Games where you can play optimally with finite memory

Patricia Bouyer\textsuperscript{1}  Stéphane Le Roux\textsuperscript{1}  Youssouf Oualhadj\textsuperscript{2}  Mickael Randour\textsuperscript{3}  Pierre Vandenhove\textsuperscript{3}

\textsuperscript{1}LSV – CNRS & ENS Paris-Saclay  \textsuperscript{2}LACL – UPEC  \textsuperscript{3}F.R.S.-FNRS & UMONS – Université de Mons

September 18, 2019

*Highlights of Logic, Games and Automata 2019*
Games where you can play optimally with finite memory

Patricia Bouyer\textsuperscript{1}  Stéphane Le Roux\textsuperscript{1}  Youssouf Oualhadj\textsuperscript{2}

Mickael Randour\textsuperscript{3}  Pierre Vandenhove\textsuperscript{3}

\textsuperscript{1}LSV – CNRS & ENS Paris-Saclay  \textsuperscript{2}LACL – UPEC

\textsuperscript{3}F.R.S.-FNRS & UMONS – Université de Mons

September 18, 2019

*Highlights of Logic, Games and Automata 2019*
Games where you can play optimally with finite memory

A sequel to the critically acclaimed blockbuster by Gimbert & Zielonka

Patricia Bouyer\textsuperscript{1}  Stéphane Le Roux\textsuperscript{1}  Youssouf Oualhadj\textsuperscript{2}
Mickael Randour\textsuperscript{3}  Pierre Vandenhove\textsuperscript{3}

\textsuperscript{1}LSV – CNRS & ENS Paris-Saclay  \textsuperscript{2}LACL – UPEC
\textsuperscript{3}F.R.S.-FNRS & UMONS – Université de Mons

September 18, 2019

Abstract. Reactive systems are often modelled as two person antagonistic games where one player represents the system while his adversary represents the environment. Undoubtedly, the most popular games in this category are parity games and their cousins (Rabin, Streett and Muller games) or games with other types of payments, like energy games. They are used extensively in model checking and verification. The complementary player may represent e.g. a player in a two player game or an attacker in a security protocol. To model reactive systems, we often need infinite memory in the player representing the system

\textcopyright\ 2019 Highlights of Logic, Games and Automata
Two-player turn-based zero-sum games on graphs

We consider finite arenas with vertex colors in \( C \). Two players: circle (\( P_1 \)) and square (\( P_2 \)). Strategies \( C^* \times V_i \rightarrow V \).

From where can \( P_1 \) ensure to reach \( v_6 \)?

How complex is his strategy?

Memoryless strategies (\( V_i \rightarrow V \)) always suffice for reachability (for both players).
When are memoryless strategies sufficient to play optimally?

Virtually always for simple winning conditions!

Examples: reachability, safety, Büchi, parity, mean-payoff, energy, total-payoff, average-energy, etc.

Can we characterize when they are?

Yes, thanks to Gimbert and Zielonka [GZ05].
Gimbert and Zielonka’s characterization

Memoryless strategies suffice for a preference relation $\preceq$ (and the induced winning conditions) if and only if

1. it is monotone,
   - Intuitively, stable under prefix addition.

2. it is selective.
   - Intuitively, stable under cycle mixing.

Example: reachability.
Gimbert and Zielonka’s corollary

If $\sqsubseteq$ is such that

- in all $\mathcal{P}_1$-arenas, $\mathcal{P}_1$ has an optimal memoryless strategy,
- in all $\mathcal{P}_2$-arenas, $\mathcal{P}_2$ has an optimal memoryless strategy (i.e., for $\sqsubseteq^{-1}$),

then both players have optimal memoryless strategies in all two-player arenas.

*Extremely useful in practice!*
Going further: finite memory

Memoryless strategies do not always suffice!

Examples:

- Büchi for \( v_1 \) and \( v_3 \) → finite (1 bit) memory.
- Mean-payoff (average weight per transition) \( \geq 0 \) on all dimensions → infinite memory!

We need a GZ equivalent for finite memory!

\[ \sim \text{For combinations, see [LPR18].} \]
A partial counter-example (lifting corollary)

Let $C \subseteq \mathbb{Z}$ and the winning condition for $\mathcal{P}_1$ be

$$\overline{TP}(\pi) = \infty \lor \exists \infty i \in \mathbb{N}, \sum_{i=0}^{n} c_i = 0$$

Both 1-player variants are finite-memory determined.

**Hint:** non-monotony is a bigger threat in two-player games.

In one-player games, *finite* memory may help.

But the two-player one is not!

$\implies \mathcal{P}_1$ needs infinite memory to win.
A new hope

Our goal

GZ-like characterization for finite-memory strategies.

Two tricks:

1. **Monotonicity as hypothesis** (cf. counter-example).
2. From selectivity to $S$-selectivity and cyclic covers for arenas.

   $\Rightarrow$ Intuitively, selectivity *modulo a memory skeleton*.

We obtain a natural GZ-equivalent for FM determinacy, including the lifting corollary (1-p. to 2-p.)!

Still some elements to flesh out.

$\Rightarrow$ Preprint writing in progress.
Thank you! Any question?
Hugo Gimbert and Wieslaw Zielonka.
Games where you can play optimally without any memory.

Stéphane Le Roux, Arno Pauly, and Mickael Randour.
Extending finite-memory determinacy by Boolean combination of winning conditions.