

Decision Support

Multicriteria decision support using rules that represent rough-graded preference relations

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Received 30 November 2005; accepted 14 March 2007

Available online 18 April 2007

Abstract

The approach described in this paper aims to support multicriteria choice and ranking of actions when the input preference information acquired from the decision maker is a graded comprehensive pairwise comparison (or ranking) of reference actions. It is based on decision-rule preference model induced from a rough approximation of the graded comprehensive preference relation among the reference actions. The set of decision rules applied to a new set of actions provides a graded fuzzy preference relation, which can be exploited by weighted-fuzzy net flow score or lexicographic-fuzzy net flow score procedure to obtain a final recommendation in terms of the best choice or of the ranking.

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Keywords: Multicriteria decision support; Decision rules; Graded preference relations; Rough sets; Ranking; Choice

1. Introduction

While decision making may sometimes be an unconscious act, it is usually followed by searching for rules that justify people's choices. The rules make evidence of a decision policy and can be used for both explanation of past decisions and recommendation of future decisions. The rules are logical statements (consequence relations) relating some conditions describing a decision situation with particular decisions. Construction of such a logical model of behavior from observation of agent's acts is a paradigm of artificial intelligence and, in particular, of inductive learning.

The set of rules representing a decision policy of an agent constitutes its *preference model*. The preference model is a necessary component of decision support systems for multicriteria choice and ranking problems.

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With respect to the basic characteristics of the preference model, two main models have been proposed in the past:

- a preference model expressed in terms of a utility function within multiple attribute utility theory (MAUT) [3,14];
- a preference model expressed in terms of procedures for building a preference relation within the outranking approach [16,19].

Both in MAUT and in the outranking approach, construction of a preference model requires that the agent – called *decision maker* (DM) – gives some *preference information*, like substitution rates among criteria within MAUT, or importance weights and thresholds of indifference, preference and veto within the outranking approach.

In this paper, we consider a third approach, different from MAUT and from the outranking approach: this is the *decision rule approach* [7,10], which follows the paradigm of artificial intelligence and inductive learning. Decision rules express preferences in a natural language in terms of “if . . . , then . . .” statements relating conditions concerning comparisons of actions on particular criteria, with conclusion being a comprehensive preference relation between these actions. For example, in a decision related to buying a house, a rule can say “if house x is preferred to house y with respect to proximity to working place, and strongly preferred with respect to the price, then x is comprehensively preferred to y ”. According to the paradigm of artificial intelligence and inductive learning, these decision rules are induced from preference information supplied by the DM in terms of some *decision examples*. It is rather certain that giving decision examples is much more natural for the DM than giving, more or less directly, the technical parameters of preference models mentioned above. Therefore, our decision model has two main advantages over the classical models: (i) the decision rules are intelligible and speak the language of the DM, (ii) the DM gives a preference information in the very natural terms of a set of exemplary decisions.

In practice, however, decision examples are often inconsistent due to hesitation of the DM, unstable character of his/her preferences and incomplete determination of the family of criteria. The inconsistencies violate the basic principle of multicriteria comparison, called *dominance* (or *Pareto*) *principle*, which has different formulation for multicriteria sorting problems, and for multicriteria choice and ranking problems.

In case of multicriteria sorting, which takes into account absolute evaluation of actions on particular criteria in decisions about their assignment to preference-ordered classes, the dominance principle says that if for two actions, x and y , action x has evaluations on all considered criteria not worse than action y , then x should be, comprehensively, weakly preferred to y .

In case of multicriteria choice and ranking, which takes into account pairwise comparisons of actions on particular criteria, in order to decide which ones are the best or how they rank from the best to the worst, the dominance principle says that if for two pairs of actions, (x,y) and (w,z) , action x is preferred to action y at least as much as action w is preferred to action z on all considered criteria, then pair (x,y) should be, comprehensively, weakly preferred to pair (w,z) , i.e. the comprehensive preference of x over y should not be less intensive than that of w over z .

The inconsistencies in the set of decision examples cannot be considered as simple error or noise – they can convey important information that should be taken into account in the construction of the DM’s preference model. Rather than correct or ignore these inconsistencies, we propose to take them into account in the preference model construction using the *rough set* concept [15,17].

The original definition of rough sets, involving a relation of indiscernibility to identify granules of objects used to build lower and upper approximations, was not able, however, to handle the inconsistencies with respect to the dominance principle. To overcome this limitation, the original version of rough set theory has been extended in two ways: (i) substituting the classical indiscernibility relation with respect to attributes by a dominance relation with respect to criteria, and (ii), substituting the data table of actions described by attributes, by a pairwise comparison table, where each row corresponds to a pair of actions described by binary relations on particular criteria, which permits approximation of a comprehensive preference relation in multicriteria choice and ranking problems. The extended rough set approach is called DRSA, which means dominance-based rough set approach [4–10,18].

We consider a finite set $A = \{x, y, z, \dots\}$ of actions evaluated by a family of criteria $G = \{g_1, \dots, g_n\}$, where for $i = 1, \dots, n$, $g_i : A \rightarrow \mathbf{R}$ such that, for all $x, y \in A$, $g_i(x) \geq g_i(y)$ means that “ x is at least as good as y with respect to criterion g_i ”. The preference information provided by the DM has the form of pairwise comparisons of some actions selected by the DM from set A : these actions, called *reference actions*, constitute a reference set $A' \subseteq A$. They should be relatively well known to the DM and such that, for some pairs of reference actions $(x, y) \in A' \times A'$, the DM is able to say how intensively x is preferred to y ; the set of pairs compared in these terms is denoted by $B \subseteq A' \times A'$. It is worth stressing that the choice of reference actions and their pairwise comparisons are two acts of the DM in which elicitation of preference information takes place. All we should do after, is to construct decision rules (preference model) compatible with this preference information – they represent the decision policy of the DM exhibited on the set of reference actions. These rules can be easily interpreted by the DM and, when approved, they can be applied to the complete set of actions in view of solving the choice or ranking problem. If a rule is non-acceptable for the DM, then it can be confronted with the pairwise comparisons supporting it, and, in consequence, the pairwise comparisons could be changed or a new pairwise comparison could be added by the DM, preventing induction of this non acceptable rule. This interactive process ends when the DM gets convinced that the preference model built from his/her preference information, as well as the consequence of its application on the whole set A , are concordant with his/her preferences.

Within this context, the preference information is represented as a pairwise comparison table (PCT) including pairs $B \subseteq A' \times A'$. In addition to evaluation on particular criteria, each pair $(x, y) \in B$ is characterized by a *comprehensive preference relation* which is graded (true or false to some grade). Using the rough set approach to the analysis of the PCT, we obtain a rough approximation of the graded preference relation by a dominance relation defined with respect to considered criteria. More precisely, the rough approximation concerns unions of graded preference relations, called upward and downward cumulated preference relations. The rough approximation is defined for a given level of consistency, changing from 1 (perfect separation of certain and doubtful pairs) to 0 (no separation of certain and doubtful pairs). The rough approximations are used to induce “if... then...” decision rules. The decision rules approved by the DM constitute a preference model of the DM. In order to recommend a best choice or a ranking of a new set of actions $M \subseteq A$, the decision rules are applied on set $M^2 = M \times M$ of pairs of these actions, inducing a specific preference structure on M .

We propose a new exploitation of this preference structure. It tends to answer the following questions:

- how to interpret the matching of one or several decision rules to a pair of actions $(u, v) \in M^2$ in terms of conclusions: u is preferred to v to some grade, or v is preferred to u to some grade?
- given a pair of actions $(u, v) \in M^2$, what is the credibility of a conclusion that u is preferred to v to some grade, or v is preferred to u to some grade?
- finally, having the above information on all the pairs of actions from M^2 , how one can build a final ranking?

Comparing to previous applications of dominance-based rough set approach to choice and ranking, this paper makes the following original contributions:

- (1) we consider a comprehensive graded preference relation: previous proposals [13,18] concentrated on two possible grades: “outranking” (when one action is at least as good as the other) and “non-outranking” (when one action is not at least as good as the other);
- (2) the decision rules have a probabilistic character due to induction based on variable consistency model of DRSA [11,18];
- (3) a credibility degree is assigned to the grade of preference for each pair of actions $(u, v) \in M^2$, taking into account the confidence level of decision rules matching the corresponding pair;
- (4) the considered preference structure on M is richer than the four-valued preference structure previously considered in dominance-based rough set approach to choice and ranking: in fact, it is a graded fuzzy preference relation of level 2; it is graded because of different grades of preference, but it is also fuzzy because of credibility degree assigned to each grade of preference;
- (5) to obtain a recommendation, we propose a weighted-fuzzy net flow score or, alternatively, a lexicographic-fuzzy net flow score exploitation procedure of the considered preference structure on M .

Moreover, in comparison to similar considerations presented in [4], in this paper, we revise and simplify the exploitation procedures. In Section 2, we define the pairwise comparison table from the decision examples given by the DM. In Section 3, we briefly sketch the variable consistency dominance-based rough set approach to the analysis of PCT, for both cardinal and ordinal scales of criteria. Section 4 is devoted to induction of decision rules and Section 5 characterizes the recommended procedures for exploitation of a graded fuzzy preference structure resulting from application of decision rules on a new set of actions. Section 6 presents an illustrative example. The last section groups conclusions.

2. Pairwise comparison table (PCT) built of decision examples

For a representative subset of reference actions $A' \subseteq A$, the DM is asked to express his/her comprehensive preferences by pairwise comparisons. In practice, he/she may accept to compare the pairs of a subset $B \subseteq A' \times A'$. For each pair $(x, y) \in B$, the comprehensive preference relation \succ assumes different grades h of intensity, hence denoted by \succ^h . Let $H \subset [-1, 1]$ be the finite set of all admitted values of h , and H^+ (resp. H^-) the subset of strictly positive (resp., strictly negative) values of h . It is assumed that $h \in H^+$ iff $-h \in H^-$. Finally, $H \equiv (H^- \cup \{0\} \cup H^+)$. In the following, $\alpha = \min\{h \in H\}$ and $\omega = \max\{h \in H\}$, with $\alpha = -\omega$.

For each pair $(x, y) \in A' \times A'$, the DM is asked to select one of the four possibilities:

- (1) action x is comprehensively preferred to action y in grade h , i.e. $x \succ^h y, h \in H^+$,
- (2) action x is comprehensively *not* preferred to action y in grade h , i.e. $x \succ^h y, h \in H^-$,
- (3) action x is comprehensively indifferent to action y , i.e. $x \succ^0 y$,
- (4) DM refuses to compare action x to action y .

Although the intensity grades are numerically valued, they may be interpreted in terms of linguistic ordinal qualifiers, for example: “*very weak preference*”, “*weak preference*”, “*strict preference*”, “*strong preference*” for $h = 0.2, 0.3, 0.7, 1.0$, respectively. A similar interpretation holds for negative values of h . Let us also note that $x \succ^h y$ does not necessarily imply $y \succ^{-h} x$ and $x \succ^0 y$ does not necessarily imply $y \succ^0 x$. Moreover, relation $x \succ^0 y$ is interpreted as indifference, rather than indetermination or incomparability, because it corresponds to situation of consciously expressed negligible preference of x over y . This expression of intensity of preference is assumed to be equally certain for all grades. Indetermination or incomparability would take place if the DM would abstain of any expression of preference with respect to the pair (x, y) , that is, if zero intensity would correspond to no certainty at all.

An $m \times (n + 1)$ pairwise comparison table S_{PCT} is then created on the base of this information. Its first n columns correspond to criteria from set G . The last, $(n + 1)$ th column of S_{PCT} , represents the comprehensive binary relations \succ^h with $h \in H$. The m rows represent pairs from B . If the DM refused to compare two reference actions, then such a pair does not appear in S_{PCT} .

In the following, we will distinguish two kinds of criteria – *cardinal* and *ordinal* ones [5]. In consequence of this distinction, for each pair of actions in S_{PCT} we have either a difference of evaluations on cardinal criteria, or pairs of original evaluations on ordinal criteria. The difference of evaluations on a cardinal criterion needs to be translated into a graded marginal intensity of preference. For any cardinal criterion $g_i \in G$, we consider a finite set $H_i \equiv (H_i^- \cup \{0\} \cup H_i^+)$ of marginal intensity grades such that for every pair of actions $(x, y) \in A \times A$ exactly one grade $h \in H_i$ is assigned.

- (1) $x \succ_i^h y, h \in H_i^+$, means that action x is preferred to action y in grade h on criterion g_i ,
- (2) $x \succ_i^h y, h \in H_i^-$, means that action x is *not* preferred to action y in grade h on criterion g_i ,
- (3) $x \succ_i^0 y$ means that action x is similar (asymmetrically indifferent) to action y on criterion g_i .

Within the preference context, the similarity relation \succ_i^0 , even if not symmetric, resembles indifference relation. Thus, in this case, we call this similarity relation “*asymmetric indifference*”. Of course, for each cardinal criterion $g_i \in G$ and for every pair of actions $(x, y) \in A \times A$, $[\exists h \in H_i^+ : x \succ_i^h y] \Rightarrow [\nexists k \in H_i^+ : y \succ_i^k x]$ as well as $[\exists h \in H_i^- : x \succ_i^h y] \Rightarrow [\nexists k \in H_i^- : y \succ_i^k x]$. Observe that the binary relation \succ_i^0 is reflexive, but neither necessar-

ily symmetric nor transitive, and \succ^h for $h \in H \setminus \{0\}$ are neither reflexive nor symmetric and not necessarily transitive. $\cup_{h \in H} \succ^h$ is not necessarily complete.

Let us observe that we consider the following natural relation between graded preference relations \succ_i^h , $h \in H_i$, and criteria g_i , $i = 1, \dots, n$ (coherence principle of graded preference relations): for all $x, y, w, z \in A$ and all $i = 1, \dots, n$:

$$[x \succ_i^h y, h \in H_i, \text{ and } g_i(w) \geq g_i(x) \geq g_i(y) \geq g_i(z)] \Rightarrow [w \succ_i^k z \text{ with } k \in H_i \text{ such that } k \geq h].$$

Consequently, PCT can be seen as decision table $S_{\text{PCT}} = \langle B, G \cup \{d\} \rangle$, where $B \subseteq A \times A$ is a non-empty set of pairwise comparisons of reference actions and d is a decision corresponding to the comprehensive pairwise comparison (comprehensive graded preference relation).

3. Rough approximation of comprehensive graded preference relations specified in PCT

Let G^N be the set of cardinal criteria, and G^O the set of ordinal criteria, such that $G^N \cup G^O = G$ and $G^N \cap G^O = \emptyset$. Moreover, for each $P \subseteq G$, we denote by P^N , P^O the same partitioning of P , i.e. $P^O = P \cap G^O$ and $P^N = P \cap G^N$. In order to define the rough approximations of comprehensive graded preference relations we need the concept of dominance relation between two pairs of actions with respect to (w.r.t.) a subset of criteria. This concept is defined below, separately for subsets of cardinal criteria and for subsets of ordinal criteria. In the case of cardinal criteria, the dominance is built on graded preference relations, and in the case of ordinal criteria, the dominance is built directly on pairs of evaluations [5].

DRSA aims at separating consistent from inconsistent preference information, so as to express certainly (P -lower approximation) or possibly only (P -upper approximation) the comprehensive graded preference relations for a pair of actions in terms of evaluations of these actions on particular criteria from set P .

3.1. Cardinal criteria

Let $P = P^N \subseteq G$ ($P \neq \emptyset$). Given $(x, y), (w, z) \in A \times A$, the pair of actions (x, y) is said to dominate (w, z) w.r.t. subset of cardinal criteria P (denoted by $(x, y)D_P(w, z)$) if x is preferred to y at least as strongly as w is preferred to z w.r.t. each $g_i \in P$. Precisely, “at least as strongly as” means “to at least the same grade”, i.e. for each $g_i \in P$ and $k \in H_i$ such that $w \succ_i^k z$, there exists $h \in H_i$ such that $h \geq k$ and $x \succ_i^h y$. Let $D_{\{i\}}$ be the dominance relation confined to the single criterion $g_i \in P$. The binary relation $D_{\{i\}}$ is a complete preorder on $A \times A$. Since the intersection of complete preorders is a partial preorder and $D_P = \bigcap_{g_i \in P} D_{\{i\}}$, then the dominance relation D_P is a partial preorder on $A \times A$. Let $R \subseteq P \subseteq G$ and $(x, y), (u, v) \in A \times A$; then the following implication holds: $(x, y)D_P(u, v) \Rightarrow (x, y)D_R(u, v)$.

Given $P \subseteq G$ and $(x, y) \in A \times A$, we define:

- a set of pairs of actions dominating (x, y) , called P -dominating set,

$$D_P^+(x, y) = \{(w, z) \in A \times A : (w, z)D_P(x, y)\},$$

- a set of pairs of actions dominated by (x, y) , called P -dominated set,

$$D_P^-(x, y) = \{(w, z) \in A \times A : (x, y)D_P(w, z)\}.$$

To approximate the comprehensive graded preference relation, we need to introduce the concept of *upward cumulated preference* (denoted by $\succ^{\geq h}$) and *downward cumulated preference* (denoted by $\succ^{\leq h}$), having the following interpretation:

- $x \succ^{\geq h} y$ means “ x is comprehensively preferred to y by at least grade h ”, i.e. $x \succ^{\geq h} y$ if $x \succ^k y$, where $h \leq k \in H$,
- $x \succ^{\leq h} y$ means “ x is comprehensively preferred to y by at most grade h ”, i.e. $x \succ^{\leq h} y$ if $x \succ^k y$, where $h \geq k \in H$.

The P -dominating sets and the P -dominated sets defined on B for all pairs of reference actions from B are “granules of knowledge” that can be used to express P -lower and P -upper approximations of cumulated preference relations $\succ^{\geq h}$ and $\succ^{\leq h}$, denoted by $\underline{P}(\succ^{\geq h})$, $\overline{P}(\succ^{\geq h})$ and $\underline{P}(\succ^{\leq h})$, $\overline{P}(\succ^{\leq h})$, respectively:

– for $h \in H$,

$$\underline{P}(\succ^{\geq h}) = \{(x, y) \in B : [D_P^+(x, y) \cap B] \subseteq \succ^{\geq h}\}, \quad \overline{P}(\succ^{\geq h}) = \left[\bigcup_{(x, y) \in \succ^{\geq h}} D_P^+(x, y) \right] \cap B,$$

– for $h \in H$,

$$\underline{P}(\succ^{\leq h}) = \{(x, y) \in B : [D_P^-(x, y) \cap B] \subseteq \succ^{\leq h}\}, \quad \overline{P}(\succ^{\leq h}) = \left[\bigcup_{(x, y) \in \succ^{\leq h}} D_P^-(x, y) \right] \cap B.$$

For all $h \in H$, $\underline{P}(\succ^{\geq h}) \subseteq \succ^{\geq h} \subseteq \overline{P}(\succ^{\geq h})$ and $\underline{P}(\succ^{\leq h}) \subseteq \succ^{\leq h} \subseteq \overline{P}(\succ^{\leq h})$. Furthermore, for all $h \in H$, $\underline{P}(\succ^{\geq h}) = B - \overline{P}(\succ^{\leq h})$ and $\underline{P}(\succ^{\leq h}) = B - \overline{P}(\succ^{\geq h})$. From the definition of the P -boundaries (P -doubtful regions) of $\succ^{\geq h}$ and of $\succ^{\leq h}$ for any $h \in H$, $\text{Bn}_P(\succ^{\geq h}) = \overline{P}(\succ^{\geq h}) - \underline{P}(\succ^{\geq h})$ and $\text{Bn}_P(\succ^{\leq h}) = \overline{P}(\succ^{\leq h}) - \underline{P}(\succ^{\leq h})$, it follows that $\text{Bn}_P(\succ^{\geq h}) = \text{Bn}_P(\succ^{\leq h})$ [5].

The concepts of the quality of approximation, reducts and core can be extended also to the approximation of cumulated preference relations. In particular, the *quality of approximation* of $\succ^{\geq h}$ and $\succ^{\leq h}$ for all $h \in H$, by $P \subseteq G$ is characterized by the coefficient

$$\gamma_P = \left| B - \left(\bigcup_{h \in H} \text{Bn}_P(\succ^{\geq h}) \right) \right| / |B| = \left| B - \left(\bigcup_{h \in H} \text{Bn}_P(\succ^{\leq h}) \right) \right| / |B|,$$

where $|\cdot|$ denotes cardinality of a set. It expresses the ratio of all pairs of actions $(x, y) \in B$ correctly assigned to $\succ^{\geq h}$ and to $\succ^{\leq h}$ by the set P of criteria, to all the pairs of actions contained in B . Each minimal subset $P \subseteq G$, such that $\gamma_P = \gamma_C$, is a *reduct* of G (denoted by $\text{RED}_{S_{\text{PCT}}}$). Let us remark that S_{PCT} can have more than one reduct. The intersection of all B -reducts is the *core* (denoted by $\text{CORE}_{S_{\text{PCT}}}$).

In fact, for induction of decision rules, we consider the variable consistency model on S_{PCT} [11,18] relaxing the definition of P -lower approximation of the cumulated preference relations $\succ^{\geq h}$ and $\succ^{\leq h}$, for any $h \in H$, such that at most $(1-l) \times 100\%$ of the pairs in P -dominating or P -dominated sets may not belong to the approximated cumulated preference relation:

$$\begin{aligned} \underline{P}^l(\succ^{\geq h}) &= \{(x, y) \in B : |D_P^+(x, y) \cap \succ^{\geq h}| / |D_P^+(x, y) \cap B| \geq l\}, \\ \underline{P}^l(\succ^{\leq h}) &= \{(x, y) \in B : |D_P^-(x, y) \cap \succ^{\leq h}| / |D_P^-(x, y) \cap B| \geq l\}, \end{aligned}$$

where $l \in (0, 1]$ is the required level of consistency.

3.2. Ordinal criteria

In the case of ordinal criteria, the dominance relation is defined directly on pairs of evaluations $g_i(x)$ and $g_i(y)$, for all pairs of actions $(x, y) \in A \times A$. Let $P = P^O$ and $P^N = \emptyset$, then, given $(x, y), (w, z) \in A \times A$, the pair (x, y) is said to dominate the pair (w, z) w.r.t. subset of ordinal criteria P (denoted by $(x, y)D_P(w, z)$) if, for each $g_i \in P$, $g_i(x) \geq g_i(w)$ and $g_i(z) \geq g_i(y)$. Let $D_{\{i\}}$ be the dominance relation confined to the single criterion $g_i \in P^O$. The binary relation $D_{\{i\}}$ is reflexive, transitive, but non-necessarily complete (it is possible that *not* $(x, y)D_{\{i\}}(w, z)$ and *not* $(w, z)D_{\{i\}}(x, y)$ for some $(x, y), (w, z) \in A \times A$). Thus, $D_{\{i\}}$ is a partial preorder. Since the intersection of partial preorders is a partial preorder and $D_P = \bigcap_{g_i \in P} D_{\{i\}}$, $P = P^O$, then the dominance relation D_P is a partial preorder.

3.3. Cardinal and ordinal criteria

If subset of criteria $P \subseteq G$ is composed of both cardinal and ordinal criteria, i.e. if $P^N \neq \emptyset$ and $P^O \neq \emptyset$, then, given $(x, y), (w, z) \in A \times A$, the pair (x, y) is said to dominate the pair (w, z) w.r.t. subset of criteria P (denoted by $(x, y)D_P(w, z)$), if (x, y) dominates (w, z) w.r.t. both P^N and P^O . Since the dominance relation w.r.t. P^N is a partial preorder on $A \times A$ and the dominance w.r.t. P^O is also a partial preorder on $A \times A$, then

also the dominance D_P , being the intersection of these two dominance relations, is a partial preorder. In consequence, all the concepts related to rough approximations introduced in Section 3.1 can be restored using this specific definition of dominance relation.

4. Induction of decision rules from rough approximations

Using the rough approximations of relations $\succ^{\geq h}$ and $\succ^{\leq h}$, defined in Section 3, it is then possible to induce a generalized description of the preference information contained in a given S_{PCT} in terms of decision rules. The syntax of these rules is based on the concept of *upward cumulated preferences w.r.t. criterion g_i* (denoted by $\succ_i^{\geq h}$) and *downward cumulated preferences w.r.t. criterion g_i* (denoted by $\succ_i^{\leq h}$), having similar interpretation and definition as for the comprehensive preference. Let also $G_i = \{g_i(x), x \in A\}$, $g_i \in G^O$, be a set of different evaluations on ordinal criterion g_i . The decision rules induced from S_{PCT} have then the following syntax:

- (1) D_{\geq} -decision rules, which are induced with the hypothesis that all pairs from $\underline{P}^I (\succ^{\geq h})$ are positive and all the others are negative learning examples:

$$\text{if } x \succ_{i1}^{\geq h(i1)} y \text{ and } \dots x \succ_{ie}^{\geq h(ie)} y \text{ and } g_{ie+1}(x) \geq r_{ie+1} \text{ and } g_{ie+1}(y) \leq s_{ie+1} \text{ and } \dots g_{ip}(x) \geq r_{ip} \text{ and } g_{ip}(y) \leq s_{ip}, \text{ then } x \succ^{\geq h} y.$$

- (2) D_{\leq} -decision rules, which are induced with the hypothesis that all pairs from $\underline{P}^I (\succ^{\leq h})$ are positive and all the others are negative learning examples:

$$\text{if } x \succ_{i1}^{\leq h(i1)} y \text{ and } \dots x \succ_{ie}^{\leq h(ie)} y \text{ and } g_{ie+1}(x) \leq r_{ie+1} \text{ and } g_{ie+1}(y) \geq s_{ie+1} \text{ and } \dots g_{ip}(x) \leq r_{ip} \text{ and } g_{ip}(y) \geq s_{ip}, \text{ then } x \succ^{\leq h} y,$$

where $P = \{g_{i1}, \dots, g_{ip}\} \subseteq G$, $P^N = \{g_{i1}, \dots, g_{ie}\}$, $P^O = \{g_{ie+1}, \dots, g_{ip}\}$, $(h(i1), \dots, h(ie)) \in H_{i1} \times \dots \times H_{ie}$ and $(r_{ie+1}, \dots, r_{ip}), (s_{ie+1}, \dots, s_{ip}) \in G_{ie+1} \times \dots \times G_{ip}$.

Since we are working with variable consistency approximations, it is enough to consider the lower approximations of the upward and downward cumulated preference relations, namely $\underline{P}^I (\succ^{\geq h})$ and $\underline{P}^I (\succ^{\leq h})$. To characterize the quality of the rules, we say that a pair of actions *supports* a decision rule ρ if it matches both the condition and decision parts of ρ . On the other hand, a pair is *covered* by a decision rule ρ as soon as it matches the condition part of ρ . Let $\text{Cover}(\rho)$ denote the set of all pairs of actions covered by rule ρ with decision part “then $x \succ^{\geq h} y$ ”. Let also remind that $\succ^{\geq h}$ denotes a subset of pairs $(w, z) \in B$, such that $w \succ^{\geq h} z$. Finally, we can define the confidence level $\eta_\rho (\succ^{\geq h})$ of D_{\geq} -decision rule ρ as

$$\eta_\rho (\succ^{\geq h}) = \frac{|\text{Cover}(\rho) \cap \succ^{\geq h}|}{|\text{Cover}(\rho)|}.$$

For D_{\leq} -decision rules, the confidence level is defined analogously.

Let us remark that the decision rules are induced from P -lower approximations whose composition is controlled by user-specified consistency level l . It seems reasonable to require that the smallest accepted confidence of the rule should not be lower than the currently used consistency level l . Indeed, in the worst case, some pairs of actions from the P -lower approximation may create a rule using all criteria from P thus giving a confidence $\eta_\rho (\succ^{\geq h}) \geq l$. The user may have a possibility of increasing this lower bound for confidence of the rule but then decision rules may not cover all pairs of actions from the P -lower approximations. Moreover, we require that each decision rule is minimal. Since a decision rule is a consequence relation, by a *minimal* decision rule we understand such a consequence relation that there is no other consequence relation with a premise of at least the same weakness and a conclusion of at least the same strength with a not worse confidence $\eta_\rho (\succ^{\geq h}) \geq l$. The induction of variable-consistency decision rules can be done using the rule induction algorithm for VC-DRSA, which can be found in [12].

5. Procedures for exploitation of a graded fuzzy preference structure resulting from application of decision rules on a new set of actions

The decision rules induced from a given S_{PCT} describe the upward and downward cumulated preference relations $\succ^{\geq h}$ and $\succ^{\leq h}$ with confidence $\eta_\rho(\succ^{\geq h}) \geq l$. After being approved by the DM, a set of decision rules covering all pairs of S_{PCT} represents a preference model of the DM who made the pairwise comparison of reference actions.

By application of the decision rules on a new subset $M^2 = M \times M \subseteq A \times A$ of pairs of actions, we get for each pair $(u, v) \in M^2$ a set of different conclusions (possibly empty) in the form of cumulated preference relations $\succ^{\geq h}$ and $\succ^{\leq h}$, where $h \in H$.

5.1. The credibility degree β and its monotonic behavior

For all pairs $(u, v) \in M^2$ we state the *credibility degree* β of conclusion $u \succ^{\geq h} v$, such that:

$$\beta(u \succ^{\geq \alpha} v) = 1 \text{ and } \beta(u \succ^{\geq h} v) = \max\{\eta_\rho(\succ^{\geq k}) : \rho \in R^{\geq}(u, v, h)\} \text{ for all } h \in H - \{\alpha\},$$

where $R^{\geq}(u, v, h)$ is the set of D_{\geq} -rules ρ matching (u, v) and concluding $x \succ^{\geq k} y$ with $k \in H$, such that $k \geq h$. The reason for stating $\beta(u \succ^{\geq \alpha} v) = 1$ is that it is certain that u is preferred to v to at least grade α which is the worst possible grade of preference. In some sense, the conclusion $u \succ^{\geq \alpha} v$ is a default one. This conclusion is always true and, therefore, it conveys no information.

Analogously, for all pairs $(u, v) \in M^2$ we state the *credibility degree* β of $u \succ^{\leq h} v$:

$$\beta(u \succ^{\leq \omega} v) = 1 \text{ and } \beta(u \succ^{\leq h} v) = \max\{\eta_\rho(\succ^{\leq k}) : \rho \in R^{\leq}(u, v, h)\} \text{ for all } h \in H - \{\omega\},$$

where $R^{\leq}(u, v, h)$ is the set of D_{\leq} -rule ρ matching (u, v) and having as conclusion “then $x \succ^{\leq k} y$ ” with $k \in H$, such that $k \leq h$. The reason for stating $\beta(u \succ^{\leq \omega} v) = 1$ is that it is certain that u is preferred to v to at most grade ω which is the best possible grade of preference. Again, $u \succ^{\leq \omega} v$ is a default conclusion.

The following lemma stems from the definition of $\beta(u \succ^{\leq h} v)$ and $\beta(u \succ^{\geq h} v)$; it expresses the monotonic behavior of the credibility degrees with respect to grade h .

Lemma 1. *For all $h_1, h_2 \in H$ and for all $(u, v) \in M^2$, the following monotonicity holds:*

- (a) $h_1 \geq h_2 \Rightarrow \beta(u \succ^{\geq h_1} v) \leq \beta(u \succ^{\geq h_2} v)$,
- (b) $h_1 \geq h_2 \Rightarrow \beta(u \succ^{\leq h_1} v) \geq \beta(u \succ^{\leq h_2} v)$.

Proof. If $\rho \in R^{\geq}(u, v, h_1)$, then the conclusion of ρ is “then $x \succ^{\geq k} y$ ” with $k \in H$, such that $k \geq h_1$ and, for hypothesis, $h_1 \geq h_2$. Thus, $\rho \in R^{\geq}(u, v, h_2)$ because $k \geq h_2$. Therefore, we have that $h_1 \geq h_2 \Rightarrow R^{\geq}(u, v, h_1) \subseteq R^{\geq}(u, v, h_2)$, and finally,

$$\beta(u \succ^{\geq h_1} v) = \max\{\eta_\rho(\succ^{\geq k}) : \rho \in R^{\geq}(u, v, h_1)\} \leq \max\{\eta_\rho(\succ^{\geq k}) : \rho \in R^{\geq}(u, v, h_2)\} = \beta(u \succ^{\geq h_2} v).$$

(b) can be proved analogously. \square

Fig. 1 shows an example of the shapes of $\beta(u \succ^{\leq h} v)$ and $\beta(u \succ^{\geq h} v)$. The monotonicity of $\beta(u \succ^{\geq h} v)$ with respect to h says that preference $u \succ^{\geq h_1} v$ is not less credible than preference $u \succ^{\geq h_2} v$, whenever $h_1 \leq h_2$. Analogous observation holds for $u \succ^{\leq h_1} v$ and for $u \succ^{\leq h_2} v$, as soon as $h_1 \geq h_2$.

For exploitation procedures, the monotonic behavior of the credibility degrees $\beta(u \succ^{\leq h} v)$ and $\beta(u \succ^{\geq h} v)$ with respect to the dominance relation D_G is particularly important. It is proved in the following lemma.

Lemma 2. *For all $(u, v), (w, z) \in M$, the following implication holds:*

$$\begin{array}{c} (u, v)D_G(w, z) \\ \Downarrow \\ \text{for all } h \in H, \beta(u \succ^{\geq h} v) \geq \beta(w \succ^{\geq h} z) \text{ and } \beta(u \succ^{\leq h} v) \leq \beta(w \succ^{\leq h} z). \end{array}$$

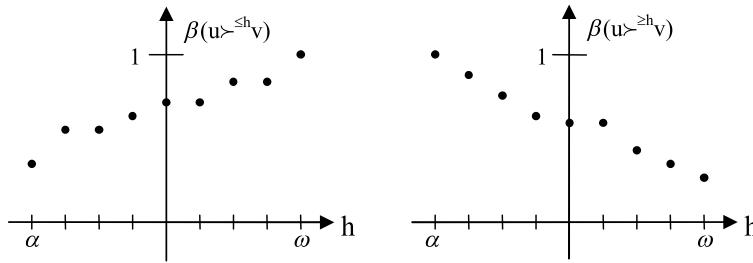


Fig. 1. Example of credibility degrees with respect to h .

Proof. For a given $h \in H$, let us consider the following D_{\geq} -decision rule ρ with $k \geq h$,

if $x \succ_{i1}^{\geq h(i1)} y$ and $\dots x \succ_{ie}^{\geq h(ie)} y$ and $g_{ie+1}(x) \geq r_{ie+1}$ and $g_{ie+1}(y) \leq s_{ie+1}$ and $\dots g_{ip}(x) \geq r_{ip}$ and $g_{ip}(y) \leq s_{ip}$, then $x \succ^{\geq k} y$.

If the pair (w, z) matches ρ , then

$w \succ_{i1}^{\geq h(i1)} z$ and $\dots w \succ_{ie}^{\geq h(ie)} z$ and $g_{ie+1}(w) \geq r_{ie+1}$ and $g_{ie+1}(z) \leq s_{ie+1}$ and $\dots g_{ip}(w) \geq r_{ip}$ and $g_{ip}(z) \leq s_{ip}$

and, therefore, the rule ρ suggests that $w \succ^{\geq k} z$.

As $(u, v)D_G(w, z)$, it is also true that

$u \succ_{i1}^{\geq h(i1)} v$ and $\dots u \succ_{ie}^{\geq h(ie)} v$ and $g_{ie+1}(u) \geq r_{ie+1}$ and $g_{ie+1}(v) \leq s_{ie+1}$ and $\dots g_{ip}(u) \geq r_{ip}$ and $g_{ip}(v) \leq s_{ip}$

and, therefore, the rule ρ suggests again that $u \succ^{\geq k} v$.

This means that if $(u, v)D_G(w, z)$ and (w, z) matches a D_{\geq} -decision rule ρ , then also (u, v) matches the same decision rule ρ , whatever $h \in H$ and $k \geq h$. Thus, for all $h \in H$, $R^{\geq}(w, z, h) \subseteq R^{\geq}(u, v, h)$ and, consequently,

$$\beta(u \succ^{\geq h} v) = \max\{\eta_{\rho}(\succ^{\geq k}) : \rho \in R^{\geq}(u, v, h)\} \geq \max\{\eta_{\rho}(\succ^{\geq k}) : \rho \in R^{\geq}(w, z, h)\} = \beta(w \succ^{\geq h} z)$$

Thus, we proved that

$$(u, v)D_G(w, z) \Rightarrow [\text{for all } h \in H, \beta(u \succ^{\geq h} v) \geq \beta(w \succ^{\geq h} z)].$$

An analogous proof holds for

$$(u, v)D_G(w, z) \Rightarrow [\text{for all } h \in H, \beta(u \succ^{\leq h} v) \leq \beta(w \succ^{\leq h} z)]. \quad \square$$

The following lemma proves the monotonic behavior of the credibility degrees with respect to the criteria evaluations.

Lemma 3. For all $u, v \in M$, the following implication holds:

$$\begin{aligned} & [g_i(u) \geq g_i(v) \text{ for all } i = 1, \dots, n] \\ & \quad \Downarrow \\ & \text{for all } h \in H, \text{ for all } z \in M, \\ & \left[\begin{aligned} & \beta(u \succ^{\geq h} z) \geq \beta(v \succ^{\geq h} z) \text{ and } \beta(u \succ^{\leq h} z) \leq \beta(v \succ^{\leq h} z) \\ & \beta(z \succ^{\geq h} u) \leq \beta(z \succ^{\geq h} v) \text{ and } \beta(z \succ^{\leq h} u) \geq \beta(z \succ^{\leq h} v) \end{aligned} \right]. \end{aligned}$$

Proof. For all $u, v, z \in M$, by definition of the dominance relation and the coherence principle of graded preference relation,

$$[g_i(u) \geq g_i(v) \text{ for all } i = 1, \dots, n] \Rightarrow [(u, z)D_G(v, z) \text{ and } (z, v)D_G(z, u)],$$

since $g_i(z) \geq g_i(z)$ for all $i = 1, \dots, n$. From Lemma 2,

$$(u, z)D_G(v, z) \Rightarrow [\text{for all } h \in H, \beta(u \succ^{\geq h} z) \geq \beta(v \succ^{\geq h} z) \text{ and } \beta(u \succ^{\leq h} z) \leq \beta(v \succ^{\leq h} z)]$$

and

$$(z, v)D_G(z, u) \Rightarrow [\text{for all } h \in H, \beta(z \succ^{\geq h} v) \geq \beta(z \succ^{\geq h} u) \text{ and } \beta(z \succ^{\leq h} v) \leq \beta(z \succ^{\leq h} u)]. \quad \square$$

For each $h \in H$, $\succ^{\geq h}$ and $\succ^{\leq h}$ are fuzzy preference relations in M . They may be represented by a pair of fuzzy preference graphs.

In order to obtain a final recommendation in terms of choice or ranking, we have to exploit the pair of sets of fuzzy preference relations (B^{\geq}, B^{\leq}) , defined as

$$B^{\geq} = \{\beta^{\geq h}, h \in H\}, B^{\leq} = \{\beta^{\leq h}, h \in H\},$$

where, for all $h \in H$, $\beta^{\geq h} : M^2 \rightarrow [0, 1]$ and $\beta^{\leq h} : M^2 \rightarrow [0, 1]$, such that for all $(u, v) \in M^2$, $\beta^{\geq h}(u, v) = \beta(u \succ^{\geq h} v)$ and $\beta^{\leq h}(u, v) = \beta(u \succ^{\leq h} v)$.

Using fuzzy preference relations (B^{\geq}, B^{\leq}) , a comprehensive score on M is a function $S : M \rightarrow \mathbf{R}$ such that for all $x \in M$

$$S(x) = K[\beta(x \succ^{\geq h} z), \beta(z \succ^{\geq h} x), \beta(x \succ^{\leq h} z), \beta(z \succ^{\leq h} x), z \in M - \{x\}, h \in H],$$

with $K : [0, 1]^{4 \times (|M|-1) \times |H|} \rightarrow \mathbf{R}$ being a function non-decreasing with values $\beta(x \succ^{\geq h} z)$ and $\beta(z \succ^{\leq h} x)$, and non-increasing with values $\beta(z \succ^{\geq h} x)$ and $\beta(x \succ^{\leq h} z)$, $z \in M - \{x\}$, $h \in H$. Of course, the greater $S(x)$, the more preferable is action $x \in M$.

Let us observe that it is also possible to define a comprehensive score relative to each grade $h \in H^+ \cup \{0\}$ on M , as a function $S^h : M \rightarrow \mathbf{R}$, such that for all $x \in M$,

$$S^h(x) = K^h[\beta(x \succ^{\geq h} z), \beta(z \succ^{\geq h} x), \beta(x \succ^{\leq -h} z), \beta(z \succ^{\leq -h} x), z \in M - \{x\}]$$

with $K^h : [0, 1]^{4 \times (|M|-1)} \rightarrow \mathbf{R}$ being a function non-decreasing with values $\beta(x \succ^{\geq h} z)$ and $\beta(z \succ^{\leq -h} x)$, and non-increasing with values $\beta(z \succ^{\geq h} x)$ and $\beta(x \succ^{\leq -h} z)$, $z \in M - \{x\}$.

Taking into account the monotonic properties of fuzzy relations B^{\geq} and B^{\leq} presented in Lemma 3, the following interesting result can be stated.

Theorem. For all $u, v \in M$,

- (a) $[g_i(u) \geq g_i(v) \text{ for all } i = 1, \dots, n] \Rightarrow S(u) \geq S(v)$,
- (b) $[g_i(u) \geq g_i(v) \text{ for all } i = 1, \dots, n] \Rightarrow [S^h(u) \geq S^h(v), h \in H^+ \cup \{0\}]$.

Proof. From Lemma 3 and the monotonicity property of function K , we get, for all $u, v \in M$,

$$[g_i(u) \geq g_i(v) \text{ for all } i = 1, \dots, n]$$

↓

$$\begin{aligned} S(u) &= K[\beta(u \succ^{\geq h} z), \beta(z \succ^{\geq h} u), \beta(u \succ^{\leq h} z), \beta(z \succ^{\leq h} u), z \in M - \{u\}, h \in H] \\ &\geq K[\beta(v \succ^{\geq h} z), \beta(z \succ^{\geq h} v), \beta(v \succ^{\leq h} z), \beta(z \succ^{\leq h} v), z \in M - \{v\}, h \in H] = S(v) \end{aligned}$$

and thus we proved point (a). Point (b) can be proved analogously. \square

Giving specific definitions to comprehensive score S , as well as comprehensive score S^h for a given $h \in H^+ \cup \{0\}$, several exploitation procedures can be proposed. In this paper, we propose two families of exploitation procedures: the weighted-fuzzy net flow score, the lexicographic-fuzzy net flow score.

5.2. The weighted-fuzzy net flow score

The weighted-fuzzy net flow score properly extends the net flow score procedure characterized in [1,2] for the basic case of a valued preference relation, and in [13] for the case of two grades of preference: “outranking” and “non-outranking”.

Because of multiple grades of intensity of preference and two ways of cumulating these grades (“at least” grade h and “at most” grade h), we ask the DM to introduce two sets of weights $w_h^{\geq} \geq 0$ and $w_h^{\leq} \geq 0$, $h \in H$, such that $w_{\alpha}^{\geq} = 0$ and $w_{\omega}^{\leq} = 0$. The weights w_{α}^{\geq} and w_{ω}^{\leq} get zero value because they correspond to default conclusions, not conveying any useful information.

On the basis of weights w_h^{\geq} and w_h^{\leq} , $h \in H$, a *comprehensive preference strength* of x over y can be calculated for each pair $(x, y) \in M^2$, as

$$P(x, y) = \sum_{h \in H} w_h^{\geq} \beta(x \succ^{\geq h} y) - \sum_{h \in H} w_h^{\leq} \beta(x \succ^{\leq h} y).$$

The following corollary concerns monotonicity of comprehensive preference strength $P(x, y)$ with respect to dominance relation D_G , whatever are the values of the weights.

Corollary 1. For all $u, v, z \in M$,

$$[g_i(u) \geq g_i(v) \text{ for all } i = 1, \dots, n] \Rightarrow [P(u, z) \geq P(v, z) \text{ and } P(z, u) \leq P(z, v)].$$

Proof. From Lemma 3 and the definition of comprehensive preference strength $P(x, y)$, we get

$$\begin{aligned} [g_i(u) \geq g_i(v) \text{ for all } i = 1, \dots, n] &\Rightarrow P(u, z) = \sum_{h \in H} w_h^{\geq} \beta(u \succ^{\geq h} z) - \sum_{h \in H} w_h^{\leq} \beta(u \succ^{\leq h} z) \\ &\geq \sum_{h \in H} w_h^{\geq} \beta(v \succ^{\geq h} z) - \sum_{h \in H} w_h^{\leq} \beta(v \succ^{\leq h} z) = P(v, z). \end{aligned}$$

Analogously, we get

$$\begin{aligned} [g_i(u) \geq g_i(v) \text{ for all } i = 1, \dots, n] &\Rightarrow P(z, u) = \sum_{h \in H} w_h^{\geq} \beta(z \succ^{\geq h} u) - \sum_{h \in H} w_h^{\leq} \beta(z \succ^{\leq h} u) \\ &\leq \sum_{h \in H} w_h^{\geq} \beta(z \succ^{\geq h} v) - \sum_{h \in H} w_h^{\leq} \beta(z \succ^{\leq h} v) = P(z, v) \end{aligned}$$

and this is what we had to prove. \square

Let us remark that the weights w_h^{\geq} and w_h^{\leq} permit tradeoffs between credibility degrees β of fuzzy preference relations relative to different grades, $\succ^{\geq h}$ and $\succ^{\leq h}$, respectively. For example, if $\succ^{\geq 0.5}$ means “at least mildly preferred” and $\succ^{\geq 0.25}$ means “at least weakly preferred”, and, moreover, the DM has specified $w_{0.5}^{\geq} = 3$ and $w_{0.25}^{\geq} = 1$, then the tradeoff means that, from the viewpoint of the comprehensive preference strength $P(x, y)$, the decrease of 1% in the credibility of the conclusion that x is “at least mildly preferred” to y can be compensated by the increase of 3% in the credibility of the conclusion that x is “at least weakly preferred” to y .

We compared preferences cumulated from different grades (“mild preference” and “weak preference”) but in the same direction (“at least”). Remark that if the directions were opposite, then even if the grades were the same, it would be justified to distinguish the weights corresponding to different directions, e.g. $w_{0.25}^{\geq} = 1$ and $w_{0.25}^{\leq} = 3$, which would mean that the decrease of 1% in the credibility of the conclusion that x is “at least weakly preferred” to y can be compensated by the increase of 3% in the credibility of the conclusion that x is “at most weakly preferred” to y . Moreover, if the values of the weights corresponding to different grades and directions, say $\succ^{\geq 0.5}$ and $\succ^{\leq 0.25}$, would be $w_{0.5}^{\geq} = 3$ and $w_{0.25}^{\leq} = 3$, this would mean that the decrease of 1% in the credibility of the conclusion that x is “at least mildly preferred” to y can be compensated by the increase of 1% in the credibility of the conclusion that x is “at most weakly preferred” to y .

Among infinitely many ways of fixing weights $w_h^{\geq} \geq 0$ and $w_h^{\leq} \geq 0$, $h \in H$, there are the following four main perspectives.

- (a) A *cautious (or pessimistic) perspective*, in which the weights assigned to weak conclusions are greater than the ones for strong conclusions. More precisely, the cautious perspective means that the comprehensive preference strength $P(x, y)$ is mainly based on credibility of preferences with relatively small degree h in relation $\succ^{\geq h}$, or large degree h in relation $\succ^{\leq h}$, i.e. for all $h \in H - \{\alpha\}$, $w_h^{\geq} \leq w_{h-1}^{\geq}$, and for all $h \in H - \{\omega\}$, $w_h^{\leq} \leq w_{h+1}^{\leq}$.

- (b) A *risky (or optimistic) perspective*, in which the weights assigned to strong conclusions are greater than the ones for weak conclusions. This perspective means that $P(x, y)$ relies mainly on the credibility of optimistic preferences, i.e. with relatively large degree h in relation $\succ \geq^h$, or small degree h in relation $\succ \leq^h$. Therefore, for all $h \in H - \{\alpha\}$, $w_h^\geq \geq w_{h-1}^\geq$, and for all $h \in H - \{\omega\}$, $w_h^\leq \geq w_{h+1}^\leq$.
- (c) A *positive perspective*, in which the positive conclusions prevail on the negative ones. In other words, this perspective means that the comprehensive preference strength $P(x, y)$ is more based on credibility of preferences $\succ \geq^h$ than on credibility of preferences $\succ \leq^h$. Thus, for all $h \in H - \{\alpha, \omega\}$, $w_h^\geq \geq w_h^\leq$.
- (d) A *negative perspective*, in which the negative conclusions get stronger weights than the positive ones. This means that $P(x, y)$ gives greater credit to credibility of preferences $\succ \leq^h$ than to $\succ \geq^h$. A DM with a negative perspective will choose the weights such that for all $h \in H - \{\alpha, \omega\}$, $w_h^\geq \leq w_h^\leq$.

On the basis of the comprehensive preference strength $P(x, y)$, we define the following weighted-fuzzy net flow score for every $x \in M$:

$$\begin{aligned}
 S^{\text{NF}}(x) &= \sum_{z \in M - \{x\}} P(x, z) - \sum_{z \in M - \{x\}} P(z, x) \\
 &= \sum_{\substack{h \in H \\ z \in M - \{x\}}} w_h^\geq [\beta(x \succ \geq^h z) - \beta(z \succ \geq^h x)] - \sum_{\substack{h \in H \\ z \in M - \{x\}}} w_h^\leq [\beta(x \succ \leq^h z) - \beta(z \succ \leq^h x)].
 \end{aligned}$$

Let us observe that the weighted-fuzzy net flow score is a particular case of the comprehensive score $S(x)$, enjoying a non-decreasing behavior with respect to $\beta(x \succ \geq^h z)$ and $\beta(z \succ \leq^h x)$ and a non-increasing one with respect to $\beta(z \succ \geq^h x)$ and $\beta(x \succ \leq^h z)$. Therefore, on the basis of the previous Theorem, we can immediately state that the weighted-fuzzy net flow score is monotonic with respect to evaluations on criteria $g_i \in G$, i.e. it is concordant with the dominance (or Pareto) principle.

From the weighted-fuzzy net flow score, as well as from any comprehensive score $S(x)$, one can build a complete preorder \succeq on M as follows: for all $u, v \in M$

$$S(u) \geq S(v) \iff u \succeq v.$$

The following corollary stemming out from the Theorem states that the weighted-fuzzy net flow score procedure is consistent with the dominance principle.

Corollary 2. For all $u, v \in M$,

$$[g_i(u) \geq g_i(v) \text{ for all } i = 1, \dots, n] \Rightarrow u \succeq v.$$

The final recommendation in ranking problems consists of the total preorder \succeq , while in choice problems, it consists of the maximal action(s) of \succeq .

5.3. The lexicographic-fuzzy net flow score

An alternative exploitation procedure can be proposed. It consists in the computation of the fuzzy net flow score $S_{\text{nf}}^h(x)$ for each grade $h \in H^+ \cup \{0\}$:

$$S_{\text{nf}}^h(x) = S_{++}^h(x) - S_{+-}^h(x) + S_{-+}^h(x) - S_{--}^h(x),$$

where

$$\begin{aligned}
 S_{++}^h(x) &= \sum_{z \in M - \{x\}} \beta(x \succ \geq^h z), & S_{+-}^h(x) &= \sum_{z \in M - \{x\}} \beta(z \succ \geq^h x), \\
 S_{-+}^h(x) &= \sum_{z \in M - \{x\}} \beta(z \succ \leq^h x), & S_{--}^h(x) &= \sum_{z \in M - \{x\}} \beta(x \succ \leq^h z).
 \end{aligned}$$

This builds up a complete preorder \succeq^h for each $h \in H^+ \cup \{0\}$, such that

$$u \succeq^h v \iff S_{\text{nf}}^h(u) \geq S_{\text{nf}}^h(v).$$

Let us observe that the fuzzy net flow score $S_{nf}^h(x)$ is a particular case of comprehensive score $S^h(x)$ relative to $h \in H^+ \cup \{0\}$, enjoying a non-decreasing behavior with respect to $\beta(x \succ^{\geq h} z)$ and $\beta(z \succ^{\leq h} x)$, and a non-increasing one with respect to $\beta(z \succ^{\geq h} x)$ and $\beta(x \succ^{\leq h} z)$.

To propose a final recommendation on the basis of the whole set of preorders \succeq^h for $h \in H^+ \cup \{0\}$, we propose to aggregate these preorders into the following two final rankings, denoted by \succeq^{LM} – called *leximax* – and \succeq^{lm} – called *leximin*.

To build the leximax final ranking, we use the following lexicographic approach:

$$u \triangleright^{LM} v \iff \exists h \in H^+ \cup \{0\} : \begin{cases} \forall k \in H^+ \cup \{0\} \text{ such that } k > h, u \succeq^k v \text{ and } v \succeq^k u, \\ \text{while } u \succeq^h v \text{ and not } v \succeq^h u \end{cases}$$

$$u \equiv^{LM} v \iff \forall h \in H^+ \cup \{0\} : u \succeq^h v \text{ and } v \succeq^h u,$$

where \triangleright^{LM} is the asymmetric part of \succeq^{LM} and \equiv^{LM} is the symmetric part of \succeq^{LM} .

To build the leximin final ranking we use the following lexicographic approach:

$$u \triangleright^{lm} v \iff \exists h \in H^+ \cup \{0\} : \begin{cases} \forall k \in H^+ \cup \{0\} \text{ such that } k < h, u \succeq^k v \text{ and } v \succeq^k u; \\ \text{while } u \succeq^h v \text{ and not } v \succeq^h u \end{cases}$$

$$u \equiv^{lm} v \iff \forall h \in H^+ \cup \{0\} : u \succeq^h v \text{ and } v \succeq^h u$$

where \triangleright^{lm} is the asymmetric part of \succeq^{lm} and \equiv^{lm} is the symmetric part of \succeq^{lm} .

Both leximax ranking and leximin ranking consider the set of preorders \succeq^h for $h \in H^+$ as providing consistent hierarchical information on the comprehensive graded preference relation. The difference between leximax ranking and leximin ranking is the following. Leximax ranking gives priority to preorders \succeq^h with high values of grade h and indeed, the preorders with lower values of h are only called to break ties from high h -grade preorders. On the contrary, leximin ranking gives priority to preorders \succeq^h with low values of grade h and, in turn, the preorders with higher values of h are only called to break ties from low h -grade preorders.

Analogously to the distinction between a risky perspective and a cautious one for the weighted-fuzzy net flow score, we can say that the leximax-fuzzy net flow score procedure corresponds to a risky (or optimistic) viewpoint, while the leximin-fuzzy net flow score procedure corresponds to a cautious (or pessimistic) viewpoint.

The final recommendation given by these procedures in ranking problems consists of the total preorder \succeq^{LM} or \succeq^{lm} ; in choice problems, it consists of the maximal action(s) of \succeq^{LM} or \succeq^{lm} .

Let us remark that one interesting feature of the leximax-fuzzy net flow score procedure and the leximin-fuzzy net flow score procedure is that they do not require any weight w_h^{\geq} and w_h^{\leq} , $h \in H$. Indeed, this is an advantage because the definition of such weights is always arbitrary to some extent.

The following corollary proves that both lexicographic procedures are monotonic with respect to the evaluations on criteria from G and, therefore, their results are consistent with the dominance principle.

Corollary 3. For all $u, v \in M$,

$$[g_i(u) \geq g_i(v) \text{ for all } i = 1, \dots, n] \Rightarrow [u \succeq^{LM} v \text{ and } u \succeq^{lm} v].$$

Proof. Let us suppose that for all $i = 1, \dots, n, g_i(u) \geq g_i(v)$. From the Theorem and because of the monotonic behavior of the aggregation $S_{nf}^h(x)$, we have, for all $h \in H^+ \cup \{0\}$,

$$S_{nf}^h(u) \geq S_{nf}^h(v).$$

This means that $u \succeq^h v$ for each grade $h \in H^+ \cup \{0\}$. Two cases are possible:

- (a) $S_{nf}^h(u) = S_{nf}^h(v)$ for each grade $h \in H^+ \cup \{0\}$. This means $u \succeq^h v$ and $v \succeq^h u$, for all grades. Thus, $u \equiv^{LM} v$ and $u \equiv^{lm} v$.
- (b) there exists at least one grade $h \in H^+ \cup \{0\}$ such that $u \succeq^h v$ and not $v \succeq^h u$. This situation can be described in two equivalent ways:

- (b1) there exists at least one grade $h \in H^+ \cup \{0\}$ such that for all $k > h$, $u \succeq^k v$ and $v \succeq^k u$, while $u \succeq^h v$ and not $v \succeq^h u$ and, therefore, $u \triangleright^{LM} v$;
- (b2) there exists at least one grade $h \in H^+ \cup \{0\}$ such that for all $k < h$, $u \succeq^k v$ and $v \succeq^k u$, while $u \succeq^h v$ and not $v \succeq^h u$ and, therefore, $u \triangleright^{lm} v$.

In both cases (a) and (b), we have $u \succeq^{LM} v$ and $u \succeq^{lm} v$. This concludes the proof. \square

6. Illustrative example

Let us consider a hypothetical case of a Belgian citizen wishing to buy a house in Poland for spending his holidays there. The selling agent approached by the customer wants to rank all the available houses to present them in a relevant order to the customer. Thereby, the latter is proposed first to have a look at a short list of seven houses (the reference actions), characterized by three criteria that seem important to the customer: (1) distance to the nearest airport, (2) price, and (3) comfort (Table 1). While the two first criteria are cardinal (expressed in km and in €, respectively), the last one is represented on a three-level ordinal scale (*Basic, Medium, Good*). The customer is then asked to give – even partially – his preferences on the set of seven proposed houses, in terms of a comprehensive graded preference relation.

The customer gives his preferences by means of the graph presented in Fig. 2, where a thin arc represents a weak preference, and a bold arc, a strong preference. Thereby, this is a comprehensive graded preference relation, with 2 positive grades of preference: weak and strong. One may observe that the customer’s preferences are allowed to be both not complete (there may exist pairs of houses without an arc; e.g., 5 and 4) and not always transitive (e.g., 6 is preferred to 4 and 4 is preferred to 3, with no preference between 6 and 3).

In order to build the PCT, differences of evaluations on cardinal criteria have been encoded in marginal graded preference relations (\succ_i^h), with $H_i = \{-1, -0.5, 0, 0.5, 1\}$, $i = 1, 2$. While comparing two alternatives, x and y , a difference in Distance criterion smaller (in absolute value) than 3 km is considered as non significant ($x \succ_1^0 y$ and $y \succ_1^0 x$). If the difference of distance is between 4 and 10 km in favor of x , then one weakly prefers

Table 1
Short list of the houses (reference actions)

Location of the house	Distance to the nearest airport (A1: [km])	Price (A2: [€])	Comfort (A3: [qualitative])
0: Poznań	3	60	Good
1: Kapalica	35	30	Good
2: Kraków	7	85	Medium
3: Warszawa	10	90	Basic
4: Wrocław	5	60	Medium
5: Malbork	50	50	Medium
6: Gdańsk	5	70	Medium

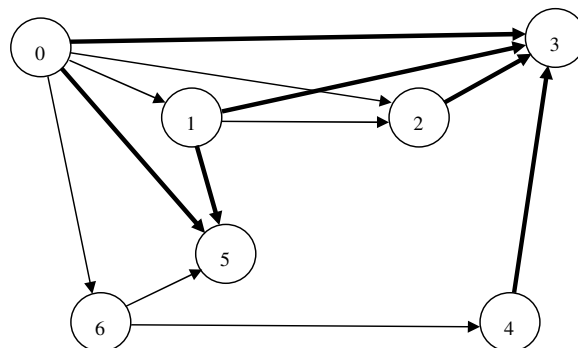


Fig. 2. Graph representation of the comprehensive graded preference relation in the set of reference actions.

Table 2
The PCT corresponding to customer's preferences on the set of reference actions

Pairs of reference actions (x, y)	h on A1: $x \succ_1^h y$	h on A2: $x \succ_2^h y$	Evaluations of (x, y) on A3	h on comprehensive preference relation: \succ^h
(0,0)	0	0	(Good; Good)	0
(0,1)	1	-0.5	(Good; Good)	0.5
(0,2)	0.5	0.5	(Good; Medium)	0.5
(0,3)	0.5	0.5	(Good; Basic)	1
(0,5)	1	0	(Good; Medium)	1
(0,6)	0	0	(Good; Medium)	0.5
(1,0)	-1	0.5	(Good; Good)	-0.5
(1,1)	0	0	(Good; Good)	0
(1,2)	-1	1	(Good; Medium)	0.5
(1,3)	-1	1	(Good; Basic)	1
(1,5)	1	0.5	(Good; Medium)	1
(2,0)	-0.5	-0.5	(Medium; Good)	-0.5
(2,1)	1	-1	(Medium; Good)	-0.5
(2,2)	0	0	(Medium; Medium)	0
(2,3)	0	0	(Medium; Basic)	1
(3,0)	-0.5	-0.5	(Basic; Good)	-1
(3,1)	1	-1	(Basic; Good)	-1
(3,2)	0	0	(Basic; Medium)	-1
(3,3)	0	0	(Basic; Basic)	0
(3,4)	-0.5	-0.5	(Basic; Medium)	-1
(4,3)	0.5	0.5	(Medium; Basic)	1
(4,4)	0	0	(Medium; Medium)	0
(4,6)	0	0	(Medium; Medium)	-0.5
(5,0)	-1	0	(Medium; Good)	-1
(5,1)	-1	-0.5	(Medium; Good)	-1
(5,5)	0	0	(Medium; Medium)	0
(5,6)	-1	0.5	(Medium; Medium)	-0.5
(6,0)	0	0	(Medium; Good)	-0.5
(6,4)	0	0	(Medium; Medium)	0.5
(6,5)	1	-0.5	(Medium; Medium)	0.5
(6,6)	0	0	(Medium; Medium)	0

x to y ($x \succ_1^{0.5} y$); finally, the preference is strong as soon as the difference is strictly greater than 10 km ($x \succ_1^1 y$). As far as the Price criterion is concerned, an absolute difference smaller than 10 leads to indifference ($x \succ_2^0 y$ and $y \succ_2^0 x$), and the weak (resp. strong) preference appears as soon as the difference is strictly greater than 10 (resp. 30). For the sake of simplicity, we have assumed in this example that the marginal graded preference relations are symmetric, e.g. $x \succ_i^{0.5} y \iff y \succ_i^{-0.5} x$. As the comfort criterion is ordinal, we have to take into account the pair of evaluations on this criterion instead of their difference. The pairwise comparison table (PCT) resulting from the above preference information is shown in Table 2.

The following 19 rules have been induced using the variable-consistency rule inducer [18], with a minimal consistency level $l = 0.85$ (within parentheses there are two numbers telling how many pairs are covered and how many pairs are supporting the corresponding rule, respectively; this allows to compute the confidence

Table 3
The set of new houses and their ranks in the final ranking

Location of the house	Distance to the nearest airport (A1: [km])	Price (A2: [€])	Comfort (A3: [qualitative])	Weighted fuzzy net flow score	Final rank
0': Kórnik	50	40	Medium	-0.1151	2
1': Rogalin	15	50	Basic	-5.4375	4
2': Lublin	8	60	Good	7.1971	1
3': Toruń	100	50	Medium	-1.6445	3

level). Moreover, to be concise in expression of rules, in the case of ordinal criterion A3, we are coding the comfort levels by 0=Basic, 1 = Medium, 2 = Good, and we are expressing the elementary conditions like $(x; y) \geq_3(2;1)$ or $(x; y) \leq_3(1;0)$, which means that on criterion A3 “x is at least Good and y is at most Medium” or “x is at most Medium and y is at least Basic”, respectively:

- if $x \succ_1^{\geq 1} y$ and $(x; y) \geq_3(2; 1)$, then $x \succ^{\geq 1} y$; (2;2)
- if $(x; y) \geq_3(1; 0)$, then $x \succ^{\geq 1} y$ (4;4)
- if $g_3(y) \geq 1$, then $x \succ^{\leq 0.5} y$ (26;24)
- if $g_3(y) \geq 2$, then $x \succ^{\leq 0.5} y$ (11;11)
- if $(x; y) \leq_3(1; 1)$, then $x \succ^{\leq 0.5} y$ (17;17)
- if $x \succ_1^{\geq 1} y$ and $x \succ_2^{\geq -0.5} y$, then $x \succ^{\geq 0.5} y$ (4;4)
- if $(x; y) \geq_3(2; 1)$, then $x \succ^{\geq 0.5} y$ (7;7)
- if $(x; y) \leq_3(1; 1)$, then $x \succ^{\leq 0} y$ (17;15)
- if $g_3(y) \geq 2$, then $x \succ^{\leq 0} y$ (11;10)
- if $g_3(x) \leq 0$, then $x \succ^{\leq 0} y$ (5;5)
- if $(x; y) \geq_3(1; 1)$, then $x \succ^{\geq 0} y$ (17;15)
- if $g_3(x) \geq 2$, then $x \succ^{\geq 0} y$ (11;10)
- if $g_3(y) \leq 0$, then $x \succ^{\geq 0} y$ (5;5)
- if $x \succ_1^{\leq -1} y$ and $x \succ_2^{\leq 0.5} y$, then $x \succ^{\leq -0.5} y$ (4;4)
- if $(x; y) \leq_3(1; 2)$, then $x \succ^{\leq -0.5} y$ (7;7)
- if $g_3(x) \geq 1$, then $x \succ^{\geq -0.5} y$ (26;24)
- if $(x; y) \geq_3(1; 1)$, then $x \succ^{\geq -0.5} y$ (17;17)
- if $x \succ_1^{\leq -1} y$ and $(x; y) \leq_3(1; 2)$, then $x \succ^{\leq -1} y$; (2;2)
- if $(x; y) \leq_3(0; 1)$, then $x \succ^{\leq -1} y$; (4;4)

Suppose that the selling agent has found four other houses, presented in Table 3, and would like to see how these houses will be ranked by the customer. He may use to this end the preference model of the customer in form of the above decision rules on the set of new houses. According to the Weighted-Fuzzy Net Flow Score exploitation procedure presented in Section 5, application of the rules on all possible pairs of the new houses results in fuzzy relations. Then, a weighted-fuzzy net flow score is computed, using the weights:

$$w_{-1}^{\geq} = 0 \quad w_{-.5}^{\geq} = 1/8 \quad w_0^{\geq} = 1/4 \quad w_{.5}^{\geq} = 1/2 \quad w_1^{\geq} = 1,$$

$$w_{-1}^{\leq} = 0 \quad w_{.5}^{\leq} = 1/16 \quad w_0^{\leq} = 1/8 \quad w_{-.5}^{\leq} = 1/4 \quad w_{-1}^{\leq} = 1/2.$$

Since for all $h \in H - \{\alpha\}, w_h^{\geq} \leq w_{h-1}^{\geq}$, and for all $h \in H - \{\omega\}, w_h^{\leq} \leq w_{h+1}^{\leq}$, we can conclude that the adopted perspective in weighting fuzzy relations $\beta(x \succ^{\geq h} y)$ and $\beta(x \succ^{\leq h} y)$ is risky (optimistic). Moreover, since for all $h \in H - \{\alpha, \omega\}, w_h^{\geq} \geq w_h^{\leq}$, we observe that the weighting perspective is also positive.

Application of the alternative leximax-fuzzy net flow score and leximin-fuzzy net flow score exploitation procedures leads to complete preorder \succeq^h in the set of new houses obtained by the (unweighted) fuzzy net flow score procedure on each grade $h \in \{0, 0.5, 1\}$. The leximax-fuzzy net flow score procedure is mainly based on the fuzzy net flow score for $h = 1$ and the corresponding complete preorder \succeq^1 . They are shown in Table 4. In fact, since one pair of actions (x, y) have the same fuzzy net flow score at grade $h = 1$, the next grade has to be investigated in order to break the tie and define the final ranking \succeq^{LM} of the new houses. The leximin-fuzzy

Table 4
The set of new houses and their ranks in the final ranking

Location of the house	Fuzzy net flow score (h = 0)	Fuzzy net flow score (h = .5)	Fuzzy net flow score (h = 1)	Leximax final rank (\succeq^{LM})	Leximin final rank (\succeq^{lm})
0': Kórnik	-0.882	0	0	2	2
1': Rogalin	-1	-2	-6	4	3
2': Lublin	3	6	6	1	1
3': Toruń	-1.118	-4	0	3	4

net flow score procedure is mainly based on the fuzzy net flow score for $h = 0$ and the corresponding complete preorder \succeq^0 . It defines the final ranking of the new houses \succeq^{lm} , which is also shown in Table 4.

In both cases, the Dominance-based Rough Set Approach gives a quite clear recommendation:

- for the *choice problem*, it suggests to *select house 2'* having the highest score,
- for the *ranking problem*, it suggests the *ranking* presented in the last column of Table 3 and in the last but one of Table 4, i.e.

$$(2') \rightarrow (0') \rightarrow (3') \rightarrow (1')$$

or in the last column of Table 4, i.e.

$$(2') \rightarrow (0') \rightarrow (1') \rightarrow (3').$$

Let us remark that the only difference due to various types of exploitation procedures is related to actions (1') and (3'). It is maybe not surprising that, in this example, the leximax-fuzzy net flow score procedure and the weighted-fuzzy net flow score procedure provide concordant results, because leximax corresponds to an optimistic attitude and the weights were also chosen according to the same attitude, while leximin corresponds to a pessimistic attitude.

7. Summary and conclusions

The presented methodology of multicriteria choice and ranking starts from acquisition of preference information, then it goes through analysis of this information using the dominance-based rough set approach (DRSA), followed by induction of decision rules from rough approximations of preference relations, and it ends with a recommendation of the best action in a set, or of a ranking of given actions.

The preference information is given by the DM in form of pairwise comparisons (or ranking) of some reference actions – comparison means specification of a grade of comprehensive preference of one reference action over another. The rough approximations of comprehensive graded preference relations prepare the ground for induction of decision rules with a warranted confidence. Upon acceptance of the DM, the set of decision rules constitutes the preference model of the DM, compatible with the pairwise comparisons of reference actions. It may then be used on a new set of actions, giving cumulated preference relations of “at least” grade h ($\succ^{\geq h}$) and “at most” grade h ($\succ^{\leq h}$), $h \in H$, with a corresponding credibility degree for each relation. Exploitation of these relations with a suitable procedure, such as the weighted-fuzzy net flow score procedure and the lexicographic-fuzzy net flow score procedure (and its two variants, the leximax and the leximin) proposed in this paper, leads to a complete preorder \succeq which is the recommended final ranking; the action(s) from the top of the ranking are the recommended best action(s).

From mathematical point of view, our proposal consists in extending a partial order of reference actions from $A' \subseteq A$ to an order of actions from A , while satisfying some consistency conditions.

Let us remark that the most interesting features of the proposed methodology are the following:

- the DM gives a preference information in very simple terms by means of a set of decision examples;
- the methodology is intelligible because each decision rule can be justified by decision examples supporting it.

We also proved that the credibilities of fuzzy preference relations $\beta(x \succ^{\geq h} y)$ and $\beta(x \succ^{\leq h} y)$ satisfy an important property of monotonicity with respect to evaluations of actions (dominance principle) and that this monotonicity is maintained in the exploitation procedures satisfying some minimal conditions of monotonicity with respect to fuzzy preference relations $\beta(x \succ^{\geq h} y)$ and $\beta(x \succ^{\leq h} y)$. More precisely, we proved that the weighted-fuzzy net flow score procedure, the leximax-fuzzy net flow score procedure and the leximin-fuzzy net flow score procedure are monotonic with respect to dominance relation, which is a fundamental property within multiple criteria decision analysis.

Acknowledgements

The research of the second author has been supported by the Italian Ministry of University and Scientific Research (MUR). The third author wishes to acknowledge financial support from the Ministry of Science and Higher Education.

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