SOS approximants for Potts crystal shapes

Joël De Coninck\textsuperscript{a}, Roman Kotecký\textsuperscript{b}, Lahoussine Laanait\textsuperscript{c} and Jean Ruiz\textsuperscript{d}

\textsuperscript{a}Université de Mons-Hainaut, Faculté des Sciences, 19 avenue Maistriau, 7000 Mons, Belgium
\textsuperscript{b}Department of Theoretical Physics, Charles University, V Holešovičkách 2, 180 00 Praha 8, Czechoslovakia
\textsuperscript{c}and Center for Theoretical Study, Charles University, Ovocný trh 3, 116 36 Praha 1, Czechoslovakia
\textsuperscript{d}Ecole Normale Supérieure, Takaddoum, Rabat, Morocco
\textsuperscript{e}Centre de Physique Théorique, CNRS Luminy Case 907, 13288 Marseille Cedex 9, France

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An approximation made when replacing the orientation dependent Potts model surface tension by the SOS one is rigorously estimated. As a result, an SOS model may be used for an explicit and precise construction of crystal and meniscus shapes for Potts models at low temperatures.

1. Introduction

The general theory of interfaces between coexisting phases of matter based on its microscopic description has been intensively studied in recent years.

Versions of solid-on-solid models [1] are often used for an approximate description of an interface. For the 2-d Ising ferromagnetic model, whose surface tension $\sigma_{\text{Ising}}(\theta)$ as a function of the orientation $\theta$ is exactly known [2], one can compare it with the SOS surface tension $\sigma_{\text{SOS}}(\theta)$. Comparing the explicit expressions, it turns out that the latter is, at low temperatures, a good approximation of the former [3]. Indeed, one has a direct control of this approximation with the help of cluster expansions [4]. Some weaker results about a similar approximation in the three-dimensional case were also obtained [5].

All this means that an interface between two phases of the Ising type may be, at low enough temperature, viewed as a simple line of separation without overhangs.

However, the situation becomes more complicated when several phases can
coexist. The interface might be in principle thick, i.e., it might contain a layer of a third phase between the considered two phases [6]. If this were the case, the interface would have to be modeled by two (or several) random walks.

Our aim in the present paper is to discuss the SOS approximation in a simple situation, where the main features of the problem are captured. Even though we restrict ourselves to the Potts model, the results can be reformulated (and proven) also for Blume–Emery–Griffiths, Clock, and other models.

In particular, we study a connection between the orientation dependent surface tension \( \sigma_\text{potts}(\theta) \) and the corresponding \( \sigma_\text{SOS}(\theta) \). This eventually allows us to consider an SOS approximation of a Potts crystal and a Potts meniscus and evaluate its precision at low temperatures.

In section 2 we introduce the SOS approximants for Potts surface tensions and present rigorous bounds on the errors and their dependence on temperature and number of states. Sections 3 and 4 are devoted to the implications concerning the use of an SOS model for the construction of Potts crystal shapes and for Potts meniscus shapes, respectively. Technical details (the proofs) are presented in two appendices.

2. SOS approximants for interfacial tensions

In this section, we show how Potts surface tensions can be approximated by SOS ones.

First we recall some definitions.

Let \( \Lambda \) be a finite set of points, of cardinality \( |\Lambda| \), in the lattice \( \mathbb{Z}^2 \). With each point \( i \) of this set we associate a spin \( \sigma_i \) that takes on its values in the set \( \{1, 2, \ldots, q\} \). We use \( \Omega_1 = \{1, 2, \ldots, q\}^\Lambda \) to denote the set of all configurations \( \sigma_1 \) on \( \Lambda \).

The Potts model is then defined by the Hamiltonian

\[
H(\sigma_1) = -\sum_{\langle i,j \rangle \subset \Lambda} J(\delta(\sigma_i, \sigma_j) - 1),
\]

where \( J \) is a positive constant and the sum is over pairs of nearest neighbours and \( \delta(\sigma_i, \sigma_j) \) is the Kronecker delta. The partition function (with free boundary conditions) is given by

\[
Z(\Lambda, \beta) = \sum_{\sigma_1 \in \Omega_1} e^{-\beta H(\sigma_1)}.
\]

Let \( \partial \Lambda \) be the boundary of \( \Lambda \),

\[
Z(\partial \Lambda, \beta) = \sum_{\sigma_1 \in \Omega_1} e^{-\beta H(\sigma_1)}.
\]
\[ \partial \Lambda = \{ i \in \Lambda : \exists j \in \Lambda^c = \mathbb{Z}^2 \setminus \Lambda, d(i, j) = 1 \} , \]

where \( d \) is the euclidean distance, and let \( \tilde{\sigma} \) be a configuration on \( \mathbb{Z}^2 \). We define the partition function with boundary condition \( \tilde{\sigma} \) as the sum of Boltzmann factors over the configurations \( \sigma_i \in \Omega_1 \) which coincide with \( \tilde{\sigma} \) on \( \partial \Lambda \),

\[ Z(\Lambda, \beta) = \sum_{\sigma_i \in \Omega_1} e^{-\beta H(\sigma_i)} , \]

Whenever \( q \) is large enough, the Potts model, in dimension two (or more), exhibits a first-order phase transition at some (inverse) transition temperature \( \beta_t \) where the mean energy is discontinuous. At this point all \( q \) ordered phases coexist with the disordered one, while above \( \beta_t \) only ordered phases coexist [7].

We denote by \( a \in \{1, \ldots, q\} \) any ordered boundary condition, i.e. \( \tilde{\sigma}_i = a \) for all \( i \). The thermodynamic limit of finite-volume Gibbs states

\[ \mu^a_1 = (Z^a(\Lambda, \beta))^{-1} \exp[-\beta H(\sigma_1)] \prod_{i\in\partial \Lambda} \delta(\sigma_i, a) \]

yields translation invariant measures, the ordered phases, while the thermodynamic limit of

\[ \mu^f_1 = (Z^f(\Lambda, \beta))^{-1} \exp[-\beta H_f(\sigma_1)] \]

yields another translation invariant measure, the disordered phase [8]. In dimension 2 the \( q \) different ordered phases actually coexist [9], whenever \( q > 4 \) and \( \beta > \log(\sqrt{q} + 1) \).

To define the orientation-dependent surface tension (interfacial free energy) between two phases, one usually enforces these phases to coexist by considering asymmetric boundary conditions that take into account the angular dependence. More precisely, we consider the following box:

\[ \Lambda = \{ x = (x^1, x^2) \in \mathbb{Z}^2 \mid 0 \leq x^1 \leq N, -M \leq x^2 \leq M \} . \]

For any unit vector \( n = (-\sin \theta, \cos \theta) \) we use \( \tilde{\sigma} \equiv (a, b, n) \) or \( (a, b, \theta) \), \( a, b \in \{1, 2, \ldots, q\} \), to denote the configuration for which \( \sigma_i = a \) if \( i \) is above or on the line that is orthogonal to \( n \) and passes through the origin, and \( \sigma_i = b \) for the remaining sites.

We introduce the ratio

\[ \Xi_{\text{Potts}}(\Lambda, \theta, \beta) = e^{\beta J} \frac{Z^{(a, b, \theta)}(\Lambda, \beta)}{Z^a(\Lambda, \beta)} , \quad (2.2) \]
whose value is independent of the choice of \( a \) and \( b (a \neq b) \) due to the symmetry of the model. Notice that already when choosing the normalization (the denominator in (2.2)), we are tacitly using the symmetry between all ordered phases. Namely, \( Z^a(A, \beta) = Z^b(A, \beta) \) for any \( A \). If there were no such symmetry, we should consider an other normalization and also, it would be more appropriate to shift the interface line into a more symmetric position (say, passing through the point \( (\frac{1}{2} N, 0) \)). See refs. [10,11] for a discussion of surface tension in a general situation. We suppose also that for a given \( \theta \) and \( N \), the height \( M \) of the rectangle is chosen sufficiently large for the interface line to hit the opposite side of the rectangle.

The surface tension between any two ordered phases, with respect to an interface of orientation \( \theta \), is defined as the thermodynamic limit

\[
\sigma_{\text{Potts}}(\theta, \beta) = -\frac{1}{\beta} \lim_{N \to \infty} \lim_{M \to \infty} \frac{\cos \theta}{N} \log \Xi_{\text{Potts}}(A, \theta, \beta).
\] (2.3)

This limit exists for any \( \theta \) such that \( |\theta| < \frac{1}{2} \pi \), it satisfies the symmetry condition \( \sigma(\frac{1}{2} \pi - \theta, \beta) = \sigma(\theta, \beta) \), and the ratio \( \sigma(\theta, \beta) / \cos \theta \) is an even convex function of the \( \tan \theta \) [11].

It is convenient to introduce also the following limit over infinite cylinders:

\[
\Xi_{\text{Potts}}(N, \theta, \beta) = \lim_{M \to \infty} \Xi_{\text{Potts}}(A, \theta, b).
\] (2.4)

The limit exists because \( \Xi(A, \theta, \beta) \) is an increasing function of \( M \) bounded from above [11].

Let us pass now to the description of the corresponding SOS model. Let \( h_0, h_1, \ldots, h_N \) be a collection of integers describing the position of the interface. The energy of the interface is assumed to be proportional to its length. Namely,

\[
E(h_0, \ldots, h_N) = J \sum_{i=1}^{N} (1 + |h_i - h_{i-1}|).
\]

We shall consider interfaces with an angle \( \theta \) with respect to the horizontal plane with the corresponding partition function

\[
\Xi_{\text{SOS}}(N, \theta, \beta) = \sum_{h_i \in \mathbb{Z}} e^{-\beta E(h_0, \ldots, h_N)} \delta(h_0, 0) \delta(h_N, [N \tan \theta]),
\] (2.5)

where \( [N \tan \theta] \) is the integer part of the number \( N \tan \theta \). The free energy of the model is
\[
\sigma_{\text{SOS}}(\theta, \beta) = -\frac{1}{\beta} \lim_{N \to \infty} \frac{\cos \theta}{N} \log \Xi_{\text{SOS}}(N, \theta, \beta).
\] (2.6)

For simplicity we shall hereafter absorb \(J\) in the temperature, i.e. put \(J = 1\).

**Theorem 1.** Whenever \(\beta\) is large enough, more precisely

\[\beta > 2 \log(q - 1) + 24 \log 3,\]

then

\[\sigma_{\text{SOS}}(\theta, \beta - \delta) \leq \sigma_{\text{Potts}}(\theta, \beta) \leq \sigma_{\text{SOS}}(\theta, \beta + \delta)\]

for any \(-\frac{1}{2} \pi < \theta < \frac{1}{2} \pi\) and any

\[\delta \geq 2 \log \left( \frac{1}{1 - 3(q - 1)^{1.12} e^{-\beta/24}} \right)\]

and

\[\delta \geq c(q - 1)^2 e^{-3\beta}\]

with a fixed constant \(c\).

**Proof.** See appendix A.

Notice that the bounds are uniform \(\theta\) with the error of order \(e^{-\text{constant}\ \beta}\) for \(\beta\) large. This clearly shows that for sufficiently large \(\beta\), the estimate

\[\sigma_{\text{Potts}}(\theta, \beta) = \sigma_{\text{SOS}}(\theta, \beta)\]

is very precise.

Since, on the other hand, the shape of a crystal is given by the Wulff construction that depends only on the values of the surface tension, it is tempting to consider the SOS approximation of a Potts crystal.

### 3. SOS approximants for crystal shapes

Let us first examine the connection between the surface tension and the corresponding crystal shape given by the Wulff construction, or its modern formulation presented by Andreev [12]. Let
\[ y = -\tan \theta, \]
\[ \tau(y, \beta) = \sigma(\theta, \beta)/\cos \theta. \]

The Legendre transform of this function is

\[ \tau^*(x, \beta) = -\sup_y [xy - \tau(y, \beta)]. \]

According to the Andreev's result, the curve

\[ \lambda y = \tau^*(-\lambda x, \beta) \]

yields directly the equilibrium crystal shape. Here \( \lambda \) is a Lagrange multiplier associated with the volume fixing constraint.

The function \( \tau^*(x, \beta) \) has a statistical mechanical equivalent \( \phi(x, \beta) \) which we introduce next. Let, as before, \( y = -\tan \theta \) and for both \( \Xi_{\text{Potts}}(N, \theta, \beta) \) and \( \Xi_{\text{SOS}}(N, \theta, \beta) \) define their Laplace transform by

\[ \Theta_{\text{Potts}}(N, x, \beta) = \sum_{N \tan \theta \in \mathcal{L}} e^{-\beta N x \tan \theta} \Xi_{\text{Potts}}(N, \theta, \beta), \]
\[ \Theta_{\text{SOS}}(N, x, \beta) = \sum_{N \tan \theta \in \mathcal{L}} e^{-\beta N x \tan \theta} \Xi_{\text{SOS}}(N, \theta, \beta). \]

In other words, we are introducing a "grand canonical" ensemble of interfaces, where the condition of a fixed slope \( \theta \) is replaced by an "external field" \( x \) (cf. [3,4,11,13]). We also introduce the corresponding free energies

\[ \phi_{\text{Potts}}(x, \beta) = -\frac{1}{\beta} \lim_{N \to \infty} \frac{1}{N} \log \Theta_{\text{Potts}}(N, x, \beta). \]
\[ \phi_{\text{SOS}}(x, \beta) = -\frac{1}{\beta} \lim_{N \to \infty} \frac{1}{N} \log \Theta_{\text{SOS}}(N, x, \beta). \]

Using a standard steepest descent method, one gets

\[ \phi_{\text{Potts}}(x, \beta) = -\sup_y [xy - \tau_{\text{Potts}}(y, \beta)] \]

and

\[ \phi_{\text{SOS}}(x, \beta) = -\sup_y [xy - \tau_{\text{SOS}}(y, \beta)]. \]

inside the domain of \( \tau^* \), i.e. for \( |x| < x_m = \lim_{y \to x} \tau(y, \beta)/y \), since the functions \( \tau(y, \beta) \) equal the infimum over \( N \) of \(-1/\beta N \log \Xi(N, \theta, \beta) \) [11].
By identification with \( \tau^* \), we infer that the Wulff shape may be viewed as the thermodynamic limit of a sequence of Laplace transforms.

The importance of this result lies in the fact that the \( \phi_{\text{SOS}} \) function can be easily computed in an explicit way:

\[
\phi_{\text{SOS}}(x, \beta) = 1 - \frac{1}{\beta} \log \frac{\sinh \beta}{\cosh \beta - \cosh \beta x}.
\]

Actually, for angles \( \theta > \frac{1}{4}\pi \) it is natural to approximate by an SOS model with the role of coordinate axes interchanged. Practically it means that we consider a symmetrized function \( \tilde{\phi}_{\text{SOS}}(x, \beta) \) defined by \( \tilde{\phi}_{\text{SOS}}(x, \beta) = \phi_{\text{SOS}}(x, \beta) \) for all \( |x| \leq x_0 \), where \( x_0 \) is the unique positive solution of the equation

\[
\phi_{\text{SOS}}(x, \beta) = x.
\]

For the remaining values of \( |x| \leq \phi_{\text{SOS}}(0, \beta) \) the function \( \tilde{\phi}_{\text{SOS}} \) is extended by symmetry.

With the help of our bounds on the surface tension we get:

**Theorem 2.** Under the hypothesis of theorem 1 one has

\[
\tilde{\phi}_{\text{SOS}}(x, \beta - 2\beta \delta_-) \leq \phi_{\text{Potts}}(x, \beta) \leq \tilde{\phi}_{\text{SOS}}(x, \beta + 2\beta \delta_+)
\]

for all \( |x| \leq \phi_{\text{SOS}}(0, \beta) \).

**Proof.** See appendix B.

This theorem states that, for sufficiently low temperatures, we have a good estimate of the Wulff shape of a Potts crystal in terms of the known Wulff shape of an SOS crystal.

### 4. SOS approximants for Potts meniscus

For \( \beta > \beta_c \), it is expected that with boundary conditions \( a \) on top and \( b \) on bottom a fixed amount of \( c \) will be located within a meniscus as indicated in fig. 1.

\( ^{*1} \) Notice that \( \phi_{\text{SOS}}(x, \beta) \) is a decreasing function of \( x \) for \( x \geq 0 \) and \( \phi_{\text{SOS}}(0, \beta) > 0 \) for \( \beta > \log (\sqrt{2} + 1) \).
This meniscus is characterized by two angles \( \theta_1 \) and \( \theta_2 \) that obey the Herring relations expressed in terms of corresponding surface tensions and their derivatives:

\[
\begin{align*}
\sigma_{ac}(\theta_1) \cos \theta_1 - \sigma'_{ac}(\theta_1) \sin \theta_1 + \sigma_{bc}(\theta_2) \cos \theta_2 - \sigma'_{bc}(\theta_2) \sin \theta_2 &= \sigma_{ab}(0), \\
\sigma_{ac}(\theta_1) \sin \theta_1 - \sigma'_{ac}(\theta_1) \cos \theta_1 + \sigma_{bc}(\theta_2) \sin \theta_2 - \sigma'_{bc}(\theta_2) \cos \theta_2 &= 0.
\end{align*}
\]

Let us first point out the connection between these relations and the Wulff construction. One starts with

\[
\phi(x) = -\sup_y \left[ yx - \tau(y) \right].
\]

If \( \tau \) is differentiable, the supremum is obtained for

\[
\phi(x) = -y^*_x x + \tau(y^*_x),
\]

where \( y^*_x \) is such that

\[
x = \tau'(y^*_x),
\]

or equivalently

\[
x = -\sigma'(\theta^*_x) \cos \theta^*_x - \sigma(\theta^*_x) \sin \theta^*_x.
\]

At this point, one has

\[
\phi(x) = \cos \theta^*_x \sigma(\theta^*_x) - \sin \theta^*_x \sigma'(\theta^*_x).
\]

Hence, the Herring relations become

\[
\phi_{ac}(x_1) + \phi_{bc}(x_2) = \phi_{ac}(0), \quad x_1 = x_2,
\]

(4.1)
using that $\phi_{ac}(0) = \tau_{ac}(0) = \sigma_{ac}(0)$ by convexity and symmetry of the function $\tau(y, \beta)$ [11]. We thus recover the double Wulff construction (see fig. 2) already pointed out in ref. [13].

As a result, a meniscus that obeys the Herring relations (4.1) is just an intersection of the two Wulff shapes with their corresponding center points on the same vertical line and at a distance $\phi_{ac}(0)$ from each other.

As a matter of fact, it turns out directly from our theorem 2 that a meniscus of the Potts model, at sufficiently low temperatures, can be well approximated by a meniscus constructed with help of SOS models.

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Appendix A. Proof of theorem 1

Proof of the lower bound

With a configuration of heights \( \{h_0, \ldots, h_N\} \) entering in the definition of the partition function (2.5) we associated, in a standard manner, an SOS-path \( \Gamma \), constructed by drawing the horizontal unit segments at heights \( h_i \) whose centers cut the vertical lines \( x_i = 0, \ldots, N \), and by joining the endpoints of these segments by vertical ones.

We use \( \mathcal{F}_{\text{SOS}} \) to denote the set of all SOS-paths. The partition function (2.5) can be written

\[
\sum_{\mathcal{F}_{\text{SOS}} \ni \Gamma : A \rightarrow B} e^{-\beta|\Gamma|},
\]

where the sum is over paths with starting point \( A = (0,0) \) and ending point \( B = (N, \lfloor N \tan \theta \rfloor) \), and where \( |\Gamma| \) is the length of the path \( \Gamma \) (recall that \( J = 1 \)).

Consider now a configuration \( \sigma \) of the Potts model, that coincides with \( \sigma = (a, b, \theta) \) on \( (\mathbb{Z}^2 \setminus \Lambda) \cup \partial \Lambda \). To each pair of nearest neighbours \((i,j)\) such that \( \sigma_i \neq \sigma_j \), we associate the unit segment passing through the centre of the segment \( ij \) and perpendicular to it. The set of such segments is then split up into maximally connected components to be called contours. Among them, there is one which is infinite, we call it interface of the configuration \( \sigma \) and denote it by \( \Gamma(\sigma) \). We shall call a set \( \Gamma \) an interface whenever there exists a configuration \( \sigma \) with \( \Gamma = \Gamma(\sigma) \), and we shall use \( |\Gamma| \) to denote the length of the part of the interface inside the box \( V \subset \mathbb{R}^2 \),

\[
V = \{x = (x^1, x^2) \in \mathbb{R}^2 \mid 0 \leq x^1 \leq N, -M \leq x^2 \leq M\}.
\]

Any interface \( \Gamma \) splits up the set \( \Lambda \) into three parts (union of connected components), to be denoted \( U, D, I \), such that \( U \) is above \( \Gamma \), \( D \) is below and the remaining part \( I \) is inside \( \Gamma \).

As a consequence we have

\[
Z^{(a,b,\theta)}(\Lambda, \beta) = \sum_{\Gamma} \sum_{\Gamma(\sigma) \ni \Gamma} e^{-\beta|\Gamma|+1} Z^{a}(U, \beta) Z^{b}(D, \beta) \prod_{m-1} Z^{m}(I_m, \beta).
\]

Here the second sum is over configurations whose only contour is \( \Gamma \). \( I_m \) is, for a given \( \sigma \), a union of those components of \( I \) for which \( \sigma_i = m \) for all \( i \in \partial I_m \), \( \bigcup_{m-1} I_m = I \). Using the symmetry of the partition function and the fact that each contribution to it is positive, we get the bound

\[
(A.1)
\]
\[ Z^{(a,b,\theta)}(\Lambda, \beta) \leq \sum_{\Gamma} \sum_{f \cap \gamma} c^{-\beta(|f'|+1)} Z^{a}(\Lambda, \beta). \]

Hence

\[ \Xi(A, \theta, \beta) = \Xi(N, \theta, \beta) \leq \sum_{\Gamma} c^{-\beta|\Gamma|} \sum_{f \cap \gamma} 1 = \sum_{\Gamma} \phi(\Gamma). \quad \text{(A.2)} \]

Here the last sum is over all interfaces \( \Gamma \) in the strip \( \{x = (x^1, x^2) \in \mathbb{R}^2 \mid 0 \leq x^1 \leq N \} \). Taking into account that each component of \( I \) is bordered by at least two horizontal bonds of \( \Gamma \), each of them shared by at most two components, we can bound the number of configurations contributing to \( \phi(\Gamma) \) and get

\[ \phi(\Gamma) \leq (q-1)^{l_h(\Gamma)} c^{-\beta|\Gamma|}, \quad \text{(A.3)} \]

where \( l_h(\Gamma) \) is the number of horizontal bonds of \( \Gamma \).

Our aim now is to compare (A.2) with

\[ \Xi_{\text{SOS}}(N, \theta, \beta') = \sum_{\Gamma \in I^{\text{SOS}}} e^{-\beta'|\Gamma|}, \quad \text{(A.4)} \]

i.e. to get an upper bound of the right hand side of (A.2). The difference between the two quantities comes from the fact that in the former the contour may contain overhangs or nonempty interiors \( I \). To isolate these excitations we decompose the interface in the following way [4] (see fig. 3).

Let us consider the set of vertical lines \( x^1 = 0, 1, \ldots, N \). Let \( m_i \) be the ordered set of integers where the vertical line \( x^1 = m_i \) intersects the interface.

![Fig. 3. Decomposition of the interface.](image)
only in 1 point \((m_i - 1 < m_i \text{ for all } i = 1, \ldots, n)\). We then divide the interface \(\Gamma\) into pieces \(\Gamma_1, \ldots, \Gamma_n\), such that \(\Gamma_i\) is the part of the interface between the vertical lines \(x^1 = m_{i-1}\) and \(x^1 = m_i\). Let \(X_i\) be the position of the interface at \(m_i : X_i = \{m_i, h_{m_i}\}, \ i = 1, \ldots, n - 1\). \(X_0 = \{0, -\frac{1}{2}\}\); \(X_n = \{N, [N \tan \theta] - \frac{1}{2}\}\). Then

\[
\sum_{\Gamma} \phi(\Gamma) = \sum_{n=1}^{N} \left( \sum_{i=1}^{n} \prod_{j=i-1}^{n-1} \sum_{\Gamma_{ij}: X_i = X_{j+1}} \phi(\Gamma_{ij}) \right). \tag{A.5}
\]

Whenever \(m_i - m_{i-1} = \Delta m = 1\), \(\Gamma_i\) is of the SOS type and

\[
\phi(\Gamma_i) = e^{-\beta |\tau|}.
\]

If not, \(m_i - m_{i-1} \geq 2\) and by construction the length \(l_h(\Gamma_i)\) of the horizontal part of \(\Gamma_i\) satisfies

\[
l_h(\Gamma_i) - 1 \geq 2|m_i - m_{i-1} - 1|. \tag{A.6}
\]

We shall now prove that for such \(\Gamma_i\)’s one can find \(\delta\) such that

\[
\sum_{\Gamma_{ij}: X_i = X_{j+1}} \phi(\Gamma_{ij}) \lesssim e^{-(\beta - \delta)l_{min}(X_{i-1}, X_i)}. \tag{A.7}
\]

Here we use \(*\) to indicate that the sum is only over \(\Gamma_i\) that are not of SOS type, i.e. such that

\[
l_h(\Gamma_i) \geq 2\Delta m - 1
\]

and introduce the minimal distance

\[
l_{min}(X_{i-1}, X_i) = m_i - m_{i-1} + |h_{m_i} - h_{m_{i-1}}| = \Delta m + \Delta h.
\]

The inequality (A.7) together with (A.5) implies the desired result. Namely, the right-hand side of (A.5) is less than (A.4) with \(\beta' = \beta - \delta\). This shows that uniformly in \(N\) and \(\theta\) we have

\[
\frac{1}{\beta N} \log \Xi_{\text{poits}}(N, \theta, \beta) \lesssim \frac{1}{\beta N} \log \Xi_{\text{SOS}}(N, \theta, \beta - \delta).
\]

Hence, it remains to prove the bound (A.7). Using \(l_v(\Gamma)\) to denote the vertical part of \(\Gamma\) and applying (A.3) and (A.6) we get
\[ \sum_{I_j : X_{i-1} \rightarrow X_i} \phi(I_j) \leq \sum_{I_j : X_{i-1} \rightarrow X_i} e^{\beta l_v(I_j)} e^{\beta l_h(I_j)} (q - 1)^{(l_h(I_j) - 1)} \]
\[ \leq e^{-(\beta - \delta) \Delta n + \Delta m} \sum_{I_j : X_{i-1} \rightarrow X_i} e^{-\delta l_v(I_j)} e^{[\beta/2 - \log(q - 1)]l_h(I_j) - 1} \]  
(A.8)

for any \( \delta \) such that \( 0 < \delta < \beta \).

To obtain an upper bound of the right-hand side of (A.8) we first replace the sum by a sum over paths starting at \( X_{i-1} \) and ending at \( X_i \). Indeed, with every \( I_j \) (viewed as a graph) we can associate an oriented path that passes through each bond of \( I_j \) at most twice. Moreover, there is an algorithm assigning this path, to be denoted \( \omega \), in a unique way (see e.g. ref. [15]). By construction we have

\[ 2 \Delta m - 1 \leq |\omega| \leq 2|I_j| \]  
(A.9)

so that

\[ \sum_{I_j} e^{-\delta l_v(I_j)} e^{-l_h(I_j) - 1} \leq e^{-\beta} \sum_{\omega} e^{(\delta - 2) l_v(\omega)} e^{(\beta/2) - l_h(\omega)} \]  
(A.10)

where \( l_v(\omega) \) and \( l_h(\omega) \) denotes the vertical and the horizontal part of \( \omega \), respectively, \( \bar{\beta} = \frac{1}{2} \beta - \log(q - 1) \), and the sum is over paths satisfying (A.9).

Now, for any \( \omega \) we consider all intersections of \( \omega \) with the vertical lines

\[ x^1 = m_{i-1}, \quad x^2 = m_{j-1} + 1, \ldots, x^k = m_i \]

ordered in a sequence \( \{P_1, \ldots, P_k\} \) according to their occurrence on the path when passing from \( P_j = X_{i-1} \) to \( P_k = X_i \) (see fig. 4).

![Fig. 4. Decomposition of the path \( \omega \).](image)
Notice that a path may intersect a vertical line twice in the same point, yielding two different points in the above sequence. We use $\omega_i$ to denote the portion of the path between $P_{i-1}$ and $P_i$ and replace the sum over paths by sums over these portions. Finally we observe that

$$\sum_{\omega} \exp\left[-\frac{1}{2} \delta l_\omega(\omega)\right] \exp\left[-\frac{1}{2} \beta l_h(\omega)\right]$$

$$= \sum_{k \geq 2 \Delta m} \prod_{j=1}^k 4 \sum_{l_j=0}^{\infty} \exp\left(-\frac{1}{2} \tilde{\beta}\right) \exp\left(-\frac{1}{2} \delta |l_j|\right) \left(2 \sum_{r=0}^{\infty} c^{\delta r}\right)$$

$$= \sum_{k \geq 2 \Delta m} \left[8 e^{-\tilde{\beta} \cdot 2} \left(1 + \frac{2 e^{-\delta / 2}}{1 - e^{-\delta / 2}}\right) \frac{1}{1 - e^{-\delta}}\right]^k$$

$$= \frac{[8 e^{-\tilde{\beta} / 2} (1 - e^{-\delta / 2})^2]^{k \Delta m} (1)}{1 - 8 e^{-\tilde{\beta} / 2} (1 - e^{-\delta / 2})^2}$$

$$\leq \frac{[8 e^{-\tilde{\beta} / 2} (1 - e^{-\delta / 2})^2]^3}{1 - 8 e^{-\tilde{\beta} / 2} (1 - e^{-\delta / 2})^2}. \quad (A.11)$$

Here, the factor 4 in front of the sum $\sum_{\omega}^{\infty}$ corresponds to four possible orientations of two horizontal halfbonds in $\omega_j$, the factor $\exp(-\frac{1}{2} \delta |l_j|)$ corresponds to those vertical bonds that appear only once in $\omega_j$, while the term $2 \sum_{r=0}^{\infty} c^{\delta r}$ takes into account the contribution of possibly doubled vertical bonds in $\omega_j$ (observe the path between the points $P_o$ and $P_j$, as well as between $P_n$ and $P_o$ in fig. 4). To get a convergence in (A.11), we are supposing that

$$8 e^{-\tilde{\beta} / 2} \frac{1}{(1 - e^{-\delta / 2})^2} < 1. \quad (A.12)$$

Whenever

$$e^{+\tilde{\beta}} \left[8 e^{-\tilde{\beta} / 2} (1 - e^{-\delta / 2})^2\right]^3 \leq 1, \quad (A.13)$$

we get from (A.8), (A.10) and (A.11) the desired inequality (A.7).

To end the proof we notice that (A.13) is equivalent to

$$\frac{8 e^{-\tilde{\beta} / 2}}{(1 - e^{-\delta / 2})^2} \left(\frac{64}{(1 - e^{-\delta / 2})^2} + 1\right) \leq 1. \quad (A.14)$$

Clearly, the inequality (A.14) implies (A.12). It is satisfied once

$$3 e^{-\tilde{\beta} / 2} \leq 1 - e^{-\delta / 2}. \quad (A.15)$$
that is indeed satisfied under the hypotheses of theorem 1, i.e. for
\[ \tilde{\beta} > 12 \log 3 \]  
(A.16)
and
\[ \delta \geq 2 \log \left( \frac{1}{1 - 3e^{-\beta_{12}}} \right). \]  
(A.17)

**Proof of the upper bound**

First, we use a low temperature cluster expansions to evaluate the partition functions in (A.1). For a configuration \( \sigma \) one introduces contours as connected components of the set of all bonds \((i, j)\) such that \( \sigma_i \neq \sigma_j \). One obtains a similar contour model as for the Ising case. Notice only that contours with three bonds meeting in one site are allowed here and that when evaluating the contribution of a contour \( \gamma \) one has to take into account, in addition to the energy term \( e^{-\beta |\gamma|} \), also the number of configurations yielding the same set of bonds. Clearly, the latter can be bounded by \( (q - 1)^{|\gamma|^2} \). Applying now any version of standard cluster expansion (e.g. ref. [16]), we get

\[ Z''(\Lambda, \beta) = \exp \left( \sum_{C \subset \Lambda} \varphi^T(C) \right), \]

where \( C \) are clusters of contours and the truncated functional \( \varphi^T \) satisfies the bound

\[ \sum_{C \subset \Lambda} |\varphi^T(C)| \leq c(q - 1)^2 e^{-4\beta}. \]

Here the sum over all clusters such that at least one of their contours passes a given site \( x \) and \( c \) is a fixed constant.

Considering now the equality (A.1), dividing both sides by \( e^{-\beta} Z''(\Lambda, \beta) \), and applying low temperature cluster expansions, we find that

\[ \Xi_{\text{Potts}}(\Lambda, \theta, \beta) = \frac{e^{\beta} Z''(\Lambda, \beta)}{Z''(\Lambda, \beta)} \]

\[ = \sum_{\Gamma} \sum_{\sigma : I(\sigma) = \Gamma} \exp \left( -\beta |\Gamma| + \sum_{C \subset \Gamma} \varphi^T(C) \right). \]

From the bound on the truncated functional \( \varphi^T \) we get
\[ \Xi_{\text{Potts}}(\Lambda, \theta, \beta) \geq \sum_{\Gamma} \sum_{\gamma(\partial \Lambda) \setminus \Gamma} \exp[-(\beta + c(q-1)^2 e^{-\beta})|\Gamma|] \]
\[ \geq \sum_{\Gamma} \exp[-(\beta + c(q-1)^2 e^{-\beta})|\Gamma|]. \]

We finally restrict the sum in the last term to SOS paths and take the limit \(M \to \infty\) to get the inequality
\[ \Xi_{\text{Potts}}(N, \theta, \beta) \geq \Xi_{\text{SOS}}(N, \theta, \beta + c(q-1)^2 e^{-\beta}), \]
which concludes the proof.

Appendix B. Proof of theorem 2

Proof of the lower bound

For any interface \(\Gamma\), we consider the decomposition into pieces \(\Gamma_i, i = 1, \ldots, n\), introduced in appendix A. Using the notations from this appendix and starting from relation (A.7), we have for any \(|x| < 1\)
\[ e^{\beta x(h_m - h_m')} \sum_{I_i: X_i \uparrow \downarrow X_i} \phi(I_i) \leq e^{\tilde{\beta} x(h_m - h_m')} e^{-(\beta - \delta') \gamma_{\text{min}}(X_{i-1} \cup X_i)} \]
\[ \leq e^{\beta x(h_m - h_m')} e^{\beta \gamma_{\text{min}}(X_{i-1} \cup X_i)} \leq e^{\beta x(h_m - h_m')} \sum_{\gamma_{\text{SOS}} \supset I_i: X_{i-1} \cup X_i} e^{\beta |I_i|}, \]
where \(\tilde{\beta} = \beta - \delta/|1 - |x||\).

Moreover when \(m_i - m_{i-1} = 1\) one has
\[ e^{\beta x(h_m - h_m')} \sum_{I_i} e^{-\tilde{\beta} |I_i|} \leq e^{\tilde{\beta} x(h_m - h_m')} \sum_{I_i} e^{\tilde{\beta} |I_i|}, \]
for any \(\tilde{\beta} \leq \tilde{\beta}'.\) From the two above relations (B.1) and (B.2) it follows
\[ \sum_{N\tan \theta \in \mathcal{L}} e^{-\beta N \tan \theta} \Xi_{\text{Potts}}(N, \theta, \beta) \leq \sum_{N\tan \theta \in \mathcal{F}} e^{-\beta' N \tan \theta} \Xi_{\text{SOS}}(N, \theta, \beta'), \]
which implies
\[ \phi_{\text{SOS}}(x, \beta - \frac{\delta}{1 - |x|}) \leq \phi_{\text{Potts}}(x, \beta) \]
for any \(|x| < 1\).
The lower bound from theorem 2 follows once we observe that we need the bound only for \( |x| \leq x_0 \) and that the solution \( x_0 \geq 0 \) of the equation

\[
\phi_{\text{SOS}}(x, \beta) = x
\]

satisfies the bound

\[
\frac{1}{1 - x_0} \leq 2\beta .
\]

Indeed, increasing the left-hand side of (B.5) we consider the solution \( x_1 \geq x_0 \) of the equation

\[
1 - \frac{1}{\beta} \log \frac{\sinh \beta}{\cosh \beta - \frac{1}{2} e^{\beta x}} = x
\]

for which, clearly, \( x_1 \geq x_0 \). Solving eq. (B.7) we get

\[
e^{\beta x_1} = e^{\beta} \frac{1 + e^{-2\beta}}{2 - e^{-2\beta}} \leq e^{\beta} \frac{1}{\sqrt{e}} .
\]

Hence \( \beta x_0 \leq \beta - \frac{1}{2} \) and thus also (B.6).

**Proof of the upper bound**

Let \( \{x_1, y_1\} \) and \( \{x_2, y_2\} \) be the coordinates of the points \( A \) and \( B \) respectively. We notice that if \( |x| \leq 1 \), then for every \( \Gamma : A \to B \) one has \( |\Gamma| \geq y_2 \) and thus

\[
e^{\beta y_2} e^{(\beta - \delta ||\Gamma||)} \geq e^{\beta y_2} e^{\beta ||\Gamma||} ,
\]

with \( \beta'' = \beta + \delta, (1 - |x|) \).

Summing both terms in the inequality (B.8) over all SOS-paths going from \( A \) to \( B \),

\[
e^{\beta y_2} \sum_{\Gamma : A \to B} e^{(\beta - \delta ||\Gamma||)} \geq e^{\beta y_2} \sum_{\Gamma : A \to B} e^{\beta'' ||\Gamma||} ,
\]

and taking into account the bound (A.18), we get

\[
\sum_{N \tan \theta \in \mathcal{E}} e^{-\beta N \tan \theta} \Xi_{\text{Potts}}(N, \theta, \beta) \geq \sum_{N \tan \theta \in \mathcal{E}} e^{-\beta'' N \tan \theta} \Xi_{\text{SOS}}(N, \theta, \beta'') .
\]

Hence
\[ \phi_{\text{Potts}}(x, \beta) \leq \phi_{\text{SOS}} \left( x, \beta + \frac{\delta}{1 - |x|} \right) \]  
\hspace{1cm} (B.11)

for any \(|x| < 1\). The upper bound in theorem 2 follows again with the help of the bound (B.6).

References