}{4}[\gamma^a, \gamma^b]_-$ and are identical for the two γ representations (5), (6).

For spinor fields, one uses the natural invariant hermitian scalar product between two spinors ϕ and ψ (with as usual $\overline{\phi} = \phi^+\gamma^0$)

$$(\phi, \psi)_{\text{hermitian}} = \int_{x=-\infty}^{+\infty} \int_{s=0}^{2\pi R} \overline{\phi}\, \psi \, d^4x \, ds \tag{10}$$

and not the invariant symplectic scalar product

$$(\phi, \psi)_{\text{symplectic}} = \int_{x=-\infty}^{+\infty} \int_{s=0}^{2\pi R} \phi^t \, D \, \psi \, d^4x \, ds \ . \tag{11}$$

The Dirac operator $\mathcal{D} = i\gamma^a \partial_a$ is symmetric for all ϕ, ψ in its domain

$$(\phi, \mathcal{D}\phi)_{\text{hermitian}} = (\mathcal{D}\phi, \phi)_{\text{hermitian}} \ , \tag{12}$$

provided the following boundary relation is satisfied

$$\left[\int_{-\infty}^{+\infty} \overline{\phi}\gamma^5\psi \, d^4x\right]_{s=2\pi R} = \left[\int_{-\infty}^{+\infty} \overline{\phi}\gamma^5\psi \, d^4x\right]_{s=0} \ . \tag{13}$$

In Appendix (A), we justify with heuristic arguments the more restrictive condition which we will impose

$$\left[\overline{\phi}\gamma^5\psi\right]_{s=2\pi R} = \left[\overline{\phi}\gamma^5\psi\right]_{s=0} \tag{14}$$

for all values of x^μ. This implies the existence of at least four linear equations among the components of the fields evaluated at $s = 2\pi R$ and $s = 0$. These boundary conditions should respect the SO(3,1) invariance in the four dimensional subpace x^μ. Hence we postulate for the boundary conditions the general form $[\psi]_{s=2\pi R} = (c_1\, \mathbb{1}_4 + c_2\gamma^5)\, [\psi]_{s=0}$ with two complex constants c_1 and c_2 ($\mathbb{1}_4$ is the unit matrix in spinor space). Introducing this form in the restriction (14), one finds that the coefficients c_1 and c_2 are expressible in terms of a real parameter ω with infinite extend and a phase angle $-\pi \le \rho < \pi$

$$\Big[\psi\Big]_{s=2\pi R} = e^{i\rho}\left(\cosh(\omega)\mathbb{1}_4 + i\sinh(\omega)\gamma^5\right)\Big[\psi\Big]_{s=0}\ . \qquad (15)$$

This is the natural set of boundary conditions in s valid for all x^μ within the hypothesis of SO(3,1) covariance. As will be seen later, this form of the boundary conditions does not imply violation of parity in any four-dimensional brane. This is due to the fact that the $\gamma^5\partial_5$ part in the Dirac equation induces a subtle natural connection between $\psi(x^\mu, s)$ and its derivative multiplied by γ^5.

3 Kaluza-Klein reduction for the Dirac equation

The Kaluza-Klein reduction of the spinor $\psi_\alpha(x^\mu, s)$ is carried out assuming the separation of variables

$$\psi_\alpha(x^\mu, s) = \sum_n \Big(F^{[n]}(s)\mathbb{1}_4 + iG^{[n]}(s)\gamma^5\Big)\psi_\alpha^{[n]}(x^\mu) \qquad (16)$$

with, for each n, two complex scalar functions of s, $F^{[n]}(s)$ and $G^{[n]}(s)$, and a spinor $\psi_\alpha^{[n]}(x^\mu)$ depending on x^μ. This form requires SO(3,1) covariance only which allows the presence of the γ^5 term. The i has been put for convenience.

Introducing this Kaluza-Klein ansatz in the Dirac equation (1), one obtains

$$\begin{aligned}&\Big(F^{[n]}\mathbb{1} - iG^{[n]}\gamma^5\Big)\Big(i\gamma^\mu\partial_\mu\psi_\alpha^{[n]}(x^\mu)\Big)\\ -&\Big(\left(MF^{[n]} - \partial_s G^{[n]}\right)\mathbb{1} + i\left(MG^{[n]} - \partial_s F^{[n]}\right)\gamma^5\Big)\Big(\psi_\alpha^{[n]}(x^\mu)\Big) = 0\ ,\end{aligned} \qquad (17)$$

while the boundary conditions (15) become

$$\begin{pmatrix} F^{[n]}(2\pi R) \\ G^{[n]}(2\pi R) \end{pmatrix} = e^{i\rho} \begin{pmatrix} \cosh(\omega) & \sinh(\omega) \\ \sinh(\omega) & \cosh(\omega) \end{pmatrix} \begin{pmatrix} F^{[n]}(0) \\ G^{[n]}(0) \end{pmatrix} . \tag{18}$$

One sees that the boundary conditions imply that the values of $F^{[n]}$ and $G^{[n]}$ at $2\pi R$ and at 0 must be related by a U(1)×SO(1,1) transformation. Remark that the usual periodic or antiperiodic boundary conditions for the spinors (allowing to closure of the s strip to a circle) correspond to the case $\omega = 0$ and $\rho = 0$ or $\rho = \pi$

$$F^{[0]}(2\pi R) = \epsilon\, F^{[0]}(0) \quad , \quad G^{[0]}(2\pi R) = \epsilon\, G^{[0]}(0) \quad \text{with } \epsilon^2 = 1 . \tag{19}$$

In a four-dimensional physical brane, we request that the spinor $\psi_\alpha^{[n]}(x^\mu)$ should satisfy the parity invariant Dirac equation

$$i\gamma^\mu \partial_\mu \psi_\alpha^{[n]}(x^\mu) = m_n \psi_\alpha^{[n]}(x^\mu) \tag{20}$$

with, by convention, $m_n \geq 0$. Then, $\psi_\alpha(x^\mu, s)$ from (16) is a solution of the Dirac equation in five dimensions if the two following coupled equations for $G^{[n]}(s)$ and $F^{[n]}(s)$ are satisfied

$$\begin{aligned} \partial_s G^{[n]}(s) &= (M - m_n) F^{[n]}(s) \\ \partial_s F^{[n]}(s) &= (M + m_n) G^{[n]}(s) . \end{aligned} \tag{21}$$

With the boundary conditions (18) taken into account, these equations lead to the determination of the allowed spinor masses m_n observable in a four dimensional subspace (20).

4 Kaluza-Klein mass towers

In this section, we treat successively the distinct cases corresponding to the bulk mass M being positive, zero or negative.

4.1 The case $M > 0$

Remember that m_n is positive by convention.

4.1.1 The subcase $m_n^2 > M^2$ $(M > 0)$

For $m_n^2 > M^2$, the solutions of (21) are

$$F^{[n]}(s) = \sqrt{m_n + M}\left(\sigma_n \sin\left(\sqrt{m_n^2 - M^2}s\right) + \tau_n \cos\left(\sqrt{m_n^2 - M^2}s\right)\right)$$
$$G^{[n]}(s) = \sqrt{m_n - M}\left(\sigma_n \cos\left(\sqrt{m_n^2 - M^2}s\right) - \tau_n \sin\left(\sqrt{m_n^2 - M^2}s\right)\right) \tag{22}$$

where σ_n and τ_n are constants and the square roots are chosen positive.

Introducing these solutions in the set of boundary conditions (18), one obtains a system of two linear homogeneous equations for σ_n and τ_n. The vanishing of the related determinant gives the mass equation for the m_n's

$$\begin{aligned}\left(\cosh(\omega)\cos\left(2\pi\sqrt{m_n^2 - M^2}\,R\right) - \cos(\rho)\right)\sqrt{m_n^2 - M^2}& \\ = M\sinh(\omega)\sin\left(2\pi\sqrt{m_n^2 - M^2}\,R\right)&\ .\end{aligned} \tag{23}$$

Note the scaling property of the equation, that it does not depend on R when the masses are expressed in units of $1/R$

$$\begin{aligned} m_n &= \frac{\overline{m}_n}{R} \\ M &= \frac{\overline{M}}{R}\ . \end{aligned} \tag{24}$$

In general, for given values of ω, ρ and $\overline{M}$, this equation has an infinite number of solutions $\overline{m}_n$, giving rise to a Kaluza-Klein tower.

Asymptotically, for large n, i.e. when $m_n >> M$, the masses in the tower are given by

$$\cos(2\pi\overline{m}_n) \approx \frac{\cos(\rho)}{\cosh(\omega)} \quad \left(\overline{m}_n >> \overline{M}\right)\ . \tag{25}$$

and become identical to the masses in the $M = 0$ tower for the same boundary parameters ρ and ω (4.2).

4.1.2 The subcase $m_1 = M$ $(M > 0)$

For $m_1 = M$, the solutions are

$$\begin{aligned} G^{[M]}(s) &= \sigma_M \\ F^{[M]}(s) &= 2M\sigma_M s + \tau_M \end{aligned} \tag{26}$$

where σ_M and τ_M are constants. Introducing these solutions in the boundary conditions (18), one finds two linear homogeneous relations in the parameters σ_M and τ_M. Defining $\overline{M}_1$ as

$$\overline{M}_1 = \frac{\cosh(\omega) - \cos(\rho)}{2\pi \sinh(\omega)} \ , \tag{27}$$

the vanishing of the determinant leads to the following condition

$$\overline{M} = \overline{M}_1 \tag{28}$$

to be satisfied by the parameters ω, ρ and $\overline{M}$ for the first mass in the tower m_1 to be equal to the bulk mass

$$m_1 = |M| \ . \tag{29}$$

4.1.3 The subcase $m_h^2 < M^2$ ($M > 0$)

For $m_h^2 < M^2$, the solutions are

$$\begin{aligned} F^{[h]}(s) &= \sqrt{M + m_h} \left(\sigma_h \sinh\left(\sqrt{M^2 - m_h^2} s\right) + \tau_h \cosh\left(\sqrt{M^2 - m_h^2} s\right)\right) \\ G^{[h]}(s) &= \sqrt{M - m_h} \left(\sigma_h \cosh\left(\sqrt{M^2 - m_h^2} s\right) + \tau_h \sinh\left(\sqrt{M^2 - m_h^2} s\right)\right) \ . \end{aligned} \tag{30}$$

Replacing these solutions into the boundary conditions, one again finds two linear homogeneous equations in σ_h and τ_h. The determinant is zero provided

$$\begin{aligned} &\left(\cosh(\omega) \cosh\left(2\pi \sqrt{M^2 - m_h^2}\, R\right) - \cos(\rho)\right) \sqrt{M^2 - m_h^2} \\ &\qquad = M \sinh(\omega) \sinh\left(2\pi \sqrt{M^2 - m_h^2}\, R\right) \ . \end{aligned} \tag{31}$$

For given values of the boundary parameters ω, ρ and of $\overline{M}$ (24), the solution of this equation, if it exists, is unique and will be the lowest mass $m_1 = m_h$ in the tower, such that

$$0 < m_1 < |M| \ . \tag{32}$$

The formula (31) is simply the analytical continuation of (23).

4.1.4 The subcase $m_h = 0$ ($M > 0$)

For $m_h = 0$, the limiting case of the equation (31), namely

$$\cosh(2\pi MR - \omega) = \cos(\rho) \ , \tag{33}$$

implies, with the definition

$$\overline{M}_2 = \frac{\omega}{2\pi} \ , \tag{34}$$

the following restrictions on the parameters ω, ρ and $\overline{M}$

$$\rho = 0 \quad , \quad \overline{M} = \overline{M}_2 \tag{35}$$

for a zero mass state to exist.

4.1.5 Summary

The results (for $M > 0$) related to the presence or absence of a first mass in the tower lower than the bulk mass M are summarized in Appendix (B).

4.2 The case $M = 0$

The case of the bulk mass $M = 0$ is obtained by letting $M \to 0$ in the relevant formulas.

4.2.1 The subcase $m_n > 0$ ($M = 0$)

For $m_n > 0$, the solutions of (21) are

$$\begin{aligned} F^{[n]}(s) &= \sigma_n \sin(m_n s) + \tau_n \cos(m_n s) \\ G^{[n]}(s) &= \sigma_n \cos(m_n s) - \tau_n \sin(m_n s) \end{aligned} \tag{36}$$

where σ_n and τ_n are constants. After introducing these solutions in the boundary conditions (18), the vanishing of the related determinant gives the mass equation for the m_n Kaluza-Klein tower

$$\cosh(\omega)\cos(2\pi m_n R) - \cos(\rho) = 0 \ . \tag{37}$$

The $\overline{m}_n$ tower is the superposition of two regular subtowers, each with spacing

$$\Delta(n+2, n) \equiv \overline{m}_{n+2} - \overline{m}_n = 1 \ . \tag{38}$$

The separation between the two subtowers is given by

$$\Delta(2n+1, 2n) \equiv \overline{m}_{2n+1} - \overline{m}_{2n} = \frac{1}{\pi} \arccos\left(\frac{\cos(\rho)}{\cosh(\omega)}\right) \ . \tag{39}$$

4.2.2 The subcase $m_1 = 0$ $(M = 0)$

To obtain a zero mass state ($m_1 = 0$), the lowest in the tower, one sees from (21) that $F^{[n]}$ and $G^{[n]}$ must be constants and hence, from (18), that the boundary conditions must be $\omega = \rho = 0$. This corresponds to the periodic boundary conditions (19) with $\epsilon = 1$, allowing the closure of the s strip to a circle.

4.3 The case $M < 0$

The case $M < 0$ is analogous to the case $M > 0$. As the main result, the mass towers are related as follows

$$\text{tower}\left\{-M, -\omega, \rho\right\} \equiv \text{tower}\left\{M, \omega, \rho\right\} . \tag{40}$$

5 Numerical evaluations

Illustrative numerical examples of spinor towers are presented in the five tables for a representative set of bulk masses $\overline{M}$ and for some chosen values of the boundary parameters ω and ρ.

As a general comment, for $\overline{M} = 0$, there are two interlaced regular subtowers, each with equal spacing $\Delta(n+2, n) = 1$ (38) and variable separation $\Delta(2n+1, 2n)$ (39) between the odd and even indexed masses.

1. In Table (1), the boundary parameters are $\omega = -1$ and $\rho = \pi/3$. Since ω is negative, the masses appearing in the tower are always larger than the bulk mass M, as it should for any value of ρ. For $M = 0$, the even-odd separation is

   $$\Delta(2n+1, 2n) \approx 0.395 . \tag{41}$$

   When the bulk mass $\overline{M}$ increases, the first masses, say the eight first masses $\overline{m}_1, \ldots, \overline{m}_8$, become closer and closer to $\overline{M}$. Already at $\overline{M} = 100$, one sees that these first masses become very densely packed just above $\overline{M}$. However, in all cases, asymptotically in n, the mass towers all tend to the mass tower corresponding to $M = 0$.

2. In Table (2), the boundary parameters are $\omega = 0$ and $\rho = \pi/3$. For $M = 0$, the even-odd separation is

   $$\Delta(2n+1, 2n) \approx 0.333 . \tag{42}$$

For increasing $\overline{M}$, the towers behave as in Table (1).

3. In Table (3), the boundary parameters are $\omega = 0$ and $\rho = 0.1$. Compared to the table (2), one sees that, for $M = 0$, the even-odd separation has become smaller

$$\Delta(2n + 1, 2n) \approx 0.032 \ . \tag{43}$$

Indeed, at the limit of $\rho = 0$, corresponding to the periodic boundary conditions (19), the two subtowers merge for $M = 0$. The mass $\overline{m}_1$ is zero while the other masses ($\overline{m}_n = n$, $n \neq 0$) are doubly degenerate. For increasing $\overline{M}$ and asymptotically, the towers behave as before.

4. In Table (4), the boundary parameters are $\omega = 2$ and $\rho = \pi/3$. For $M = 0$, the even-odd separation is

$$\Delta(2n + 1, 2n) \approx 0.458 \ . \tag{44}$$

For $\overline{M} < \overline{M}_1$ (27), $\overline{m}_1$ is larger than $\overline{M}$. For $\overline{M} > \overline{M}_1$, it is smaller than $\overline{M}$. Disregarding the exceptional $\overline{m}_1$, the towers behave as before for increasing $\overline{M}$ and asymptotically.

5. In Table (5), the boundary parameters are $\omega = 2$ and $\rho = 0$. For $M = 0$, the even-odd separation is

$$\Delta(2n + 1, 2n) \approx 0.414 \ . \tag{45}$$

For $\overline{M} < \overline{M}_1$, $\overline{m}_1$ (27) is larger than $\overline{M}$. For $\overline{M} > \overline{M}_1$, it is smaller than $\overline{M}$. For $\overline{M} = \overline{M}_2$ (34), there is a zero mass state in the tower ($\overline{m}_1 = 0$). Here again, disregarding the exceptional $\overline{m}_1$, the towers behave as before for increasing $\overline{M}$ and asymptotically.

6 Conclusions

In this article, we have carefully analysed the procedure of generation of Kaluza-Klein mass towers of four dimensional spinor fields, starting from a massive or massless five dimensional free Dirac field propagating in a flat bulk space with its fifth dimension compactified on a strip or on a circle.

Requiring the five dimensional Dirac operator to be symmetric, we have deduced the set of all the allowed boundary conditions. The natural invariant hermitian scalar product and the SO(3,1) invariance in the x^μ subspace were taken into account. The boundary conditions depend in a subtle way on the properties of the γ^5 matrix and are expressible in terms of two free parameters.

The Kaluza-Klein reduction is conducted in such a way that the spinor fields in four dimensions, which are related to a given bulk spinor field, obey the ordinary parity invariant Dirac equation. Requiring SO(3,1) covariance, it turns out that the γ^5 matrix plays also an essential role in the separation of variables. Notwithstanding, the presence of γ^5 does not spoil the parity conservation.

The equations whose solutions provide the Kaluza-Klein mass towers have been written in full generality. A few numerical examples are presented and discussed.

This work will be extended to the expected more realistic case of spinor fields propagating in five dimensional warped spaces, in line with our recent model of scalar fields living in warped spaces without [3] or with [4] metric singularities.

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A Heuristic justification of (15)

In this appendix, we justify with plausibility and simplicity arguments our derivation of the form (15) for the general boundary conditions for the Dirac fields.

The integrated boundary condition (13)

$$\left[\int_{-\infty}^{+\infty} \overline{\phi}\gamma^5\psi\, d^4x\right]_{s=2\pi R} = \left[\int_{-\infty}^{+\infty} \overline{\phi}\gamma^5\psi\, d^4x\right]_{s=0} \tag{46}$$

should lead to linear relations between the fields $\psi(x^\mu, s)$ evaluated at the edges of the s domain, namely at $s = 2\pi R$ and $s = 0$. Given $\psi(y^\mu, 0)$ for all y^μ, $\psi(x^\mu, 2\pi R)$ would then be related to it by the most general linear relation

$$\psi_\alpha(x^\mu, 2\pi R) = \int_{-\infty}^{+\infty} C_{\alpha\beta}(x^\mu, y^\nu)\, \psi_\beta(y^\nu, 0)\, d^4y \tag{47}$$

where $C_{\alpha\beta}(x^\mu, y^\nu)$ is a complex 4×4 matrix of functions depending on the eight coordinates.

If this boundary condition is to be covariant under the space-time SO(3,1) (subgroup of SO(4,1)) transformations $\psi'(x'^\mu, s)_\alpha = S_{\alpha\beta}\, \psi_\beta(x^\mu, s)$, the matrix $C_{\alpha\beta}$ must in particular commute with S and hence is restricted to a combination of the unit and γ^5 matrices

$$C(x^\mu, y^\nu) = C_1(x^\mu, y^\nu)\mathbb{1}_4 + iC_2(x^\mu, y^\nu)\gamma^5 \tag{48}$$

where $C_1(x^\mu, y^\nu)$ and $C_2(x^\mu, y^\nu)$ are two complex invariant functions or distributions which depend essentially on the invariant distance $(x-y)^2$ between the points x^μ and y^μ (the i is for convenience).

Introducing this form (48), (47), valid both for ψ and for ϕ, in the condition (46), one finds

$$\begin{aligned}
\int_{-\infty}^{+\infty} \Big(C_1^*(x,y)C_1(x,z) - C_2^*(x,y)C_2(x,z)\Big) d^4x &= \delta^4(y-z) \\
\int_{-\infty}^{+\infty} \Big(C_1^*(x,y)C_2(x,z) - C_2^*(x,y)C_1(x,z)\Big) d^4x &= 0\ .
\end{aligned} \tag{49}$$

The natural solution is expressible in terms of invariant δ distributions

$$C_i(x^\mu, y^\nu) = c_i\, \delta^4(x^\mu - y^\mu) \tag{50}$$

with the two constants being $c_1 = e^{i\rho}\sinh(\omega)$ and $c_2 = e^{i\rho}\cosh(\omega)$, leading through (47) to the final form (15). More general solutions of (49) are probably not very useful.

B Summary of the results for $M > 0$

Let us summarize the results for the position of the lowest mass m_1 in a tower relative to the bulk mass $M > 0$.

1. For $\omega < 0$, there is no mass $m_h < M$ (see (31)) and hence $m_1 > M$.

2. For $\omega = 0$ and $\rho \neq 0$, there is no mass $m_h < M$. Indeed, the condition resulting from (31) $(\cosh(2\pi\sqrt{\overline{M}^2 - \overline{m}_h^2}) = \cos(\rho))$ has no solution. Hence $m_1 > M$.

3. For $\omega = 0$ and $\rho = 0$, the lowest mass in the tower is $m_1 = M_M = M$.

4. For $\omega > 0$ and $\overline{M} < \overline{M}_3$,

$$\overline{M}_3 = \frac{\cosh(\omega) - 1}{2\pi\sinh(\omega)} \tag{51}$$

$$\overline{M}_4 = \frac{\cosh(\omega) + 1}{2\pi\sinh(\omega)} , \tag{52}$$

   there is no mass $m_h < M$. Hence $m_1 > M$.

5. For $\omega > 0$, $\overline{M}_3 \leq \overline{M} \leq \overline{M}_4$ and $-\rho_1 < \rho < \rho_1$,

$$\rho_1 = \arccos\Big(\cosh(\omega) - 2\pi\sinh(\omega)\overline{M}\Big) \qquad (0 < \rho_1 < \pi) , \tag{53}$$

   there is a mass $m_h < M$, which is the lowest mass ($m_1 = m_h$) in the tower. For $\rho = \rho_1$, $m_1 = M = M_1$ (27).

6. For $\omega > 0$, $\overline{M} > \overline{M}_4$, there is a mass $m_h < M$ for any value of ρ. The lowest mas m_1 is always smaller than M.

7. A mass $m_0 = 0$ exists provided that the boundary conditions belong to the case

$$\rho = 0 \quad , \quad \omega = 2\pi\overline{M} . \tag{54}$$

   This mass is the lowest mass in the tower.

Table 1: Mass towers for $\omega = -1$ and $\rho = \frac{\pi}{3}$

Case $\omega = -1$, $\rho = \frac{\pi}{3}$ For very large n, the mass towers converges toward the $\overline{M} = 0$ tower								
$\overline{M}$	$\overline{m}_1$	$\overline{m}_2$	$\overline{m}_3$	$\overline{m}_4$	$\overline{m}_5$	$\overline{m}_6$	$\overline{m}_7$	$\overline{m}_8$
0	0.198	0.803	1.198	1.803	2.198	2.803	3.198	3.803
0.1	0.267	0.823	1.212	1.812	2.205	2.809	3.203	3.807
0.2	0.346	0.854	1.234	1.827	2.218	2.818	3.211	3.814
0.3	0.431	0.895	1.264	1.847	2.234	2.831	3.223	3.824
0.4	0.519	0.943	1.301	1.871	2.256	2.848	3.238	3.836
0.5	0.609	0.998	1.344	1.901	2.281	2.867	3.255	3.851
0.6	0.700	1.059	1.392	1.935	2.310	2.890	3.276	3.868
0.7	0.793	1.126	1.446	1.974	2.343	2.917	3.300	3.888
1	1.077	1.346	1.632	2.112	2.464	3.013	3.387	3.961
2	2.049	2.210	2.413	2.758	3.048	3.502	3.835	4.346
10	10.012	10.0486	10.106	10.193	10.292	10.429	10.566	10.751
100	100.001	100.005	100.011	100.02	100.031	100.045	100.061	100.08

Table 2: Mass towers for $\omega = 0$ and $\frac{\pi}{3}$

Case $\omega = 0$, $\rho = \frac{\pi}{3}$ For very large n, the mass towers converge toward the $\overline{M} = 0$ tower								
$\overline{M}$	$\overline{m}_1$	$\overline{m}_2$	$\overline{m}_3$	$\overline{m}_4$	$\overline{m}_5$	$\overline{m}_6$	$\overline{m}_7$	$\overline{m}_8$
0	0.167	0.833	1.167	1.833	2.167	2.833	3.167	3.833
0.1	0.194	0.839	1.171	1.836	2.169	2.835	3.168	3.835
0.2	0.260	0.857	1.184	1.844	2.176	2.840	3.173	3.837
0.3	0.343	0.886	1.205	1.858	2.187	2.849	3.181	3.845
0.4	0.433	0.924	1.233	1.877	2.203	2.862	3.192	3.854
0.5	0.527	0.972	1.269	1.900	2.224	2.877	3.206	3.866
0.6	0.623	1.027	1.312	1.929	2.248	2.896	3.223	3.880
0.7	0.720	1.088	1.361	1.963	2.277	2.919	3.243	3.897
1	1.014	1.302	1.537	2.088	2.386	3.005	3.321	3.962
10	10.001	10.035	10.068	10.167	10.232	10.394	10.489	10.710
100	100.0001	100.0034	100.0068	100.0168	100.0235	100.0401	100.0501	100.0734

Table 3: Mass towers for $\omega = 0$ and $\rho = 0.1$

Case $\omega = 0$, $\rho = 0.1$ For very large n, the mass towers converge to the $M = 0$ tower								
$\overline{M}$	$\overline{m}_1$	$\overline{m}_2$	$\overline{m}_3$	$\overline{m}_4$	$\overline{m}_5$	$\overline{m}_6$	$\overline{m}_7$	$\overline{m}_8$
0	0.016	0.984	1.016	1.984	2.016	2.984	3.016	3.984
0.1	0.101	0.989	1.021	1.987	2.018	2.986	3.018	3.985
0.2	0.201	1.004	1.035	1.994	2.026	2.991	3.023	3.989
0.3	0.3004	1.029	1.059	2.007	2.038	2.999	3.031	3.995
0.4	0.4003	1.062	1.092	2.024	2.055	3.011	3.042	4.004
0.5	0.5003	1.104	1.132	2.046	2.077	3.026	3.052	4.015
0.6	0.6002	1.153	1.180	2.073	2.103	3.044	3.075	4.029
0.7	0.7002	1.208	1.234	2.104	2.134	3.065	3.096	4.045
1	1.0001	1.403	1.426	2.222	2.250	3.147	3.177	4.108
2	2.00006	2.229	2.243	2.817	2.840	3.592	3.619	4.458
5	5.000025	5.096	5.102	5.379	5.391	5.823	5.839	6.393
10	10.000013	10.048	10.051	10.195	10.201	10.436	10.445	10.764
100	100.0000013	100.0048	100.0052	100.0197	100.0203	100.0445	100.0455	100.0793

Table 4: Mass towers for $\omega = 2$ and $\rho = \frac{\pi}{3}$

Case $\omega = 2$, $\rho = \frac{\pi}{3}$ $\overline{M}_1 = 0.143\dots$ (27), $\overline{M}_3 = 0.121\dots$ (51), $\overline{M}_4 = 0.209\dots$ (52) For very large n, the mass towers converge to the $M = 0$ tower									
$\overline{M}$	$\rho_1(53)$	$\overline{m}_1$	$\overline{m}_2$	$\overline{m}_3$	$\overline{m}_4$	$\overline{m}_5$	$\overline{m}_6$	$\overline{m}_7$	$\overline{m}_8$
0		0.229	0.771	1.229	1.771	2.229	2.771	3.229	3.771
0.1		0.167	0.757	1.220	1.765	2.224	2.768	3.226	3.769
0.12		0.155	0.756	1.220	1.765	2.224	2.767	3.225	3.768
$\overline{M}_3$	0	0.155	0.756	1.220	1.765	2.224	2.767	3.225	3.768
0.13	0.644	0.150	0.756	1.220	1.765	2.224	2.767	3.225	3.768
0.14	0.962	0.145	0.755	1.220	1.765	2.224	2.767	3.225	3.768
$\overline{M}_1$	ρ	$\overline{M}_1$	0.755	1.219	1.765	2.224	2.767	3.225	3.768
0.15	1.220	0.140	0.755	1.219	1.764	2.224	2.767	3.225	3.768
0.20	2.490	0.117	0.756	1.220	1.765	2.224	2.767	3.226	3.768
$\overline{M}_4$	π	0.114	0.757	1.220	1.765	2.224	2.768	3.226	3.768
0.5		0.118	0.834	1.271	1.798	2.251	2.788	3.244	3.784
1		0.264	1.166	1.502	1.960	2.384	2.894	3.336	3.862
2		0.532	2.074	2.268	2.577	2.912	3.334	3.727	4.199
10		2.658	10.013	10.051	10.116	10.203	10.319	10.453	10.616
100		26.585	100.0012	100.005	100.011	100.020	100.031	100.045	100.061

Table 5: Mass towers for $\omega = 2$ and $\rho = 0$

Case $\omega = 2$, $\rho = 0$								
$\overline{M}_1 = \overline{M}_3 = 0.121\ldots$ (27), (51), $\overline{M}_2 = 1/\pi = 0.318\ldots$ (34)								
For very large n, the mass towers converge to the $M = 0$ tower								
$\overline{M}$	$\overline{m}_1$	$\overline{m}_2$	$\overline{m}_3$	$\overline{m}_4$	$\overline{m}_5$	$\overline{m}_6$	$\overline{m}_7$	$\overline{m}_8$
0	0.207	0.793	1.207	1.793	2.207	2.793	3.207	3.793
0.05	0.171	0.785	1.202	1.789	2.204	2.791	3.205	3.791
0.1	0.136	0.779	1.199	1.787	2.203	2.789	3.204	3.790
$\overline{M}_1(=\overline{M}_3)$	$\overline{M}_1$	0.778	1.198	1.786	2.202	2.789	3.204	3.790
0.15	0.102	0.777	1.198	1.786	2.202	2.789	3.204	3.790
0.2	0.069	0.778	1.199	1.787	2.202	2.789	3.204	3.790
$\overline{M}_2$	0	0.793	1.211	1.793	2.208	2.793	3.208	3.793
0.4	0.043	0.813	1.225	1.802	2.216	2.799	3.213	3.798
1	0.262	1.173	1.489	1.977	2.366	2.913	3.316	3.882
10	2.658	10.013	10.052	10.117	10.203	10.321	10.451	10.619
100	26.580	100.001	100.005	100.011	100.020	100.031	10.045	100.061