Parity-violating vertices for spin-3 gauge fields

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The problem of constructing consistent parity-violating interactions for spin-3 gauge fields is considered in Minkowski space. Under the assumptions of locality, Poincaré invariance, and parity non-invariance, we classify all the nontrivial perturbative deformations of the Abelian gauge algebra. In space-time dimensions $n = 3$ and $n = 5$, deformations of the free theory are obtained which make the gauge algebra non-Abelian and give rise to nontrivial cubic vertices in the Lagrangian, at first order in the deformation parameter $g$. At second order in $g$, consistency conditions are obtained which the five-dimensional vertex obeys, but which rule out the $n = 3$ candidate. Moreover, in the five-dimensional first-order deformation case, the gauge transformations are modified by a new term which involves the second de Wit-Freeman connection in a simple and suggestive way.

I. INTRODUCTION

One of the most intriguing open questions in classical field theory is the construction of consistent interactions among massless fields of spin (or helicity) $s$ greater than 2. It is generally believed that, when perturbatively deforming Fronsdal’s massless spin-$s$ quadratic Lagrangian [1], the only first-order vertex is cubic and contains $s$ derivatives. This is what comes out of manifestly covariant analyzes [2–6] and a light-front approach [7]. Moreover, in the aforementioned works, it is found that the massless spin-3 field (more generally, the odd-spin fields) carries a spin-3 field (more generally, the odd-spin fields) carries a color, the fields taking value in an internal anticommutative, invariant-normed algebra. This implies that there must be (self-interacting) multiplets of spin-3 fields, analogously to what happens in Yang-Mills theories for spin-1 gauge fields.

In a recent paper [8], the problem of introducing consistent interactions among spin-3 gauge fields has been carefully analyzed in Minkowski space-time $\mathbb{R}^{n-1,1}$ ($n > 3$) using Becchi-Rouet-Stora-Tyutin-(BRST)-cohomological methods. Under the assumptions of locality, Poincaré and parity invariance, all the perturbative, consistent deformations of the Abelian gauge algebra have been determined, together with the corresponding deformations of the quadratic action, at first order in the deformation parameter. On top of the covariant cubic vertex of [2], a new cubic vertex is found which also corresponds to a non-Abelian gauge algebra related to an internal, noncommutative, invariant-normed algebra (like in Yang-Mills’s theories). This new cubic vertex brings in five derivatives of the field: it is of the form $\mathcal{L}_1 \sim g_{[abc]}(h^a \partial^2 h^b \partial^3 h^c + h^a \partial h^b \partial^2 h^c)$.

In the present paper, we determine the non-Abelian consistent deformations of the free spin-3 gauge theory in Minkowski space-time $\mathbb{R}^{n-1,1}$, relaxing the parity invariance constraint of [8]. In other words, we look for all non-Abelian deformations of the free theory such that the corresponding first-order vertices involve the completely antisymmetric Levi-Civita density $\varepsilon^{\mu_1 \cdots \mu_5}$, like in Chern-Simons theories. As we show explicitly in Sec. IV, such first-order parity-breaking nontrivial deformations exist only in space-time dimensions $n = 3$ and $n = 5$.

Following the cohomological procedure of [8], we first classify all the possible first-order deformations of the spin-3 gauge algebra that contain one Levi-Civita antisymmetric density (these are the “$a_2$ terms” in the notation of [8]). We find two such deformations that make the algebra non-Abelian, in dimensions $n = 3$ and $n = 5$. Then, we investigate whether these algebra-deforming terms give rise to consistent first-order vertices. Very interestingly, both of the algebra-deforming terms do lead to nontrivial deformations of the quadratic Lagrangian. The first one is defined in dimension $n = 3$, for spin-3 gauge fields that take value in an internal, anticommutative, invariant-normed algebra $\mathcal{A}$, while the second one is defined in a space-time of dimension $n = 5$ for fields that take value in a commutative, invariant-normed internal algebra $\mathcal{B}$. However, as we demonstrate, consistency conditions at second order in $\mu$ imply that the algebras $\mathcal{A}$ and $\mathcal{B}$ must also be nilpotent of order three and associative, respectively. In turn, this means that the $n = 3$ parity-breaking deformation is trivial while the algebra $\mathcal{B}$ is a...
II. FREE THEORY

The local action for a collection \( \{ h^a_{\mu \nu \rho} \} \) of \( N \) noninteracting completely symmetric massless spin-3 gauge fields in flat space-time is [1]

\[
S_0[h^a_{\mu \nu \rho}] = \sum_{a=1}^{N} \int \left[ -\frac{1}{2} \partial_\sigma h^a_{\mu \nu \rho} \partial^\sigma h^a_{\mu \nu \rho} 
+ \frac{3}{2} \partial^\rho h^a_{\mu \rho \sigma} \partial_\mu h^a_{\nu \sigma \rho} 
+ \frac{3}{2} \partial_\mu h^a_{\nu \rho \sigma} \partial^\rho h^a_{\mu \sigma \rho} 
+ \frac{3}{4} \partial_\mu h^a_{\nu \rho \sigma} \partial_\nu h^a_{\mu \rho \sigma} - 3 \partial_\mu h^a_{\nu \rho \sigma} \partial_\rho h^a_{\mu \sigma \nu} \right] \, d^nx,
\]

where \( h^a_{\mu \nu \rho} := \eta^{\sigma \rho} h^a_{\mu \nu \rho} \). The Latin indices are internal indices taking \( N \) values. They are raised and lowered with the Kronecker delta \( \delta_{ab} \). The Greek space-time indices are raised and lowered with the “mostly plus” Minkowskian metric \( \eta_{\mu \nu} \). The free action (1) is invariant under the linear gauge transformations \( \delta h^a_{\mu \nu \rho} = \delta_{\lambda} h^a_{\mu \nu \rho} \), where \( \eta^{\mu \nu} \lambda^a_{\mu \nu} \equiv 0 \). The gauge transformations are Abelian and irreducible. Curved (respectively square) brackets on indices denote strength-one complete symmetrization (respectively antisymmetrization) of the indices.

The field equations read \( \delta S_0[h^a_{\mu \nu \rho}] \equiv G^a_{\mu \nu \rho} = 0 \), where \( G^a_{\mu \nu \rho} := F^a_{\mu \nu \rho} - \frac{1}{2} \eta_{\mu \nu} F^a_{\rho} \) and \( F^a_{\mu \nu \rho} \) is the Fronsdal gauge-invariant tensor \( F^a_{\mu \nu \rho} := \Box h^a_{\mu \nu \rho} - 3 \partial^\rho h^a_{\mu \nu \rho} + 3 \partial_\mu h^a_{\nu \rho \sigma} - 3 \partial_\nu h^a_{\mu \rho \sigma} + 3 \partial_\sigma h^a_{\nu \rho \mu} \). We denote \( F_\mu = \eta^{\nu \rho} F_{\mu \nu \rho} \). The gauge symmetries enable one to get rid of some components of \( h^a_{\mu \nu \rho} \), leaving it on-shell with \( N^3 \) independent physical components, where \( \frac{n(n^2 - 3n + 12)}{6} \).

Notice that there is no propagating physical degree of freedom in \( n = 3 \).

An important object is the curvature gauge-invariant tensor [12–14]

\[
K^a_{\alpha \mu \beta \rho \gamma \nu} := 8 \partial_{\gamma} \partial_{\nu} [a h^a_{\mu \gamma \nu}] - S^a_{\beta \mu \nu | \gamma | \alpha} - S^a_{\gamma \rho | \alpha | \beta \mu \nu}, \tag{2}
\]

antisymmetric in the three pairs \( \alpha \mu, \beta \nu, \) and \( \gamma \rho \). Actually, the spin-3 curvature is invariant under gauge transformations where the parameters \( \lambda^a_{\mu \nu} \) are not constrained to be traceless.

Its importance, apart from gauge invariance with unconstrained gauge parameters, stems from the fact that the field equations \( G^a_{\mu \nu \rho} = 0 \) in Fronsdal’s constrained approach are dynamically equivalent [15,16] to the following field equation in the unconstrained approach:

\[
\eta^{\alpha \beta} K^a_{\alpha \mu \beta \rho \gamma \nu} = 0. \tag{3}
\]

There exists another field equation for completely symmetric gauge fields in the unconstrained approach, which also involves the curvature tensor but is nonlocal [17] (see also [18]). The equivalence between both unconstrained field equations was proved in [15]. One of the advantages of the nonlocal field equation of [17] is that it can be derived from an action principle. The Eq. (3) is obtained from the general \( n \)-dimensional bosonic mixed symmetry case [15] by specifying to a completely symmetric rank-3 gauge field and is [16] a generalization of Bargmann-Wigner’s equations in \( n = 4 \) [19]. However, it cannot be directly obtained from an action principle. For a recent work in direct relation to [17,18], see [20].

Notice that when \( n = 3 \), the Eq. (3) implies that the curvature vanishes on-shell, which reflects the “topological” nature of the theory in the corresponding dimension. This is similar to what happens in 3-dimensional Einstein gravity, where the vacuum field equations \( R_{\mu \nu} := R^a_{\mu \nu a} = 0 \) imply that the Riemann tensor \( R^a_{\mu \nu a} \) is zero on-shell. The latter property derives from the fact that the conformally-invariant Weyl tensor identically vanishes in dimension 3, allowing the Riemann tensor to be expressed entirely in terms of the Ricci tensor \( R_{\mu \nu} \). Those properties are a consequence of a general theorem (see [21], p. 394) which states that a tensor transforming in an irreducible representation of \( O(n) \) identically vanishes if the corresponding Young diagram is such that the sum of the lengths of the first two columns exceeds \( n \).

Accordingly, in dimension \( n = 3 \) the curvature tensor \( K^a_{\alpha \mu \beta \rho \gamma \nu} \) can be written [14] as

\[
K^a_{\alpha \mu \beta \rho \gamma \nu} = \frac{1}{3} \left( S^a_{\alpha \mu \beta \rho \gamma \nu} + S^a_{\beta \rho \gamma \nu \alpha} + S^a_{\gamma \nu \alpha \beta \rho \mu} + S^a_{\rho \mu \alpha \beta \gamma \nu} + S^a_{\mu \alpha \beta \gamma \nu \rho} + S^a_{\nu \beta \rho \gamma \mu \alpha} \right), \tag{4}
\]
where the tensor \( S^a_{\alpha\mu}[\nu] \) is defined, in dimension \( n = 3 \), by
\[
S^a_{\alpha\mu}[\nu] = 2\delta_{(\alpha}(F^\mu_{\beta\nu)} - \frac{1}{2}(2\delta_{(\alpha} F^\mu_{\beta\nu)} - \partial_\rho F^\mu_{(\alpha} \eta_{\beta\nu)} - \partial_\nu F^\mu_{(\alpha} \eta_{\beta\rho)}) - \partial_\nu F^\mu_{\alpha\beta} \eta_{\nu\rho} - \partial_\beta F^\mu_{\alpha\nu} \eta_{\rho\nu} - \partial_\nu F^\mu_{\alpha\beta} \eta_{\rho\beta} \].

It is antisymmetric in its first two indices and symmetric in its last two indices. For the expression of \( S^a_{\alpha\mu}[\nu] \) in arbitrary dimension \( n \geq 1 \), see [14] where the curvature tensor \( K^a_{\alpha\beta\nu}[\rho] \) is decomposed under the (pseudo-)orthogonal group \( O(n - 1, 1) \). The latter reference contains a very careful analysis of the structure of Fronsdal’s spin-3 gauge theory, as well as an interesting “topologically massive” spin-3 theory in dimension \( n = 3 \).

### III. BRST FORMULATION

In the present section we summarize the BRST-cohomological procedure [10,11] where consistent couplings define deformations of the solution of the master equation. For more details, we refer, in particular, to [8] where the BRST approach is applied to the spin-3 gauge theory at hand.

#### A. Basic assumptions

We assume, as in the traditional perturbative Noether deformation procedure, that the deformed action can be expressed as a power series in a coupling constant \( g \), the zeroth-order term in the expansion describing the free theory: \( S = S_0 + gS_1 + \mathcal{O}(g^2) \).

We require that the deformed Lagrangian be invariant under the Poincaré group, but explicitly breaks parity symmetry by the presence of a completely antisymmetric Levi-Civita density \( e^{\mu_1...\mu_n} \) in the deformed Lagrangian.

We reject trivial deformations arising from field-redefinitions that reduce to the identity at order \( g^0 \) and compute only consistent deformations, in the sense that the deformed theory possesses the same number of (possibly deformed) independent gauge symmetries, reducibility identities, etc., as the system we started with. Finally, crucial in the cohomological approach [10,11] is the locality requirement: the deformed action \( S_{\phi}[\phi] = S_0[\phi] + gS_1[\phi] + \ldots \) must be a local functional. The deformations of the gauge transformations, etc., must be local functions, as well as the allowed field redefinitions.

A local function of some set of fields \( \phi^i \) is a smooth function of the fields \( \phi^i \) and their derivatives \( \partial \phi^i, \partial^2 \phi^i, \ldots \), up to some finite order, say \( k \), in the number of derivatives. Such a set of variables \( \phi^i, \partial \phi^i, \ldots, \partial^2 \phi^i \) will be collectively denoted by \( [\phi^i] \). Therefore, a local function of \( \phi^i \) is denoted by \( f([\phi^i]) \). A local \( p \)-form (\( 0 \leq p \leq n \)) is a differential \( p \)-form the components of which are local functions. A local integral is the integral of a local \( n \)-form.

#### B. BRST spectrum and differential

According to the general rules of the BRST-antifield formalism, for the spin-3 gauge theory under consideration, the spectrum of fields (including ghosts) and antifields together with their respective ghost and antighost numbers is given by [8]

(a) the fields \( h^a_{\mu\nu\rho} \), with ghost number \( 0 \) and antighost number \( 0 \);
(b) the ghosts \( C^a_{\mu\nu} \), with ghost number \( 1 \) and antighost number \( 0 \);
(c) the antifields \( h^a_{\ast\mu\nu\rho} \), with ghost number \( -1 \) and antighost number \( 1 \);
(d) the antifields \( C^a_{\ast\mu\nu} \), with ghost number \( -2 \) and antighost number \( 2 \).

The Grassmannian parity of the (anti)fields is given by their ghost number modulo two.

The BRST differential \( s \) of the free theory is generated by the functional
\[
W_0 = S_0[h^a] + \int (3h^a_{\ast\mu\nu\rho}\partial_\mu C^0_{\nu\rho} + m^a_{\gamma\nu} dx \partial_\mu C^0_{\gamma\nu}).
\]

More precisely, \( W_0 \) is the generator of the BRST differential \( s \) of the free theory through \( sA = (W_0, A)_{a,b} \), where the antibracket \( (\ , \) \) is defined by \( (A, B)_{a,b} = sA \times sB = \frac{1}{12} \varepsilon_{\mu\nu\rho\sigma} \partial_\mu A\partial_\nu B\partial_\rho C - \frac{1}{12} \varepsilon_{\mu\nu\rho\sigma} \partial_\mu B\partial_\nu A\partial_\rho C \).

In the context of the free spin-3 gauge theory, the BRST-differential \( s \) decomposes into \( s = s + \gamma \). The Koszul-Tate differential \( \delta \) decreases the antighost number by one unit, while the first piece \( \gamma \), the differential along the gauge orbits, leaves it unchanged. Both \( \delta \) and \( \gamma \), and consequently the differential \( s \), increase the ghost number by one unit.

The action of the differentials \( \delta \) and \( \gamma \) gives zero on all the fields of the formalism except in the few following cases:

\[
\delta h^a_{\ast\mu\nu\rho} = C^a_{\mu\nu\rho},
\]
\[
\delta C^a_{\ast\mu\nu} = -3\left( \partial_\rho h^a_{\ast\mu\nu\rho} - \frac{1}{n} \varepsilon^{\mu\nu\rho\sigma} \partial_\sigma h^a_{\ast\mu\nu\rho} \right),
\]
\[
\gamma h^a_{\mu\nu\rho} = 3\partial_\mu C^a_{\nu\rho}.
\]

#### C. BRST deformation

As shown in [10], the usual Noether procedure can be reformulated within a BRST-cohomological framework. Any consistent deformation of the gauge theory corresponds to a solution \( W = W_0 + gW_1 + g^2W_2 + \mathcal{O}(g^3) \) of the deformed master equation \( (W, W)_{a,b} = 0 \). Consequently, the first-order nontrivial consistent local deformations \( W_1 = \int a^\alpha dx \) are in one-to-one correspondence with elements of the cohomology \( H^{n,0}(s|d) \) of the zeroth order BRST differential \( s = (W_0, \cdot) \) modulo the total derivative \( d \), in maximum form-degree \( n \) and in ghost number
0. That is, one must compute the general solution of the cocycle condition

$$s a^{n,0} + db^{n-1,1} = 0,$$  \hspace{1cm} (5)$$

where $$a^{n,0}$$ is a topform of ghost number zero and $$b^{n-1,1}$$ a $$(n - 1)$$-form of ghost number one, with the understanding that two solutions of (5) that differ by a trivial solution should be identified $$a^{n,0} \sim a'^{n,0} + x p^{n-1,1} + dq^{n-1,1}$$ as they define the same interactions up to field redefinitions. The cocycles and coboundaries $$a, b, p, q, \ldots$$ are local forms of the field variables (including ghosts and antifields).

The corresponding second-order interactions $$W_2$$ must satisfy the consistency condition $$s W_2 = - \frac{i}{2}(W_1, W_1)_{a,b}$$. This condition is controlled by the local BRST cohomology group $$H^{n,1}(s|d)$$.  

D. Cohomology of $$\gamma$$

The groups $$H^s(\gamma)$$ have recently been calculated [22] for massless spin-$$s$$ gauge fields represented by completely symmetric (and double traceless when $$s > 3$$) rank $$s$$ tensors. In the special case $$s = 3$$, the result reads:

**Proposition 1**—The cohomology of $$\gamma$$ is isomorphic to the space of functions depending on

(a) the antifields $$h^{a, \mu \nu \rho}, C^a_{\mu \nu}$$ and their derivatives, denoted by $$[\Phi^a]$$,

(b) the curvature and its derivatives $$[K^a_{\mu \rho \nu \lambda}], [F^a_{\mu \nu}],[F^a_{\nu \rho}]$$,

(c) the symmetrized derivatives $$\delta_{(a_1 \ldots a_l} F^a_{\mu \nu \rho)}$$ of the Fronsdal tensor,

(d) the ghosts $$C_{\mu \nu}$$ and the traceless parts of $$\delta_{(a} C^a_{\mu \nu)}$$ and $$\delta_{(a} C_{\mu \nu \rho)}$$.

Thus, identifying with zero any $$\gamma$$-exact term in $$H(\gamma)$$, we have $$\gamma \phi = 0$$ if and only if $$f = f([\Phi^a], [K^a_{\mu \rho \nu \lambda}], [F^a_{\mu \nu}],[F^a_{\nu \rho}])$$ where $$F^a_{\mu \nu}$$ stands for the completely symmetrized derivatives $$\delta_{(a_1 \ldots a_l} F^a_{\mu \nu \rho)}$$ of the Fronsdal tensor, while $$\bar{\omega}^a_{\alpha \mu \nu}$$ denotes the traceless part of $$T^a_{\alpha \rho \nu} := \delta_{\{a} C^a_{\mu \nu \rho \}}$$ and $$\bar{U}^a_{\alpha \mu \nu}$$ the traceless part of $$U^a_{\alpha \mu \nu} := \delta_{\{a} C^a_{\mu \nu \rho \}}$$.

Let us introduce some useful standard notations and make some remarks.

Let $$\omega^I$$ be a basis of the space of polynomials in the $$C^a_{\mu \nu}$$, $$\bar{\omega}^a_{\alpha \mu \nu}$$ and $$\bar{U}^a_{\alpha \mu \nu}$$ (since these variables anticommute, this space is finite-dimensional). If a local form $$\alpha$$ is $$\gamma$$-closed, we have

$$\gamma a = 0 \Rightarrow a = \alpha_{j}([\Phi^a], [K], \{F\}) \omega^I(C^a_{\mu \nu}, \bar{T}^a_{\alpha \mu \rho \nu}, \bar{U}^a_{\alpha \mu \rho \nu}) + \gamma b,$$

If $$\alpha$$ has a fixed, finite ghost number, then $$\alpha$$ can only contain a finite number of antifields. Moreover, since the local form $$\alpha$$ possesses a finite number of derivatives, we find that the $$\alpha_j$$ are polynomials. Such a polynomial $$\alpha_j([\Phi^a], [K], \{F\})$$ will be called an invariant polynomial.

**Remark 1**: The Damour-Deser identity [14] $$\eta_{\alpha \beta} K^a_{\alpha \beta \rho \nu} \gamma = 2 \delta_{\{a} F^a_{\rho \nu} \gamma$$ implies that the derivatives of the Fronsdal tensor are not all independent of the curvature tensor $$K$$. Therefore only the completely symmetrized derivatives of $$F$$ appear in Proposition 1, while the derivatives of the curvature $$K$$ are not restricted. However, in the sequel we will assume that every time the trace $$\eta_{\alpha \beta} K^a_{\alpha \beta \rho \nu}$$ appears, we substitute $$2 \delta_{\{a} F^a_{\rho \nu}$$ for it. We can then write $$\alpha_j([\Phi^a], [K], \{F\})$$ instead of the inconvenient notation $$\alpha_j([\Phi^a], [K], \{F\})$$.

**Remark 2**: Proposition 1 must be slightly modified in the special $$n = 3$$ case. As we said in the introduction, the curvature tensor $$K$$ can be expressed in terms of the partial derivatives of the Fronsdal tensor, see (4). Moreover, the ghost variable $$\bar{U}^a_{\alpha \mu \rho \nu}$$ identically vanishes because it possesses the symmetry of the Weyl tensor. Thus, in dimension $$n = 3$$ we have

$$\gamma a = 0 \Rightarrow a = \alpha_{j}([\Phi^a], [K], \{F\}) \omega^I(C^a_{\mu \nu}, \bar{T}^a_{\alpha \mu \rho \nu}) + \gamma b.$$  \hspace{1cm} (6)$$

Another simplifying property in $$n = 3$$ is that the variable $$\bar{T}^a_{\alpha \mu \rho \nu}$$ can be replaced by its dual

$$\bar{T}^a_{\beta \alpha \rho \nu} := \varepsilon^{\mu \nu \alpha \beta} \bar{T}^a_{\alpha \mu \rho \nu} \quad (\bar{T}^a_{\mu \nu \rho \lambda} = - \frac{1}{2} \varepsilon^{\mu \nu \alpha \beta} \bar{T}^a_{\alpha \rho \nu \lambda})  \hspace{1cm} (7)$$

which is readily seen to be symmetric and traceless, as a consequence of the symmetries of $$\bar{T}^a_{\alpha \mu \rho \nu}$$:

$$\bar{T}^a_{\alpha \beta \rho \nu} = \bar{T}^a_{\beta \alpha \rho \nu}, \quad \eta^{\alpha \beta} \bar{T}^a_{\alpha \beta \rho \nu} = 0.$$  \hspace{1cm} (8)$$

We now remind [8] the definition of a differential which plays an important role in the classification of the consistent first-order deformations of $$W_0$$.

**Definition (differential D)**: The action of the differential $$D$$ on $$h^{a, \mu \nu \rho}, h_{a, \mu \nu \rho}, C^{a, \mu \nu}, \ldots$$ and all their derivatives is exactly the same as the action of the total derivative $$d$$, but its action on the ghosts is given by

$$D C^a_{\mu \nu} = \frac{4}{3} dx^a \bar{T}^a_{\mu \nu \rho \lambda}, \quad D \bar{T}^a_{\mu \nu \rho \lambda} = dx^a \bar{U}^a_{\mu \nu \rho \lambda},$$

$$D(\partial_{(p} C^a_{\mu \nu)}) = 0, \quad D(\partial_{(p \mu} C^a_{\nu \rho)}) = 0 \quad \text{if} \quad p \geq 2.  \hspace{1cm} (9)$$

The operator $$D$$ coincides with $$d$$ up to $$\gamma$$-exact terms. It follows from the definitions that $$\partial_0 \omega^I = A^I_j \omega^I$$ for some constant matrix $$A^I_j$$ that involves $$dx^a$$ only.

The differential $$D$$ is associated with a grading.

**Definition (D-degree)**: The number of $$\bar{T}^a_{\alpha \mu \nu \rho}$$’s plus twice the number of $$\bar{U}^a_{\alpha \mu \nu \rho}$$’s is called the $$D$$-degree. It is bounded because there is a finite number of $$\bar{T}^a_{\alpha \mu \nu \rho}$$’s and $$\bar{U}^a_{\alpha \mu \nu \rho}$$’s, which are anticommuting.

The operator $$D$$ splits as the sum of an operator $$D_1$$ that raises the $$D$$-degree by one unit, and an operator $$D_0$$ that leaves it unchanged. The differential operator $$D_0$$ has the same action as $$d$$ on $$h^{a, \mu \nu \rho}, h_{a, \mu \nu \rho}, C^{a, \mu \nu}, \ldots$$ and all their
derivatives, and gives 0 when acting on the ghosts. The operator $D_\gamma$ gives 0 when acting on all the variables but the ghosts on which it reproduces the action of $D$.

**E. Cohomological group $H^2_\gamma(\delta|d)$ and $H^2_\gamma(\delta|d, H(\gamma))$, $k \geq 2$**

We first mention a result proved in [22–24] and needed later.

**Proposition 2**—A complete set of representatives of $H^2_\gamma(\delta|d)$ is given by the antifields $C_{a}\mu\nu$, up to explicitly $x$-dependent terms. In other terms,

$$
\delta a^2 + db^2 = 0,
\quad a^2 \sim a_1^2 + \delta c^2 + dc_1^2
$$

$$
\begin{cases}
\lambda^2 = L_{\mu\nu}(x)C_{a}\mu\nu d^2 x + \delta b^2 + db_1^2,
L_{\mu\nu}(x) = \lambda_{\mu\nu} + P_{\mu\nu}(x).
\end{cases}
$$

The tensors $\lambda_{\mu\nu}$ and $P_{\mu\nu}$ are symmetric and traceless in the indices $(\mu, \nu)$. The tensor $\lambda^2$ is constant whereas the tensor $P_{\mu\nu}(x)$ depends on the coordinates $x^\mu$ explicitly.

Then, we recall that the cohomological groups $H^2_\gamma(\delta|d)$ vanish for $k > 2$, which is a consequence of the fact that the free theory at hand is linear and possesses no reducibility (cf. Theorem 9.1 of [25]).

Finally, the following result is a cornerstone of the cohomological deformation procedure:

**Proposition 3**—For the free spin-3 gauge theory of Fronsdal, we have $H^2_\gamma(\delta|d, H(\gamma)) \equiv 0$, $k > 2$, and the non-trivial elements of $H^2_\gamma(\delta|d, H(\gamma))$ are the same as for $H^2_\gamma(\delta|d)$.

Essentially, using Proposition 2 and $H^2(\delta|d) \equiv 0$ ($k > 2$), the proof of Proposition 3 goes as follows. It consists in showing that, if the invariant polynomial $a^k_{\nu} (k \geq 2)$ is $d$-trivial modulo $d$, $a^k_{\nu} = \mu_{k+1} + d\nu_{k-1}$, then one can always choose $\mu_{k+1}$ and $\nu_{k-1}$ to be invariant.

Proposition 3 is proved in [8] for $n > 3$. We now demonstrate that it also holds for $n = 3$. Without recalling all the details, let us point out the place where, in Sec. 4.6.2 of [8], the proof of Proposition 3 must be adapted to $n = 3$. It is when one makes use of the projector on the symmetries of the Weyl tensor. In [8] we have the equations (4.45) and (4.46)

$$
\begin{align}
Y_{\mu\nu} & = \delta_{\alpha} T_{k+1}^{\alpha\gamma} + \eta_{\mu\nu} T_{k+1}^{\alpha\gamma} + \delta(\ldots), \\
T_{k+1}^{\alpha\gamma} & \equiv \partial_{\alpha} T_{\mu\nu}^{\gamma} = Y_{\mu\nu} - \eta_{\mu\nu} \eta_{\lambda\beta} Y_{\lambda\beta} + \delta(\ldots),
\end{align}
$$

which are instrumental in order to obtain

$$
Y_{\mu\nu} = \partial_{\alpha} T_{\mu\nu}^{\gamma} - \eta_{\mu\nu} T_{\lambda\beta}^{\gamma} + \delta(\ldots),
$$

where $Y_{\mu\nu}$ is completely symmetric (the subscript denotes the antighost number), $T_{\mu\nu}^{\gamma}$ is antisymmetric in $(\beta, \alpha)$ and $(\mu, \nu)$, satisfies $S_{k+1}^{\alpha\beta}[\mu\nu] = 0 \equiv \eta_{\gamma\mu} S_{k+1}^{\alpha\beta}[\mu\nu]$. The tensor $\Psi_{k+1}$ possesses the symmetries of the curvature $K_{\alpha\beta}[\mu\nu]$ and $G_{\alpha\beta\gamma}$ is the second-order differential operator appearing in the equations of motion 0 = $G_{\mu\nu} = G_{\mu\nu}^{\alpha\beta} K_{\alpha\beta}$. Finally, we have

$$
\begin{align}
\hat{X}_{\alpha\beta\gamma} & = \left( \frac{2}{n-2} \right) \Psi_{\alpha\beta\gamma} \left( -S_{\mu\nu}^{\alpha\beta\gamma} + \frac{1}{n} \eta_{\sigma\tau} S_{\mu\nu}[\rho\kappa+1] + S_{\mu\nu}[\mu\nu][\rho\kappa+1] \right),
\end{align}
$$

where $\Psi_{\alpha\beta\gamma} = \psi_{\alpha\beta\gamma}$ projects on completely symmetric rank-3 tensors. The tensors $Y_{\mu\nu}$, $T_{\mu\nu}$, $S_{\mu\nu}[\rho\kappa+1]$, and $\Psi_{\alpha\beta\gamma}$ are invariant.

In order to obtain the important Eq. (11), one had to project $\hat{X}_{\alpha\beta\gamma}^{\alpha\beta\gamma}$ on the symmetries of the Weyl tensor [8]. In dimension 3, this gives zero identically.

**IV. PARITY-BREAKING DEFORMATIONS**

In this section, we first compute all possible parity-breaking and Poincaré-invariant first-order deformations of the Abelian spin-3 gauge algebra. We find that such deformations exist in three and five dimensions. We then
The last equation implies that, modulo trivial terms, \( a_2 = \alpha_l \omega^l \), where \( \alpha_l \) is an invariant polynomial and the \( \{ \omega^l \} \) provide a basis of the polynomials in \( C_{\mu \nu} T_{\mu \nu \rho \sigma} U_{\mu \nu \rho \sigma} \) (see Sec. III). Let us stress that, as \( a_2 \) has ghost number zero and antifield number two, \( \omega^l \) must have ghost number two. Then, acting with \( \gamma \) on (16) and using the triviality of \( d \), one gets that \( b_1 \) should also be an element of \( H(\gamma) \), i.e., modulo trivial terms, \( b_1 = \beta_1 \omega^l \), where the \( \beta_1 \) are invariant polynomials.

We further expand \( a_2 \) and \( b_1 \) according to the \( D \)-degree defined in Sec. III D:

\[ a_2 = \sum_{i=0}^{M} a_i^2 = \sum_{i=0}^{M} \alpha_i \omega^i, \quad b_1 = \sum_{i=0}^{M} b_i = \sum_{i=0}^{M} \beta_i \omega^i, \]

where \( a_i^2, b_i, \) and \( \omega^i \) have \( D \)-degree \( i \). The Eq. (16) then reads

\[ \sum_i \delta(\alpha_i \omega^i) + \sum_i D(\beta_i \omega^i) = \gamma(\ldots), \]

or equivalently

\[ \sum_i \delta(\alpha_i \omega^i) + \sum_i D(\beta_i \omega^i) + \sum_i \beta_i A_{i+1}^{i+1} = \gamma(\ldots), \]

where \( A_{i+1}^{i+1} \omega^i = D \omega^i \), which implies

\[ \delta \alpha_i + d \beta_i + \beta_{i-1} A_{i+1}^{i+1} = 0 \] (18)

for each \( D \)-degree \( i \), as the elements of the set \( \{ \omega^i \} \) are linearly independent nontrivial elements of \( H(\gamma) \).

The next step is to analyze the Eq. (18) for each \( D \)-degree. The results depend on the dimension, so we split the analysis into the cases \( n = 3, 4, 5, 6 \), and \( n > 5 \).

**Dimension 3**

(a) degree zero: In \( D \)-degree 0, the Eq. (18) reads \( \delta \alpha_0 + d \beta_0 = 0 \), which implies that \( \alpha_0 \) belongs to \( H_2(\delta d) \). In antifield number 2, this group has nontrivial elements given by Proposition 2, which are proportional to \( C_{\mu \nu}^{a \mu \nu} \). The requirement of translation-invariance restricts the coefficient of \( C_{\mu \nu}^{a \mu \nu} \) to be constant. On the other hand, in \( D \)-degree 0 and ghost number 2, we have \( \omega^0 = C_{\mu \nu}^{a \mu \nu} C_{\mu \nu}^{a \mu \nu} \). To get a parity-breaking but Lorentz-invariant \( \alpha_0 \), a scalar quantity must be build by contracting \( \omega^0, C_{\mu \nu}^{a \mu \nu} \), the tensor \( C_{\mu \nu}^{a \mu \nu} \) and a product of \( \eta_{\mu \nu} \)'s. This cannot be done because there is an odd number of indices, so \( a_0^2 \) vanishes: \( a_0^2 = 0 \). One can then also choose \( b_0 = 0 \).

(b) degree one: We now analyze Eq. (18) in \( D \)-degree 1. It reads \( \delta \alpha_1 + d \beta_1 = 0 \) and implies that \( \alpha_1 \) is an element of \( H_2(\delta d) \). Therefore the only parity-breaking and Poincaré-invariant \( \alpha_1 \) that can be built is \( \alpha_1 = \sum_{i=0}^{M} a_i^2 \), where \( a_i \) is an invariant polynomial.

Indeed, it should have the structure \( \varepsilon C^{CT} \vec{T} \) (or \( \varepsilon C^{CT} \), up to trivial terms), contracted with \( \eta \)'s. In an equivalent way, it must have the structure \( C^{CT} \vec{T} \), contracted with \( \eta \), where the variable \( \vec{T} \) was introduced in Eq. (7). Because of the symmetry properties (8) of \( \vec{T} \) which are the same as the symmetries of \( C_{\mu \nu}^{a \mu \nu} \) and \( C_{\mu \nu}^{a \mu \nu} \), there is only one way of contracting \( \vec{T}, C \), and \( C \) together: \( f^{a \mu \nu} C_{\mu \nu}^{a \mu \nu} \delta \vec{T} \). Of course, no Schouten identity (see the appendix) can come into play because of the number and the symmetry of the fields compos-
ing \( f_{ab}^{\mu} C_{a}^{\alpha \nu} C_{\mu \nu}^{\beta} T_{\nu} \). The latter term is proportional to \( a_{1}^{2} = f_{ab}^{\mu} e^{\mu \nu \rho} C_{\rho}^{\alpha \beta} C_{\nu}^{\beta} T_{\nu} \), up to trivial terms. One can now easily compute that

\[
-b_{1}^{2} a_{1}^{2} \omega_{1} = 3 f_{bc}^{\mu} e^{\mu \nu \rho} (h_{a}^{\alpha \beta} - \frac{1}{3} \eta_{\alpha \beta} h_{a}^{\mu})
\]

\[
\times (\frac{2}{3} T_{\nu}^{\alpha \beta} C_{\mu}^{\beta} T_{\nu}^{\alpha})_{\nu}^{\beta} x^{\nu} dx^{\nu} dx^{\tau}
\]

\[
= 2 f_{ab}^{\mu} e^{\mu \nu \rho} (h_{a}^{\alpha \beta})
\]

\[
- \frac{2}{3} \eta_{\alpha \beta} h_{a}^{\mu} T_{\alpha \beta}^{\mu} x^{\nu} dx^{\nu} dx^{\tau}.
\]

The latter equality holds up to irrelevant trivial \( \gamma \)-exact terms. It is obtained by using the fact that there are only two linearly independent scalars having the structure \( e^{\mu \nu \rho} T_{\nu} \). They are \( e^{\mu \nu \rho} h_{a}^{\alpha \beta} T_{\nu}^{\alpha \beta} \) and \( e^{\mu \nu \rho} h_{a}^{\alpha \beta} T_{\nu}^{\alpha \beta} \).

To prove this, it is again easier to use the dual variable \( T \) instead of \( T^{\alpha \beta} \). One finds that the linearly independent terms with the structure \( e^{\mu \nu \rho} T_{\nu} \) are \( a_{1}^{2} \) and \( b_{1}^{2} \).

Since the expression for \( b_{1}^{2} a_{1}^{2} \omega_{1} \) is not \( \delta \)-exact modulo \( d \) it must vanish: \( f_{ab}^{\mu} = f_{ab}^{\mu} \). One then gets that \( a_{1}^{2} = 0 \) and \( b_{1}^{2} = 0 \).

\( \delta \)-degree one: The equation in D-degree 1 reads \( \delta \alpha_{1}^{(a)} + d \beta_{1}^{(a)} = 0 \). The nontrivial part of \( \alpha_{1}^{(a)} \) has the same form as in D-degree 0. It is however impossible to build a nontrivial Lorentz-invariant \( a_{1}^{2} \) because \( \omega_{1} \sim CT \) has an odd number of indices. So \( a_{1}^{2} = 0 \) and \( b_{1}^{2} = 0 \).

\( \delta \)-degree two: In D-degree 2, the equation \( \delta \alpha_{2} + d \beta_{2} = 0 \) must be studied. Once again, one has \( \alpha_{2} = k_{bc}^{\mu} e^{\mu \nu \rho} C_{\rho}^{\alpha \beta} d^{\alpha} x^{\rho} \). There are two sets of \( \omega_{1}^{\mu} \) that are \( \delta \)-invariant \( \delta T_{\nu}^{\alpha \beta} \). The latter equality is obtained using Schouten identities (see Appendix A).
degree three

(b) degree four: The Eq. (18) reads

\[ \delta \alpha_{i4} + d \beta_{i4} = 0 \]

\( \) However, it is impossible to build a nonvanishing Lorentz-invariant \( a_2 \) because in a product \( C^\ast C \tilde{T} \) there are not enough indices that can be antisymmetrized to be contracted with the Levi-Civita density. So \( \alpha_{i4} \) and \( \beta_{i4} \) can be chosen to vanish.

(c) degree two: The Eq. (18) reads

\[ \delta \alpha_{i2} + d \beta_{i2} = 0 \]

\( \) Once again, there is no way to build a Lorentz-invariant \( a_2 \) because of the odd number of indices. So \( \alpha_{i2} = 0 \) and \( \beta_{i2} = 0 \).

(d) degree three: In D-degree 3, the equation is

\[ \delta \alpha_{i3} + d \beta_{i3} = 0 \]

\( \) This gives rise to an \( a_2 \) of the form "geometric L". There is only one non-trivial Lorentz-invariant object of this form:

\[ a_2 = g_{bc}^\mu \epsilon^{\mu \nu \rho \sigma} \dot{\gamma}_{\nu \rho \beta} \dot{\gamma}^{\mu \nu \beta} \sigma \tau d^3 x. \]

\( \) It is equal to (14) modulo a \( \gamma \)-exact term. One has

\[ b_{i3} = \beta_{i3} \omega^{i3} = -3 g_{bc}^\mu \epsilon^{\mu \nu \rho \sigma} \left( h_{\mu \nu}^a - \frac{1}{5} \eta_{\mu \nu} h^a \right) \]

\( \times \dot{T}_{\nu \rho \beta} \dot{\gamma}^{\mu \nu \beta} \sigma \tau d^3 x. \]

\( \) The coefficient of \( \omega^{i3} \) is \( \dot{U} \dot{U} \) cannot be \( \delta \)-exact modulo \( D \) unless it vanishes, which implies that

\[ g_{bc}^a = g_{bc}^a. \]

One is left with the equation

\[ \delta \alpha_{i3} + d \beta_{i3} = 0, \]

\( \) but once again it has no Lorentz-invariant solution because of the odd number of indices to be contracted. So \( \alpha_{i3} = 0 \) and \( \beta_{i3} = 0. \)

(f) degree higher than four: There is again no \( a_i \) for \( i > 4 \), for the same reasons as in four dimensions.

Dimension \( n > 5 \)

No new \( a_2 \) arises because it is impossible to build a nonvanishing parity-breaking term by contracting an element of \( H^2(\delta [d]), i.e. C^\ast C, \) two ghosts from the set \( \{ C^\mu, \tilde{T}^\mu \rho \sigma, \dot{\gamma}^{\mu \nu \rho \sigma} \} \), an epsilon-tensor \( \epsilon^{\mu \nu \rho \sigma} \) and metrics \( \eta_{\mu \nu}. \)

Let us finally notice that throughout this proof we have acted as if \( \alpha_1's \) trivial in \( H^2(\delta [d]) \) lead to trivial \( a_2's. \) The correct statement is that trivial \( a_2's \) correspond to \( \alpha_1's \) trivial in \( H^2(\delta [d], H(\gamma)) \) (see e.g. [9] for more details).

However, both statements are equivalent in this case, since both groups are isomorphic (Proposition 3).

This ends the proof of Theorem 1.

B. Deformation in 3 dimensions

In the previous section, we determined that the only nontrivial first-order deformation of the free theory in three dimensions deforms the gauge algebra by the term (13). We now check that this deformation can be consistently lifted and leads to a consistent first-order deformation of the Lagrangian. However, we then show that obstructions arise at second order, i.e. that one cannot construct a corresponding consistent second-order deformation unless the whole deformation vanishes.

1. First-order deformation

A consistent first-order deformation exists if one can solve Eq. (15) for \( a_0 \), where \( a_1 \) is obtained from Eq. (16). The existence of a solution \( a_1 \) to Eq. (16) with \( a_2 = a_2^3 \) is a consequence of the analysis of the previous section. Indeed, the \( a_3's \) of Theorem 1 are those that admit an \( a_3 \) in Eq. (16). Explicitly, \( a_1 \) reads, modulo trivial terms,

\[ a_1 = f_{[bc]}^\alpha \epsilon^{\mu \nu \rho \sigma} \left( 3 h_{\alpha}^{a \beta \lambda} - \frac{1}{5} \eta_{\mu \nu} h_{\alpha}^{a \lambda} h^a \right) \theta_{\alpha \mu \lambda} T_{\nu \rho \beta} + \frac{1}{2} C^b_{\alpha \mu \lambda} \delta_{[\rho} h_{\sigma \lambda]} + \frac{1}{2} h_{\alpha \mu} \tilde{T}^b_{\lambda \sigma \rho} h_{\nu}^c + h_{\alpha \mu} C^a_{\nu} \left( -\frac{1}{2} \delta^{\lambda \sigma} h_{\lambda \alpha \rho} + \delta_{(\lambda} h_{\sigma \rho)} \right) d^3 x. \]

On the contrary, a new condition has to be imposed on the structure function for the existence of an \( a_0 \) satisfying Eq. (15). Indeed a necessary condition for \( a_0 \) to exist is that

\[ \delta_{aa} f_{[bc]}^d = f_{[abc]}, \]

which means that the corresponding internal anticommutative algebra \( \mathcal{A} \) is endowed with an invariant norm. The internal metric we use is \( \delta_{ab} \), which is positive-definite.

The condition is also sufficient and \( a_0 \) reads, modulo trivial terms,
To prove these statements about $a_0$, one writes the most general $a_0$ with two derivatives, that is Poincaré-invariant but breaks the parity symmetry. One inserts this $a_0$ into the equation to solve, i.e., $\delta a_1 + \gamma a_0 = db_0$, and computes the $\delta$ and $\gamma$ operations. One takes an Euler-Lagrange derivative of the equation with respect to the ghost, which removes the total derivative $db_0$. The equation becomes $rac{\delta}{\delta a_1} (\delta a_1 + \gamma a_0) = 0$, which we multiply by $C\beta$. The terms of the equation have the structure of linear combinations of a set of linearly independent quantities, which is not obvious as there are Schouten identities relating them (see Appendix A 4). One can finally solve the equation for the arbitrary coefficients in $a_0$, yielding the above results.

2. Second-order deformation

Once the first-order deformation $W_1 = \int (a_0 + a_1 + a_2)$ of the free theory is determined, the next step is to investigate whether a corresponding second-order deformation $W_2$ exists. This second-order deformation of the master equation is constrained to obey $sW_2 = -\frac{1}{2} (W_1, W_1)_{a.b.}$ (see Sec. III C). Expanding both sides according to the antighost number yields several conditions. The maximal antighost number condition reads

$$\frac{1}{2} (a_2, a_2)_{a.b.} = \gamma c_2 + \delta c_3 + df_2,$$

where $W_2 = \int d^3x (c_0 + c_1 + c_2 + c_3)$ and $antigh (c_i) = \iota$. It is easy to see that the expansion of $W_2$ can be assumed to stop at antighost number 3 and that $c_3$ may be assumed to be invariant. The calculation of $(a_2, a_2)_{a.b.}$, where $a_2 = \int_{[bc]} e^{\mu \nu \rho} C^a \mu C^b \rho \partial \partial c \sigma$, gives

$$0 \equiv C^a \mu \tilde{T}^c \mu \tilde{T}^d$$

in order to substitute in Eq. (20) the expression of $C^{a \sigma} C^b \mu \tilde{T}^c \mu \tilde{T}^d$, in terms of the other summands appearing in Eq. (21). Consequently, the following expression for $(a_2, a_2)_{a.b.}$ contains only linearly independent terms:

$$\begin{align*}
(a_2, a_2)_{a.b.} &= \frac{g^a}{\delta a_2} \frac{g^b}{\delta a_2} \bigg[ \frac{1}{2} C^{c \sigma \tau} C^b \mu \tilde{T}^c \mu \tilde{T}^d, \tilde{T}^d \bigg] - \frac{1}{3} \tilde{T}^{c \mu} \tilde{T}^{b \nu} T_{\mu}^{a, \nu} - \frac{1}{3} \tilde{T}^{b \nu} \tilde{T}^{c \mu} T_{\nu}^{a, \mu}
&\quad + \frac{1}{3} \tilde{T}^{c \mu} \tilde{T}^{b \nu} T_{\nu}^{a, \mu} - \frac{1}{3} \tilde{T}^{b \nu} \tilde{T}^{c \mu} T_{\mu}^{a, \nu}
&= \gamma \mu + d \nu + f^{a \mu}_{bc} f^{b \nu}_{ed} C^{c \sigma \tau} C^{d \mu \nu} \tilde{T}^{e \rho} \tilde{T}^{d \rho} \tilde{T}^{e \rho} \tilde{T}^{d \rho} \tilde{T}^{e \rho} \tilde{T}^{d \rho}
&\quad + C^{c \mu \nu} \tilde{T}^{b \nu} \tilde{T}^{c \mu} - \frac{1}{3} C^{c \mu \nu} \tilde{T}^{b \nu} \tilde{T}^{c \mu} - \frac{1}{3} C^{c \mu \nu} \tilde{T}^{b \nu} \tilde{T}^{c \mu}.
\end{align*}$$
\[(a_2, a_2)_{a.h} = \gamma \mu + d \nu + C^{\epsilon \mu \nu \rho} [f^a_{(bc)} f_{dea} \epsilon^{bpa} \tilde{T}_c^{\rho \alpha} \tilde{T}_d^{\beta \mu}] + \frac{1}{a^2 b c} f_{dea} C^\mu \nu \rho \omega \tilde{T}_\alpha^{\nu} \tilde{T}_\beta^{\rho} \tilde{T}_\gamma^{\omega} + 2 \alpha \epsilon_{(b) d e c} \epsilon^{a \beta \gamma} \tilde{T}_c^{\beta \rho} \tilde{T}_d^{\gamma} \nu \\
+ \frac{1}{a^2 b c} f_{dea} C^\mu \nu \rho \omega \tilde{T}_\alpha^{\nu} \tilde{T}_\beta^{\rho} \tilde{T}_\gamma^{\omega} + \frac{1}{a^2 b c} f_{dea} C^\mu \nu \rho \omega \tilde{T}_\alpha^{\nu} \tilde{T}_\beta^{\rho} \tilde{T}_\gamma^{\omega} \]

where we used that the structure constants of \( \mathcal{A} \) obey \( f_{abc} = \delta_{ad} f^d_{bc} = f_{(abc)} \).

Therefore, the above expression is a \( \gamma \)-coboundary modulo \( d \) if and only if \( f^a_{bc} f_{dea} = 0 \), meaning that the internal algebra \( \mathcal{A} \) is nilpotent of order three. In turn, this implies that \( f^a_{bc} = 0 \) and the deformation is trivial.

C. Deformation in 5 dimensions

Let us perform the same analysis for the candidate in five dimensions.

1. First-order deformation

First, \( a_1 \) must be computed from \( a_2 \) (given by (14)), using the equation \( \delta a_2 + \gamma a_1 + db_1 = 0 \):

\[
\delta a_2 = -3g_{(bc)}^a \epsilon^{\mu \nu \rho \sigma \tau} \partial_\nu h^a_{\alpha \mu} \partial_\rho h^a_{\beta \mu} \partial_\sigma h^a_{\gamma \mu} \partial_\tau h^a_{\delta \mu} d^5x
- \partial \partial h^a_{\alpha \mu} \partial_\nu h^a_{\beta \mu} \partial_\rho h^a_{\gamma \mu} \partial_\sigma h^a_{\delta \mu} \partial_\tau h^a_{\mu} d^5x
+ \partial \partial h^a_{\alpha \mu} \partial_\nu h^a_{\beta \mu} \partial_\rho h^a_{\gamma \mu} \partial_\sigma h^a_{\delta \mu} \partial_\tau h^a_{\mu} d^5x.
\]

We recall that it is a consequence of Theorem 1 that \( g_{bc}^a \) is symmetric in its lower indices, thereby defining a commutative algebra. It is not an assumption, it comes from consistency. Therefore the first term between square bracket vanishes because of the symmetries of the structure constants \( g_{bc}^a \) of the internal commutative algebra \( \mathcal{B} \). We finally obtain, modulo trivial terms,

\[
a_1 = \frac{1}{2} g_{(bc)}^a \epsilon^{\mu \nu \rho \sigma \tau} \partial_\nu h^a_{\alpha \mu} \partial_\rho h^a_{\beta \mu} \partial_\sigma h^a_{\gamma \mu} \partial_\tau h^a_{\delta \mu} d^5x
- 2 \partial h^a_{\alpha \mu} \partial_\nu h^a_{\beta \mu} \partial_\rho h^a_{\gamma \mu} \partial_\sigma h^a_{\delta \mu} d^5x.
\]

The element \( a_1 \) gives the first-order deformation of the gauge transformations. By using the definition of the generalized de Wit-Freundman connections [12], we get the following simple expression for \( a_1 \):

\[
a_1 = g_{(bc)}^a \epsilon^{\mu \nu \rho \sigma \tau} C^\mu \nu \rho \sigma \tau \Gamma_{\alpha \beta \gamma} d^5x,
\]

where \( \Gamma_{\alpha \beta \gamma} \) is the second spin-3 connection

\[\text{2. Second-order deformation}\]

The next step is the equation at order 2: \( (W_1, W_1)_{a.b} = -2W_2 \). In particular, its antighost 2 component reads \( (a_2, a_2)_{a.b} = \gamma c_2 + \partial_\mu f^a_{\mu} \). The left-hand side is directly
computed from Eq. (14):

\[
(a_2, a_2)_{a,b} = -g^{a}_{bc}g^{d(e}e^{f}e^{g}e_{\mu}^{p\sigma\tau}\delta_{\tau}^{(\mu}\delta^{(\nu}g^{a)}_d \\
\times (4\partial_{\tau}\mu\nu\rho\sigma\tau\delta_{\tau}^{(\mu}\delta^{(\nu}g^{a)}_d + 2\partial_{\tau}\mu\nu\rho\sigma\tau\delta_{\tau}^{(\mu}\delta^{(\nu}g^{a)}_d \\
\times \partial_{\tau}\mu\nu\rho\sigma\tau\delta_{\tau}^{(\mu}\delta^{(\nu}g^{a)}_d \\
= -12\delta^{a}_{b(c}g^{d)e\mu\nu\rho\sigma\tau\delta_{\tau}^{(\mu}\delta^{(\nu}g^{a)}_d \mu\nu\rho\sigma \\
+ \gamma c_2 + \delta_{\mu}f_2.
\]

The first term appearing in the right-hand side of the above equation is a nontrivial element of \( H(\gamma|d) \). Its vanishing implies that the structure constants \( g_{(abc)} \) of the commutative invariant-normed algebra \( \mathcal{B} \) must obey the associativity relation \( g_{(b(c|d|e)a) = 0. \) As for the spin-2 deformation problem (see [9], Secs. 5.4 and 6), this means that, modulo redefinitions of the fields, there is no cross-interaction between different kinds of spin-3 gauge fields provided the internal metric in \( \mathcal{B} \) is positive-definite— which is demanded by the positivity of energy. The cubic vertex \( a_0 \) can thus be written as a sum of independent self-interacting vertices, one for each field \( h_{\mu\nu\rho}, a = 1, \ldots, N. \) Without loss of generality, we may drop the internal index \( a \) and consider only one single self-interacting spin-3 gauge field \( h_{\mu\nu\rho} \).

V. CONCLUSION

By relaxing the parity the complexification imposed in [8], the present work completes the classification of the consistent non-Abelian perturbative deformations of Fronsdal’s spin-3 gauge theory, under the assumptions of Poincaré invariance and locality.

In [8] the first-order cubic vertex of Berends, Burgers, and van Dam [2] was recovered. However, the latter vertex leads to inconsistencies when continued to second order in the coupling constant, as was shown in several places [3,5,6]. A new first-order non-Abelian deformation, leading to a cubic vertex, was also found in [8]. It is defined in space-time dimension \( n \geq 5 \) and passes the second-order test where the vertex of [2] shows an inconsistency.

In the present paper, by explicitly breaking parity invariance, we obtained two more consistent non-Abelian first-order deformations, leading to a cubic vertex in the Lagrangian. The first one is defined in \( n = 3 \) and involves a multiplet of gauge fields \( h_{\mu\nu\rho} \), taking values in an internal, anticommutative, invariant-normed algebra \( \mathcal{A} \). The second one lives in a space-time of dimension \( n = 5 \), the fields taking value in an internal, commutative, invariant-normed algebra \( \mathcal{B} \). Taking the metrics which define the inner product in \( \mathcal{A} \) and \( \mathcal{B} \) positive-definite (which is required for the positivity of energy), the \( n = 3 \) candidate gives rise to inconsistencies when continued to perturbation order two, whereas the \( n = 5 \) one passes the test and can be assumed to involve only one kind of self-interacting spin-3 gauge field \( h_{\mu\nu\rho} \), bearing no internal “color” index.

Remarkably, the cubic vertex of the \( n = 5 \) deformation is rather simple. Furthermore, the Abelian gauge transformations are deformed by the addition of a term involving the second de Wit-Freedman connection in a straightforward way, cf. Eq. (23). The relevance of this second generalized Christoffel symbol in relation to a hypothetical spin-3 covariant derivative was already stressed in [5].

It is interesting to compare the results of the present spin-3 analysis with those found in the spin-2 case first studied in [26]. There, two parity-breaking first-order consistent non-Abelian deformations of Fierz-Pauli theory were obtained, also living in dimensions \( n = 3 \) and \( n = 5 \). The massless spin-2 fields in the first case bear a color index, the internal algebra \( \mathcal{A} \) being commutative and further endowed with an invariant scalar product. In the second, \( n = 5 \) case, the fields take value in an anticommutative, invariant-normed internal algebra \( \mathcal{B} \). It was further shown in [26] that the \( n = 3 \) first-order consistent deformation could be continued to all orders in powers of the coupling constant, the resulting full interacting theory being explicitly written down\(^2\). However, it was not determined in [26] whether the \( n = 5 \) candidate could be continued to all orders in the coupling constant. Very interestingly, this problem was later solved in [29], where a consistency condition was obtained at second order in the deformation parameter, viz the algebra \( \mathcal{B} \) must be nilpotent of order three. Demanding positivity of energy and using the results of [26], the latter nilpotency condition implies that there is actually no \( n = 5 \) deformation at all: the structure constant of the internal algebra \( \mathcal{B} \) must vanish [29]. Stated differently, the \( n = 5 \) first-order deformation candidate of [26] was shown to be inconsistent [29] when continued at second order in powers of the coupling constant, in analogy with the spin-3 first-order deformation written in [2].

In the present spin-3 case, the situation is somehow the opposite. Namely, it is the \( n = 3 \) deformation which shows inconsistencies when going to second order, whereas the \( n = 5 \) deformation passes the first test. Also, in the \( n = 3 \) case the fields take values in an anticommutative, invariant-normed internal algebra \( \mathcal{A} \) whereas the fields in the \( n = 5 \) case take value in a commutative, invariant-normed algebra \( \mathcal{B} \). However, the associativity condition deduced from a second-order consistency condition is obtained for the latter case, which implies that the algebra \( \mathcal{B} \) is a direct sum of one-dimensional ideals. We summarize the previous discussion in Table I.

\(^2\)Since the deformation is consistent, starting from \( n = 3 \) Fierz-Pauli, the complete \( n = 3 \) interacting theory of [26] describes no propagating physical degree of freedom. On the contrary, the topologically massive theory in [27,28] describes a massive graviton with one propagating degree of freedom (and not two, as was erroneously typed in [26]; N. B. wants to thank S. Deser for having pointed this out to him).
It would be of course very interesting to investigate further the $n=5$ deformation exhibited here, since if the deformation can be consistently continued to all orders in powers of the coupling constant, this would give the first consistent interacting Lagrangian for a higher-spin gauge field. We hope to come back to this issue in the near future.

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APPENDIX A: SCHOUTEN IDENTITIES

The Schouten identities are identities due to the fact that in $n$ dimensions the antisymmetrization over any $n+1$ indices vanishes. These identities obviously depend on the dimension and relate functions of the fields. As it is important to have a real basis when solving equations, this appendix is devoted to finding a basis for various sets of functions, depending on their structure and the number of dimensions. We consider Poincaré-invariant and parity-breaking functions, and we are interested mainly in the $\gamma$-nontrivial quantities. The fields may have an internal index but we only write it when necessary.

1. Functions of the structure $\varepsilon C^s \hat{T} \hat{T} \hat{T}$ in $n=4$

In order to achieve the four-dimensional study of the algebra deformation in $D$-degree 2, a list of the Schouten identities is needed for the functions of the structure $\varepsilon C^s \hat{T} \hat{T} \hat{T}$. The space of these functions is spanned by

$$T_{1}^{bc} = \varepsilon^{\mu \nu \rho \sigma} C_{\mu \nu}^{a} \hat{T}^{b}_{\rho \sigma} \hat{T}^{c}_{\rho \sigma},$$

$$T_{2}^{bc} = \varepsilon^{\mu \nu \rho \sigma} C_{\mu \nu}^{a} \hat{T}^{b}_{\rho \sigma} \hat{T}^{c}_{\rho \sigma},$$

$$T_{3}^{bc} = \varepsilon^{\mu \nu \rho \sigma} C_{\mu \nu}^{a} \hat{T}^{b}_{\rho \sigma} \hat{T}^{c}_{\rho \sigma},$$

There are two Schouten identities. Indeed, one should first notice that all Schouten identities are linear combinations of identities with the structure $\delta^{[\alpha \beta \gamma \delta \varepsilon]} \varepsilon^{\mu \nu \rho \sigma} C_{\mu \nu}^{a} \hat{T}^{b}_{\rho \sigma} \hat{T}^{c}_{\rho \sigma} = 0$, where the indices $\alpha \beta \gamma \delta \varepsilon$ are contracted with the indices of the ghosts and where $\delta^{[\alpha \beta \gamma \delta \varepsilon]} = [\alpha \beta \gamma \delta \varepsilon]$. Furthermore, there are only two independent identities of this type:

$$\delta^{[\alpha \beta \gamma \delta \varepsilon]} \varepsilon^{\mu \nu \rho \sigma} C_{\mu \nu}^{a} \hat{T}^{b}_{\rho \sigma} \hat{T}^{c}_{\rho \sigma} \lambda = 0,$$

$$\delta^{[\alpha \beta \gamma \delta \varepsilon]} \varepsilon^{\mu \nu \rho \sigma} C_{\mu \nu}^{a} \hat{T}^{b}_{\rho \sigma} \hat{T}^{c}_{\rho \sigma} \lambda = 0.$$

Expanding the product of $\delta$, one finds that the first identity implies that $T_{1}^{bc}$ is symmetric: $T_{1}^{bc} = T_{1}^{(bc)}$, while the second one relates $T_{2}^{bc}$ and $T_{3}^{bc}$.

So, in four dimensions, a basis of the functions with the structure $\varepsilon C^s \hat{T} \hat{T}$ is given by $T_{1}^{(bc)}$ and $T_{3}^{(bc)}$.

2. Functions of the structure $\varepsilon h \hat{U} \hat{T} \hat{T}$ in $n=4$

These functions appear in the study of the algebra deformation in $D$-degree 3, $n=4$. They are completely generated by the following terms:

$$T_{1} = \varepsilon^{\mu \nu \rho \sigma} h_{\mu \nu}^{a \alpha \beta \gamma} \hat{T}^{b}_{\rho \sigma} \hat{T}^{c}_{\rho \sigma},$$

$$T_{2} = \varepsilon^{\mu \nu \rho \sigma} h_{\mu \nu}^{a \alpha \beta \gamma} \hat{T}^{b}_{\rho \sigma} \hat{T}^{c}_{\rho \sigma},$$

$$T_{3} = \varepsilon^{\mu \nu \rho \sigma} h_{\mu \nu}^{a \alpha \beta \gamma} \hat{T}^{b}_{\rho \sigma} \hat{T}^{c}_{\rho \sigma}.$$
Indeed, they imply that $T_1^{[bc]} + T_2^{[bc]} = 0$ and $T_2^{[bc]} = T_2^{(bc)}$, which can be satisfied only if $T_1^{[bc]} = T_2^{(bc)} = 0$.

4. Functions of the structure $\varepsilon C \partial^3 hh$ and $\varepsilon C \partial^3 h \partial h$ in $n = 3$

These functions appear when solving $\delta a_1 + \gamma a_0 = db_0$ in Sec. IV B. In generic dimension ($n > 4$), there are, respectively, 45 and 130 independent functions in the sets $\varepsilon C \partial^3 hh$ and $\varepsilon C \partial^3 h \partial h$. In three dimensions, there are 108 Schouten identities relating them, which leave only 67 independent functions. We have computed all these identities and the relations between the 108 dependent functions and the 67 independent ones. However, given their numbers, they will not be reproduced here.