BÉZOUT DOMAINS AND LATTICE-VALUED MODULES.

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ABSTRACT. Let B be a commutative Bézout domain B and let $MSpec(B)$ be the maximal spectrum of B. We obtain a Feferman-Vaught type theorem for the class Mod_B of B-modules. We analyse the definable sets in terms, on one hand, of the definable sets in the classes $Mod_{B_{\mathcal{M}}}$, where $B_{\mathcal{M}}$ ranges over the localizations of B at $\mathcal{M} \in MSpec(B)$, and on the other hand, of the constructible subsets of $MSpec(B)$. When B has good factorization, it allows us to derive decidability results for the class Mod_B, in particular when B is the ring $\widetilde{\mathbb{Z}}$ of algebraic integers or the one of rings $\widetilde{\mathbb{Z}} \cap \mathbb{R}$, $\widetilde{\mathbb{Z}} \cap \mathbb{Q}_p$.

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1. Introduction

Let B be a commutative Bézout domain with 1 and let $B^\star := B \setminus \{0\}$. Let $MSpec(B)$ be the space of maximal ideals of B endowed with the Zariski topology. (Basic closed sets are $V(a) := \{\mathcal{M} \in MSpec(R) : a \in \mathcal{M}\}$ and constructible subsets of $MSpec(B)$ are the elements of the Boolean algebra generated by the basic closed subsets.)

In the class Mod_B of all B-modules M, we will describe the definable subsets of M in terms of the definable sets in each localization $M_{\mathcal{M}}$, $\mathcal{M} \in MSpec(B)$, and of constructible subsets of $MSpec(B)$. This description of definable subsets can be seen as a Feferman-Vaught type result and this is the content of our main Theorem 5.6. We will work in a definable expansion of the language of B-modules, adding to the usual language of modules unary predicates for submodules indexed by the group of divisibility $\Gamma(B)$ of B.

A key intermediate step in the proof of Theorem 5.6 is a positive quantifier elimination result in that expansion of the language (showing that any positive primitive formula is equivalent to a conjunction of atomic formulas) in the class Mod_B when B is a valuation domain (Theorem 3.2). Even though this result was essentially known, in order to apply it when B is a Bézout domain, we need to control what happens when we localize B at maximal ideals.

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For $a, b \in B^\star$, recall that the *Jacobson radical relation* $a \in rad(b)$ holds if a belongs to every maximal ideal that contains b and denote by $gcd(a, b)$ the greatest common divisor of a, b.

In [6, 2.10], L. van den Dries and A. Macintyre defined the property for a Bézout domain B to have *good factorisation*, namely if for all $a,\ b \in B^\star$, there are $c,\ a_1 \in B$ such that $a = c.a_1$ with $gcd(c, b) = 1$ and $b \in rad(a_1)$. Then they observed that in case B has good factorization, the constructible subsets of $MSpec(B)$ are either basic closed sets or basic open sets [6, Lemma 2.12]. Furthermore, they show that one can encode the properties of the Boolean algebra of constructible subsets of $MSpec(B)$ using the Jacobson radical relation.

Then, under the further assumption that B has good factorization, we consider questions of decidability for the class Mod_B. First we make explicit in our context the property for a countable ring B to be *effectively given* [25, Chapter 17]; we call the resulting assumption on B, *assumption (EF)* (Definition 4.1).

We show that the class Mod_B is decidable (Proposition 5.8), under the following hypotheses on B: B is countable and it satisfies assumption (EF), it has good factorization, each quotient $B/\mathcal{M}$ is infinite, for $\mathcal{M} \in MSpec(B)$, and the Jacobson radical relation is recursive.

Bézout domains with good factorization include the class of good Rumely domains introduced by L. van den Dries and A. Macintyre [6]. Even though, their work takes place in the context of rings, there are similarities in their approach and the one we are taking. They show that the class of good Rumely domains admits quantifier elimination in the language of rings extended by a family of radical relations [6, Theorem 3.14]. There are three main ingredients in their proof: Rumely local-global principle, the quantifier elimination result for non-trivial valuation rings with algebraically closed fraction fields (following from a theorem of A. Robinson) and the fact that the constructible subsets of the maximal spectrum of a such ring form an atomless Boolean algebra. They axiomatize the class of good Rumely domains and retrieve the former result of van den Dries [5] on the decidability of the ring of algebraic integers.

The plan of the paper is as follows. In section 2, we recall the basic notions of the model theory of modules (or abelian structures) that we will use and the properties of the group of divisibility $\Gamma(B)$ of a Bézout domain.

In section 3, for A a valuation domain, we revisit a quantifier elimination result in the class Mod_A, adding to the module language unary predicates for certain pp definable submodules, following the approach of Bélair–Point [2, Proposition 4.1]. This result was essentially known but we need the additional property that for A of the form $B_{\mathcal{M}}$, where $B_{\mathcal{M}}$ is the localization of a Bézout domain B at a maximal ideal, to any pp $\mathcal{L}_{B_{\mathcal{M}}}$-formula one can associate a constructible subset of $MSpec(B)$ over which the elimination is uniform.

In section 4, we give a direct proof of a decidability result in the case of valuation domains with infinite residue field. This was previously done in [22] in the case of an archimedean value group and later extended in [14] in the general case (and furthermore without the assumption on the residue field). The proof given here is more algebraic than the one in [22] which is of a more geometrical nature. We apply the result in section 5 in the case of Bézout domains.

In section 5, we first relate the property of having good factorization for B to properties of $MSpec(B)$ and note that such ring is *adequate*, a better known property. Then we prove our main theorem, a Feferman-Vaught type result for the class Mod_B, which takes a simpler form in the case where B has good factorization.

We derive a decidability result when B is an effectively given countable Bézout domain where the Jacobson radical relation is recursive, assuming that the quotient $B/\mathcal{M}$ is infinite for any maximal ideal $\mathcal{M}$. Note that L. Gregory observed that if the theory of all B-modules is decidable, then the prime radical relation is recursive [14, Lemma 3.2]. For good Rumely domains, the prime radical relation and the Jacobson radical relation *rad* always coincide [27] and so our hypothesis on the Jacobson radical relation is justified in view of Gregory's result.

In section 6, we apply our decidability result to the case where B is an effectively given good Rumely domain e.g. $\widetilde{\mathbb{Z}}$ and to the cases $\widetilde{\mathbb{Z}} \cap \mathbb{R}$, $\widetilde{\mathbb{Z}} \cap \mathbb{Q}_p$ [28].

In the last subsection, we discuss the case when B is either the ring of holomorphic functions over $\mathbb{C}$ or the integral closure of that ring. Of course in this case, the ring is uncountable (and so the language of modules is uncountable), but also it is not known to satisfy the Rumely local-global principle [6, 5.6]. However since these rings has good factorization, we still have a manageable description of definable subsets in that class, using Corollary 5.7.

Then, in the last section, we introduce the notion of ℓ-valued B-modules, in view of future work. When B is a valuation domain, we get back the more classical notion of valued modules [3], [8].

Finally let us note that since this paper was submitted other works on decidability of the theory of modules over Bézout domains appeared; see for instance *Decidability of the theory of modules over Bézout domains with infinite residue fields, arXiv:1706:08940* by L. Gregory, S. L'Innocente, G. Puninskiĭ and C. Toffalori.

2. Preliminaries

Throughout the paper, all our rings B will be commutative integral domains with 1. Let $B^\star := B \setminus \{0\}$. Then B is *Bézout* if every finitely generated ideal is principal, equivalently if for any $a, b \in B^\star$, there exist $c, u, v, a_1, b_1 \in B$, such that $c = a.u + b.v$ and $a = c.a_1$, $b = c.b_1$ (the Bézout relations). We set $c = gcd(a, b)$ and we denote by $(a : b) := a_1$, a generator of the ideal $(a) : (b) := \{u \in B : \ b.u \in (a)\}$; note that these two elements c, a_1 of B are defined up to an invertible element. We can also define the least common multiple of two elements a, b, denoted by $lcm(a, b)$. It is easily checked that $a.b = gcd(a, b).lcm(a, b).u$, where u is an invertible element of B.

In the next two subsections, we will quickly review some basic facts on the one hand the group of divisibility of a Bézout domain and on the other hand abelian structures.

2.1. **Group of divisibility.**

Let B be a Bézout domain and denote by $Q(B)^\times$ the multiplicative group of the field of fractions of B and by U the subgroup of units (equivalently of invertible elements) of B.

Recall that the *group of divisibility* $\Gamma(B)$ of B is the quotient of $Q(B)^\times$ by U, more formally $\Gamma(B) = (Q(B)^\times/U, \cdot, 1)$, where 1 denotes the neutral element and is the coset $1 \cdot U$. Denote by $\Gamma^+(B) := \{a.U : a \in B^\star\}$. One can define a partial order on $\Gamma(B)$ by: $a.U \leq b.U$ iff $b.a^{-1} \in B$. Endowed with this partial order, $\Gamma(B)$ becomes a partially ordered group [12, 1.2]. Note that $\Gamma^+(B)$ corresponds to the submonoid of positive elements of $\Gamma(B)$ (namely those bigger than or equal to 1). Recall that since B is Bézout, this order is a lattice order and so $\Gamma(B)$ is an abelian lattice-ordered group (in short, ℓ-group) [9, Chapter 3, Proposition 4.5]. For $a,\ b \in B^\star$, we can explicitly define the lattice operations on the set of positive elements $\Gamma^+(B)$ as follows: $a.U \wedge b.U := gcd(a,b).U$ and $a.U \vee b.U = lcm(a,b).U$. One then extends the lattice operations to the elements of $Q(B)^\times$.

Definition 2.1. [33, Definition 1] Let D be an integral domain, $(\Gamma, \wedge, \cdot, 1)$ be an ℓ-group and set $\bar{\Gamma} := \Gamma \cup \{\infty\}$ with $\infty \cdot a = \infty \geq a$ for $a \in \bar{\Gamma}$. Then the map $v :\ D \to \bar{\Gamma}$ is an *ℓ-valuation* if it satisfies the properties (1) up to (3) below: for all $a,\ b \in D$,

(1) $v(a+b) \geq v(a) \wedge v(b)$,
(2) $v(a.b) = v(a).v(b)$,
(3) $v(1) = 1$ and $v(0) = \infty$.

One can show that every ℓ-valuation on an integral domain D has a unique extension to its field of fractions $Q(D)$ [33, Corollary to Proposition 2].

When B is a Bézout domain, the map $v : B^\star \to \Gamma(B) : a \mapsto a.U$ is an ℓ-valuation on B. For $\mathcal{M} \in MSpec(B)$, the space of maximal ideals of B, we denote by $B_{\mathcal{M}}$ the localization of B at $\mathcal{M}$. Let $U_{\mathcal{M}}$ denote the subgroup of invertible elements of $((B_{\mathcal{M}})^\star, \cdot, 1)$. Since B is a Bézout domain, $B_{\mathcal{M}}$ is a valuation domain and $\Gamma(B_{\mathcal{M}})$ is an abelian totally ordered group.

It is well-known that any ℓ-group is isomorphic to a subdirect product of totally-ordered abelian groups [12, Lemmas 3.2.5, 3.5.4]. In our framework, it is useful for us to describe that isomorphism as follows. We will indicate that we consider a subdirect product by using $\prod^s$.

Lemma 2.2. *The map $f :\ a.U \to (a.U_{\mathcal{M}})_{\mathcal{M} \in MSpec(B)}$, with $a \in B$, induces an isomorphism between the lattice-ordered monoid $(\Gamma^+(B), ., \wedge, 1)$ and $\prod^s_{\mathcal{M} \in MSpec(B)}(\Gamma(B_{\mathcal{M}})^+, ., \wedge, 1)$, where $(\Gamma^+(B_{\mathcal{M}}), ., \wedge, 1)$, $\mathcal{M} \in MSpec(B)$ are totally ordered monoids. It can be extended to an isomorphism of lattice-ordered groups, that we will still denote by f, from $(\Gamma(B), ., \wedge, 1)$ to $\prod^s_{\mathcal{M} \in MSpec(B)}(\Gamma(B_{\mathcal{M}}), ., \wedge, 1)$.* □

In particular, we denote by $f_{\mathcal{M}} : \Gamma(B)^+ \to \Gamma(B_{\mathcal{M}})^+ :\ a.U \to a.U_{\mathcal{M}}$ with $a \in B^\star$. This induces an ℓ-valuation $v_{\mathcal{M}}$ on $B_{\mathcal{M}}$, with values in $\Gamma(B_{\mathcal{M}})^+$ as follows: let $a \in B$, $s \in B \setminus \mathcal{M}$, then $v_{\mathcal{M}}(a.s^{-1}) := a.s^{-1}.U_{\mathcal{M}} = a.U_{\mathcal{M}} = f_{\mathcal{M}}(a.U)$.

2.2. **Abelian structures.**

We will consider the class of (right) B-modules endowed with a family of subgroups. Let $\mathcal{L}_B := \{+, -, 0, \cdot a; a \in B\}$ be the language of B-modules, where $\cdot a$ denotes scalar multiplication by a, and let $\mathcal{L}_{B,V}$ be the language of B-modules expanded by a set $\{V_\delta; \delta \in \Delta\}$ of unary predicates (namely unary relation symbols); i.e. $\mathcal{L}_{B,V} := \mathcal{L}_B \cup \{V_\delta; \delta \in \Delta\}$ (and Δ is some index set). (When the ring B is clear from the context, we will drop the subscript B).

Given a B-module M , we will consider its expansion M_V by a prescribed family of subgroups $V_\delta(M)$, $\delta \in \Delta$; this is an instance of an *abelian structure* [7], [25, Chapter 3, 3A]. E. Fisher in his thesis had extended to the class of abelian structures most of the classical results on the model theory of theory of modules. At the beginning of [34], M. Ziegler pointed out that most results in the model theory of modules still hold in this larger setting. Later, in [17, section 1.9], T. Kucerǎ and M. Prest described a way to view any abelian structure as a module over a certain path algebra associated with the language. This point of view has the advantage of staying in the classical framework of modules but the disadvantage of changing the ring.

We are interested in describing the definable subsets of such $\mathcal{L}_V$-structure M_V and we will use the Baur–Monk pp elimination theorem, namely any $\mathcal{L}_V$-formula is equivalent to a boolean combination of pp formulas and invariant sentences [16, Appendix A.1]. Moreover, such pp elimination is effective and uniform in the class of $\mathcal{L}_V$-structures.

Let $p^+(x)$ (respectively $p^-(x)$) be a set of pp formulas (in one variable, in $\mathcal{L}_V(A)$, where A is a set of parameters) such that any pp formula either belong to p^+ or to $p^-(x)$ (but not to both) and such that given any finite subset E of formulas in $p^+(x)$ there is a module M_V (containing A) and an element $m \in M_V$ such that $\phi(m)$ holds in M_V for any $\phi \in E$. We will call p^+ a pp type. Denote by $\neg p^-(x)$ the collection of negations of formulas in $p^-(x)$. Then a type $p(x)$ (in one variable) is of the form $p^+ \cup \neg p^-(x)$ and we say it is realized in a structure M_V if there is an element m in M_V such that $\phi(m)$ (respectively $\neg\phi(m)$) holds, for any $\phi \in p^+$ (respectively for any $\phi \in p^-$).

A *pure-injective* abelian structure M_V is an abelian structure where every pp type is realized (with parameters in A with $|A| \leq |\mathcal{L}_V|$ [34, Theorem 3.1]); M_V is indecomposable when one cannot decompose it as $M_1 \oplus M_2$, with M_1, M_2 non-zero. With a type p (over the empty set), one may associate a unique minimal pure-injective structure $H(p) = H(m)$ [34, Theorem 3.6] with $m \in H(p)$ such that $p^+(x)$ is exactly the set of pp-formulas satisfied by m in $H(p)$. One says that p is indecomposable if $H(p)$ is indecomposable.

A basic result in the model theory of modules which has been adapted to the setting of abelian structures is the following: any abelian structure M_V is elementarily equivalent to a direct sum of pure-injective indecomposable abelian structures [34, Corollary 6.9]. This result (for classical B-modules) has led M. Ziegler to associate with the class of B-modules, a topological space Zg_B (the Ziegler spectrum) whose points are isomorphism types of non-zero pure-injective indecomposable modules and basic open sets (denoted by $[\phi/\psi]$, where ϕ, ψ are pp formulas in one free variable with $\psi \to \phi$) consisting of the (isomorphism types of) pure-injective indecomposable modules M_V where the index of $\psi(M_V)$ in $\phi(M_V)$ is strictly bigger that 1 [34, Corollary 6.13]. This space is quasi-compact.

Recall that two pp-formulas $\phi,\ \psi$ are said to be equivalent if in all modules M, $\phi(M)=\psi(M)$. Denote by $L(B)$ the set of pp $\mathcal{L}$-formulas quotiented by this equivalence relation. One can show that $L(B)$ forms a lattice [23, Section 2] and in case B is Bézout, this lattice is distributive [23, Fact 2.4]. On that lattice of pp formulas, we will consider the *duality functor* D which transforms a formula in the category of left B-modules to a formula in the category of right B-modules [25, chapter 8]. Since B is commutative, it will allow us to simply the description of Zg_B.

It has been observed that the group $\Gamma(B)$ of divisibility of B reflects (some of) the model-theoretic properties of the class of B-modules. For instance in [23, Theorem 7.1], they show that $L(B)$ has no width if and only if $\Gamma(B)$ contains a densely ordered subchain.

When $B,\ B'$ are two valuation domains (and so $\Gamma(B),\ \Gamma(B')$ are totally ordered abelian groups, also called the value groups), L. Gregory observed that the Ziegler spectra $Zg_B,\ Zg_{B'}$ are homeomorphic if and only if the value groups $\Gamma(B),\ \Gamma(B')$ are isomorphic [14, Corollary 3.3].

2.3. **Prüfer domains.** Recall that a positive primitive (pp) $\mathcal{L}_V$-formula $\phi(\bar{x})$ with $\bar{x}:=(x_1,\dots,x_n)$, $n\geq 1$, is an existential formula of the form:

$$\exists\bar{y}\ (\bar{y}.A=\bar{x}.C\ \&\ \bigwedge_{\delta\in\Delta} V_\delta(\bar{y}\bar{x}.C_\delta)),$$

where $\bar{y}=(y_1,\dots,y_m)$, $m\geq 1$, A is a $m\times k$-matrix, C a $n\times k$ matrix, $k\geq 1$, C_δ a $(n+m)\times 1$ matrix, all with coefficients in B and Δ a finite subset of Γ. By $\phi(M)$ we denote the submodule of M^n consisting of the tuples $\bar{u}$ of elements of M^n such that for some $\bar{w}\in M^m$ we have $\bar{w}.A=\bar{u}.C\ \&\ \bigwedge_{\delta\in\Delta} V_\delta(\bar{w}\bar{u}.C_\delta)$. Let $\phi(x),\ \psi(x)$ be two pp formulas in one free variable, then an invariant sentence is a sentence of the form $(\phi/\psi)>n$ that expresses that the index of the subgroup $\psi(M)\cap\phi(M)$ in the subgroup $\phi(M)$ is strictly greater than n, $n\in\mathbb{N}$.

Let us recall a former result of G. Puninskiĭ [21] on the special equivalent form of pp formulas over a Bézout domain B (or more generally over Prüfer domains).

Any pp $\mathcal{L}_B$-formula $\phi(\bar{x})$ is equivalent to the pp $\mathcal{L}_B$-formula: $\exists\bar{y}\ (\bar{y}.S=\bar{x}\ \&\ \bar{y}.\bar{r}=0)$, where $\bar{y}=(y_1,\dots,y_m)$, S is a $m\times n$-matrix, $\bar{r}$ a $m\times 1$ matrix, all with coefficients in B.

In view of this result, given a B-module M, $\delta\in\Gamma(B)^+$ and $a\in B$ with $v(a)=\delta$, we define $V_\delta(M):=\{m\in M:\ \exists y\in M\ m=y.a\}$. This is well-defined since for $a,\ b\in B^\star$ with $v(a)=v(b)$, we have $a.b^{-1}\in B$. Since B is commutative, $V_\delta(M)$ is not only a pp definable subgroup but an B-submodule. Finally note that the formula $V_{v(a)}(m.b)$ is either equivalent to the formula $m=m$ if $v(b)\geq v(a)$ or if $v(b)<v(a)$ to the formula $\exists y_1\exists y_2\exists z\ (m=y_1+y_2\ \&\ y_1.b=0\ \&\ y_2=z.a.b^{-1})$. (Indeed, in case $v(b)<v(a)$ and $V_{v(a)}(m.b)$ holds, there exists $n\in M$ such that $m.b=n.a=n.a.b^{-1}.b$ and so $m=m-n.a.b^{-1}+n.a.b^{-1}$ with $(m-n.a.b^{-1}).b=0$. In the formalism used in [22], we will abbreviate $\exists y\ y.(a.b^{-1})=x$ by $a.b^{-1}|x$ and we will write the formula $V_{v(a)}(m.b)$ as $(b.x=0)+(a.b^{-1}|x)$, or equivalently $(b.x=0)+V_{v(a).v(b)^{-1}}(x)$.

Example 2.3. *Let $\mathcal{H}(\mathbb{C})$ be the ring of holomorphic functions over $\mathbb{C}$. A key ingredient is the Weierstrass factorization theorem* [30, Theorem 15.10]*. One defines the functions $E_0(z) = 1 - z,\ E_p(z) = (1 - z).exp\{z + z^2 + \cdots + \frac{z^p}{p}\}$. Letting $(z_n)_{n\in\omega}$ be a sequence of elements of $\mathbb{C} \setminus \{0\}$ (possibly with repetitions) such that $|z_n| \to \infty$, the infinite product $P(z) := \prod_{n=1}^{\infty} E_{n-1}(\frac{z}{z_n})$ belongs to $\mathcal{H}(\mathbb{C})$, the zeroes of P are exactly the z_n's and if z_n occurs m times in $P(z)$, then z_n is a zero of $P(z)$ of multiplicity m* [30, Theorem 15.9].

The ring $\mathcal{H}(\mathbb{C})$ is a Bézout domain [15, Theorem 1], [30, Theorem 15.15]*. Let us describe its group of divisibility. Given any element $f \in \mathcal{H}(\mathbb{C})$, we define the multiplicity function $\mu_f : \mathbb{C} \to \mathbb{N}$ sending $z \in \mathbb{C}$ to the multiplicity of z as a zero of f. Set $\daleth^+ := \{\mu_f : \ f \in \mathcal{H}(\mathbb{C})\}$. We have that $\mu_{f.g} = \mu_f + \mu_g$; $\daleth^+$ forms a commutative monoid (w.r.to $+$) and can be endowed with a partial order: $\mu \leq \nu$ iff $\forall z\ \mu(z) \leq \nu(z)$, for $\mu,\ \nu \in \daleth^+$. This partial order reflects the divisibility relation in $\mathcal{H}(\mathbb{C})$: $f|g$ in $\mathcal{H}(\mathbb{C})$ iff $\mu_f \leq \mu_g$ and f is invertible iff $\mu_f = 0$. Denote by $\daleth$ the group generated by $\daleth^+$; it is easy to see that $(\daleth, \leq)$ is an ℓ-group and that it is isomorphic to $\Gamma(\mathcal{H}(\mathbb{C}))$.*

Using the above description of the group of divisibility of $\mathcal{H}(\mathbb{C})$ and [23, Theorem 7.1]*, one can easily see that the lattice of pp formulas over $\mathcal{H}(\mathbb{C})$ has no width* [23, Example 6.3]*. Indeed, choose two elements $f,\ g \in \mathcal{H}(\mathbb{C})$ with the same infinite (discrete) subset of zeroes: $Z(f) = Z(g) = \{z_n : \ n \in \mathbb{N}^\star\}$ and such that $\mu_g(z_n) = 2n.\mu_f(z_n)$. Then using Weierstrass factorization theorem recalled above, there exists $h \in \mathcal{H}(\mathbb{C})$ such that $\mu_h(z_n) := \mu_f(z_n) + \lfloor \frac{\mu_g(z_n) - \mu_f(z_n)}{2} \rfloor$. Then $\mu_f < \mu_h < \mu_g$ and the strict inequality holds simply because $\lim_{n\to\infty} \mu_g(z_n) - \mu_f(z_n) = +\infty$, and so we may re-apply the same procedure to both: (μ_g, μ_h), (μ_h, μ_f).*

3. Quantifier elimination for valuation domains

Let B be a Bézout domain and let $\Gamma(B)$ be its group of divisibility. Recall that the space $MSpec(B)$ of maximal ideals of B is endowed with the Zariski topology where a basis of closed subsets is given by $V(a) := \{\mathcal{M} \in MSpec(B) : \ a \in \mathcal{M}\}$. The *constructible* subsets of $MSpec(B)$ are the elements of the Boolean algebra generated by the basic closed subsets.

We have the following relationship between the lattice generated by these basic closed subsets of $MSpec(B)$ and the lattice of principal ideals of B. Let $a, b \in B^\star$. Then $V(a) \cap V(b) = V(gcd(a, b))$, $V(a) \cup V(b) = V(a.b)$, $V(a) \setminus V(b) \subset V((a : b))$.

Let $\mathcal{L}_V$ be the language of B-modules expanded with a set of unary predicates V_δ indexed by the submonoid Γ^+ of positive elements of the group $\Gamma := \Gamma(B)$. Let M be an B-module and for $\delta = v(a)$, set $V_\delta(M) := \{m \in M : \ \exists n \in M\ m = n.a\}$, where $a \in B^\star$.

In the theorem below, we consider a pp $\mathcal{L}_V$-formula $\exists \mathbf{x}\, \phi(\mathbf{x}, \mathbf{z})$ in the language $\mathcal{L}_V$, where $\phi(\mathbf{x}, \mathbf{z})$ is a conjunction of atomic formulas and we will show that this formula is equivalent in any $B_\mathcal{M}$-module to a conjunction of atomic formulas. In case of the pure module language, that property is called *positive quantifier elimination* (or *elim-Q^+* following the terminology of [25, page 319]) and implies structural properties on indecomposable pure-injective modules [25, Corollary 16.7]. We could have applied Puninskiĭ's result on Prüfer domains (recalled in subsection 2.2) and the fact that over a valuation domain a matrix is conjugated to a diagonal matrix. However since we are ultimately interested in describing definable subsets in the class of B-modules (see section 5), we will consider classes of modules over valuation domains $B_{\mathcal{M}}$, where $\mathcal{M}$ varies over $MSpec(B)$. So we will use that for a given pp $\mathcal{L}_{B_{\mathcal{M}}}$-formula, the elimination is uniform on a certain constructible subset of $MSpec(B)$. This is why we chose to give a self-contained and direct proof of that result.

We will use the isomorphism between $(\Gamma(B)^+, ., \wedge, 1)$ and the subdirect product $\prod^s_{\mathcal{M}\in MSpec(B)}(\Gamma(B_{\mathcal{M}})^+, ., \wedge, 1)$ (Lemma 2.2). Let us denote the image of $\delta \in \Gamma(B)$ in $\Gamma(B_{\mathcal{M}})$ by $\delta_{\mathcal{M}}$. Note that if $\delta_{\mathcal{M}} := v_{\mathcal{M}}(a)$, $a \in B^\star$, and M is a $B_{\mathcal{M}}$-module, we get that $V_\delta(M) = V_{\delta_{\mathcal{M}}}(M)$.

Remark 3.1. [5, section 1] In the localization $B_{\mathcal{M}}$, we have that $a|b$ iff $v(a) \leq v(b)$ iff $(a : b) \notin \mathcal{M}$ iff $\mathcal{M} \notin V((a : b))$.

Since $B_{\mathcal{M}}$ is a valuation domain, we have either $a|b$ or $b|a$, in other words $\emptyset = V((a : b)) \cap V((b : a))$.

Proof: Suppose that $(a : b) \notin \mathcal{M}$, then in $B_{\mathcal{M}}$ express $b = a.(a : b)^{-1}.(b : a)$. Now, suppose that in $B_{\mathcal{M}}$, $a|b$. Therefore $gcd^{B_{\mathcal{M}}}(a, b) = a$, so $(a : b) \in U_{\mathcal{M}}$, which exactly mean that $(a : b) \notin \mathcal{M}$. □

Theorem 3.2. *Let B be a Bézout domain. Then given any pp $\mathcal{L}_{B,V}$-formula $\phi(\mathbf{z})$, there exists finitely many constructible subsets $C_{\phi,k}$ of $MSpec(B)$ such that for any $\mathcal{M} \in C_{\phi,k}$, $\phi(\mathbf{z})$ is equivalent to a conjunction $\chi_k(\mathbf{z})$ of atomic $\mathcal{L}_{B_{\mathcal{M}},V}$-formulas, in the classes $Mod_{B_{\mathcal{M}}}$ of all $B_{\mathcal{M}}$-modules, $k \in K$, K finite.*

In particular, any pp $\mathcal{L}_{B,V}$-formula $\phi(z)$ (in one variable) is equivalent (in $Mod_{B_{\mathcal{M}}}$) to a formula of the form

$$z.a = 0 \ \& \ \bigwedge_{i=1}^{n} V_{\delta_i}(z.b_i),$$

for some a, $b_i \in B_{\mathcal{M}}$, $\delta_i \in \Gamma^+_{\mathcal{M}}$, $1 \leq i \leq n$, with $\delta_1.v_{\mathcal{M}}(b_1)^{-1} > \cdots > \delta_n.v_{\mathcal{M}}(b_n)^{-1}$ and $v_{\mathcal{M}}(b_1) > \cdots > v_{\mathcal{M}}(b_n)$.

Proof: We will proceed by induction on the number of existential quantifiers. We start with the existential $\mathcal{L}_{B,V}$-formula $\phi(\mathbf{z}) := \exists x_n \cdots \exists x_1 \ \psi_1(x_1, x_2, \cdots, x_n, \mathbf{z})$, where $\psi_1(x_1, x_2, \cdots, x_n, \mathbf{z})$ is a conjunction of atomic (c.a.) $\mathcal{L}_{B,V}$-formulas. We consider the innermost existential quantifier and the formula $\exists x_1 \ \psi_1(x_1, x_2, \cdots, x_n, \mathbf{z})$. We want to find a finite covering of $MSpec(B)$ by constructible subsets $C_{1,j}$ and finitely many c.a. $\mathcal{L}_{B_{\mathcal{M}},V}$-formulas $\psi_{2,j}(x_2, \cdots, x_n, \mathbf{z})$ such that for any $\mathcal{M}$ in $C_{1,j}$, $\exists x_1 \ \psi_1(x_1, x_2, \cdots, x_n, \mathbf{z})$ is equivalent to $\psi_{2,j}(x_2, \cdots, x_n, \mathbf{z})$ in $Mod_{B_{\mathcal{M}}}$.

We proceed inductively as follows. Set $C_{0,1} := MSpec(B)$, $J_0 := \{1\}$. On the constructible subset $C_{\ell,j}$, $j \in J_\ell$, $n-1 \geq \ell \geq 0$, we consider the existential formula $\exists x_{\ell+1} \ \psi_{\ell+1,j}(x_{\ell+1}, \cdots, x_n, \mathbf{z})$, where $\psi_{\ell+1,j}$ is a c. a. $\mathcal{L}_{B_{\mathcal{M}},V}$-formula, and we show that there is a finite covering of $C_{\ell,j}$ by constructible subsets $C_{\ell+1,j'}$, $j' \in J_{\ell+1}$ such that for any $\mathcal{M} \in C_{\ell+1,j'}$, this formula $\exists x_{\ell+1} \ \psi_{\ell+1,j}(x_{\ell+1}, \cdots, x_n, \mathbf{z})$ is equivalent in $C_{B_{\mathcal{M}}}$, to a formula $\psi_{\ell+2,j'}(x_{\ell+2}, \cdots, x_n, \mathbf{z})$, where $\psi_{\ell+2,j'}$ is a c.a. $\mathcal{L}_{B_{\mathcal{M}},V}$-formula.

At the last step, we obtain J_n, constructible subsets $C_{n,j}$, $j \in J_n$ and corresponding c.a. $\mathcal{L}_{B_{\mathcal{M}},V}$-formula $\psi_{n+1,j}(\mathbf{z})$.

For ease of notation, we set $x := x_{\ell+1}$ and $\mathbf{y} := (x_{\ell+1}, \cdots, x_n, \mathbf{z})$, $C_\ell := C_{\ell,j}$ and $\psi_{\ell+1}(x, \mathbf{y}) := \psi_{\ell+1,j}(x, \mathbf{y})$.

Among atomic $\mathcal{L}_V$-formulas, we have formulas of the form $V_\delta(x.r - u)$, where $r \in B$ and $\delta \in \Gamma^+$ that we will abbreviate as $x.r \equiv_\delta u$ ("congruence relations"). The outline of the proof is similar to [2, Proposition 4.1].

First let us show that we can always assume that we have at most one equation involving x.

Indeed, consider $x.r_0 = t_0(\mathbf{y})$ & $x.r_1 = t_1(\mathbf{y})$. For every $\mathcal{M} \in C_\ell$, either $v_{\mathcal{M}}(r_0) \geq v_{\mathcal{M}}(r_1)$, or $v_{\mathcal{M}}(r_0) \leq v_{\mathcal{M}}(r_1)$. W.l.o.g. assume we are in the second case. So, $r_1.r_0^{-1} \in B_{\mathcal{M}}$ and the above conjunction is equivalent to: $x.r_0 = t_0(\mathbf{y})$ & $t_0(\mathbf{y}).r_1.r_0^{-1} = t_1(\mathbf{y})$. So we will subdivide C_ℓ into two subsets according to whether $v_{\mathcal{M}}(r_0) \geq v_{\mathcal{M}}(r_1)$ holds.

So we may reduce ourselves to consider c.a. $\mathcal{L}_{B_{\mathcal{M}},V}$-formulas $\psi_{\ell+1}(x, \mathbf{y})$ of the form:

$$(\star) \quad \text{either } x.r_0 = t_0(\mathbf{y}) \ \& \ \bigwedge_{i=1}^{n} V_{\delta_i}(x.r_i - t_i(\mathbf{y})) \ \& \ \theta(\mathbf{y}),$$

$$\text{or } \bigwedge_{i=1}^{n} V_{\delta_i}(x.r_i - t_i(\mathbf{y})) \ \& \ \theta(\mathbf{y}),$$

where $r_i \in B_{\mathcal{M}}$, $\theta(\mathbf{y})$ is a c.a. $\mathcal{L}_{B_{\mathcal{M}},V}$-formula, the $t_i(\mathbf{y})$, $0 \leq i \leq n$, are $\mathcal{L}_{B_{\mathcal{M}}}$-terms, and $\delta_i \in \Gamma^+_{\mathcal{M}}$.

Consider $\exists x\, \psi_{\ell+1}(x, \mathbf{y})$. It suffices to show that we can find finitely many c.a. $\mathcal{L}_{B_{\mathcal{M}},V}$-formulas and a finite covering of C_ℓ by finitely constructible subsets $C_{\ell+1,j}$ such that any such existential formula is equivalent to one of these formulas on $C_{\ell+1,j}$.

Before eliminating the existential quantifier, we make a series of reductions which lead us to break up C_ℓ into finitely many subsets according to whether certain valuational inequalities hold among the indices of the predicates or the coefficients occurring in the terms. To avoid too many indices after each reduction we rename C_ℓ each of the subsets we obtained from C_ℓ.

First we examine whether $v_{\mathcal{M}}(r_0) > v_{\mathcal{M}}(r_i)$, for $1 \leq i \leq n$. Indeed, suppose that $v_{\mathcal{M}}(r_0) \leq v_{\mathcal{M}}(r_i)$, for some i, say $i = 1$. Then, we write $r_1 = r_0.(r_1.r_0^{-1}) \in B_{\mathcal{M}}$. So, we replace in $(\star)$ the congruence relation $V_{\delta_1}(x.r_1 - t_1(\mathbf{y}))$ by $V_{\delta_1}(t_0(\mathbf{y}).r_1.r_0^{-1} - t_1(\mathbf{y}))$. So we are left with congruence relations $V_{\delta_i}(x.r_i - t_i(\mathbf{y}))$ with $v_{\mathcal{M}}(r_0) > v_{\mathcal{M}}(r_i)$.

We also break up C_ℓ according to finitely many subsets according to whether $v_{\mathcal{M}}(r_i) < \delta_{i\mathcal{M}}$, $1 \leq i \leq n$. Indeed suppose for instance that $v_{\mathcal{M}}(r_i) \geq \delta_{i\mathcal{M}}$. Then we replace the congruence condition $x.r_i \equiv_{\delta_i} t_i$ by $t_i \equiv_{\delta_i} 0$.

Now let us order the set $\{\delta_{i\mathcal{M}} \cdot v_{\mathcal{M}}(r_i^{-1});\ 1 \leq i \leq n\}$. By re-indexing we may assume that $\delta_{1\mathcal{M}} \cdot v_{\mathcal{M}}(r_1)^{-1} \geq \delta_{2\mathcal{M}} \cdot v_{\mathcal{M}}(r_2)^{-1} \geq \cdots \geq \delta_{n\mathcal{M}} \cdot v_{\mathcal{M}}(r_n)^{-1}$. Again we break up C_ℓ into finitely many subsets according to which such conjunctions of inequalities hold.

First, we will assume that there is one equation present in $\phi(x, \mathbf{y})$ and we proceed by induction on the number of congruence conditions. If there are none, we replace the pp formula $x.r_0 = t_0$ by $V_{v_{\mathcal{M}}(r_0)}(t_0)$.

Consider the system (1):

$$(1)\ :\ x.r_0 = t_0,\ x.r_1 \equiv_{\delta_1} t_1,\ \cdots,\ x.r_n \equiv_{\delta_n} t_n$$

with $v_{\mathcal{M}}(r_0) > v_{\mathcal{M}}(r_i)$, $t_0 = t_0(\mathbf{y}), t_i = t_i(\mathbf{y})$, $1 \le i \le n$.

We claim that in any $B_{\mathcal{M}}$-module M, with $\mathcal{M} \in C_\ell$, system (1) is equivalent to the following system (2) :

$$(2)\ :\ x.r_1 = t_1,\ t_1.r_0.r_1^{-1} \equiv_{\delta_1 \cdot v(r_0.r_1^{-1})} t_0,\ x.r_2 \equiv_{\delta_2} t_2,\ \cdots,\ x.r_n \equiv_{\delta_n} t_n$$

(1) $\to$ (2)
Let $x \in M$ satisfy (1). So $x.r_1 = t_1 + n$ for some $n \equiv_{\delta_1} 0$. Let $s_1 \in B_{\mathcal{M}}$ be such that $v_{\mathcal{M}}(s_1) = \delta_1$. By definition of the predicate V_{δ_1} and the assumption that $V_{\delta_1}(n)$, there exists $u' \in M$ such that $u'.s_1 = n$ and since $v_{\mathcal{M}}(r_1) < \delta_1$, we have $u'.(s_1.r_1^{-1}).r_1 = n$ with $v_{\mathcal{M}}(s_1.r_1^{-1}) > 1$ (and so $s_1.r_1^{-1} \in B_{\mathcal{M}}$). Set $u := u'.(s_1.r_1^{-1})$ and let $y := x - u$. So we get that $y.r_1 = t_1$ and $y.r_0 = t_0 - u.r_1.(r_0.r_1^{-1}) = t_0 - n.(r_0.r_1^{-1}) = y.r_1.(r_0.r_1^{-1}) = t_1.(r_0.r_1^{-1})$. Therefore, $V_{\delta_1 \cdot v(r_0.r_1^{-1})}(t_0 - t_1.r_0.r_1^{-1})$. Consider the other congruence conditions: if we replace x by y, then for $i \ge 2$, $y.r_i = x.r_i - u'.(s_1.r_1^{-1}).r_i$ with $V_{\delta_i}(u'.(s_1.r_1^{-1}).r_i)$.
(2) $\to$ (1)
Let y satisfy (2), namely $y.r_1 = t_1$. Consider $y.r_0 = y.r_1.(r_0.r_1^{-1}) = t_1.(r_0.r_1^{-1})$. Since $V_{\delta_1 \cdot v(r_0.r_1^{-1})}(t_0 - t_1.r_0.r_1^{-1})$ holds, we have $y.r_0 \equiv_{\delta_1 \cdot v(r_0.r_1^{-1})} t_0$. Let $n \in M$ be such that $y.r_0 = t_0 + n$ with $n \equiv_{\delta_1 \cdot v(r_0.r_1^{-1})} 0$. Let $s_1 \in B_{\mathcal{M}}$ with $v_{\mathcal{M}}(s_1) = \delta_1$. Since $V_{\delta_1 \cdot v(r_0.r_1^{-1})}(n)$, there exists an element $n' \in M$ with $n'.s_1.(r_0.r_1^{-1}) = n$, namely $(n'.s_1.r_1^{-1}).r_0 = n$. Set $x = y - n'.s_1.r_1^{-1}$. Since $v_{\mathcal{M}}(s_1) = \delta_1$, $V_{\delta_1}(n'.s_1)$ holds, and similarly since we have $v_{\mathcal{M}}(s_1.r_1^{-1}.r_i) \ge \delta_i$, $V_{\delta_i}(n'.s_1.r_1^{-1}.r_i)$ holds, $i \ge 2$.

Second, we will consider the case where there are only congruence relations in the system. We will reduce to the previous case, making x occurring in a non-trivial equation (see (4)). Let M be any $B_{\mathcal{M}}$-module, with $\mathcal{M} \in C_\ell$.

Consider the following system (3):

$$(3)\ :\ x.r_1 \equiv_{\delta_1} t_1,\ \cdots,\ x.r_n \equiv_{\delta_n} t_n$$

Since $v_{\mathcal{M}}(r_1) < \delta_1$, we may replace the congruence condition by a divisibility condition. Indeed, $x.r_1 = t_1 + u$ with $u \equiv_{\delta_1} 0$. Let $s_1 \in B_{\mathcal{M}}$ be such that $v_{\mathcal{M}}(s_1) = \delta_1$ and let $u' \in M$ such that $u'.s_1 = u$. Set $y := x - u'.s_1.r_1^{-1}$ and we check that for any $i \ge 2$, $V_{\delta_i}(u'.s_1.r_1^{-1}.r_i)$.

So system (3) is equivalent to a system (4) of the form:

$$(4)\ :\ x.r_1 = t_1,\ x.r_2 \equiv_{\delta_2} t_2,\ x.r_3 \equiv_{\delta_3} t_3,\ \cdots,\ x.r_n \equiv_{\delta_n} t_n.$$

Again we may assume that $v_{\mathcal{M}}(r_1) > v_{\mathcal{M}}(r_i)$, for all $i \geq 2$ (otherwise we may eliminate x in the corresponding congruence relation). So we proceed as in the first case with one less congruence relation and we conclude by induction.

Finally we consider a c.a. $\mathcal{L}_{B_{\mathcal{M}},V}$-formula with one free variable of the form $\bigwedge_{j\in J} x.a_j = 0$ & $\bigwedge_{i\in I} V_{\delta_j}(x.b_i)$. We proceed as in the beginning of the proof to reduce ourselves to at most one annihilator condition comparing the values $v_{\mathcal{M}}(a_i)$ in $\Gamma^+_{\mathcal{M}}$.

For the congruence conditions, as before we may assume that $v_{\mathcal{M}}(b_i) < \delta_i$, otherwise we remove that the corresponding congruence relation. Then we compare in $(\Gamma(B_{\mathcal{M}}), \leq)$ the elements $\delta_i.v_{\mathcal{M}}(b_i)^{-1}$ and so we assume that $\delta_1.v_{\mathcal{M}}(b_1)^{-1} \geq \cdots \geq \delta_n.v_{\mathcal{M}}(b_n)^{-1}$. Let $i \neq 1$. First note that if $v_{\mathcal{M}}(b_1) \leq v_{\mathcal{M}}(b_i)$, we express $x.b_i = x.b_1.(b_1^{-1}.b_i)$. So if $V_{\delta_1}(x.b_1)$ holds, then $V_{\delta_i}(x.b_i)$ holds. Now assume that $\delta_1.v_{\mathcal{M}}(b_1)^{-1} = \delta_i.v_{\mathcal{M}}(b_i)^{-1}$, $v_{\mathcal{M}}(b_1) > v_{\mathcal{M}}(b_i)$ and $V_{\delta_i}(x.b_i)$. Then since $x.b_1 = x.b_i.b_1.b_i^{-1}$, we have $V_{\delta_1}(x.b_1)$. So, proceeding in a similar way for all indices, we may assume that the congruence conditions are of the form: $\bigwedge_{i=1}^{n} V_{\delta_j}(x.b_i)$ with $\delta_1.v_{\mathcal{M}}(b_1)^{-1} > \cdots > \delta_n.v_{\mathcal{M}}(b_n)^{-1}$ and $v_{\mathcal{M}}(b_1) > \cdots > v_{\mathcal{M}}(b_n)$. □

Remark 3.3. From the proof of the above theorem, we see that the constructible subsets $C_{\phi,k}$ occurring in the statement are of the form $V(a)^c \cap V(b)$, for $a, b \in B$ that can be obtained from the formula ϕ and the operations $\cdot$ and gcd (in B).

4. Axiomatization

Since we are interested in decidability results for theories of modules, we will axiomatize the theories of modules that we will consider. We will start with considering modules over any Bézout ring and then we will apply the results of the preceding section on valuation domains.

For a ring R, when one considers the decidability problem for the theory of R-modules, it is reasonable to assume from the start, certain effectivity properties of the ring R. In particular one can ask: assuming that the theory of R-modules is decidable, which effectivity properties does it imply on the ring operations? One usually assume that R is *effectively given.* This notion has been discussed in length in [24], [25, Chapter 17] and specifically for valuation domains in: [22, Section 3], [14, Definition 3.1]). In particular if a valuation domain A is effectively given, there is an algorithm which given a, $b \in A$ decides whether $a|b$ [22, Remark 3.2].

Definition 4.1. Let B be a Bézout domain, we will call assumption (EF) on B the following: B is a countable ring, it can be enumerated as $(r_n : n \in \omega)$ in such a way, there are algorithms to perform the following operations: given a, $b \in B$, produce $a + b$, $-a$, $a.b$, decide whether $a = b$ or not and the relation $\{(a, b) \in R^2 : a|b\}$ is recursive.

This implies (as in [22, Section 3]) there is an algorithm which decides whether an element $a \in B$, decide whether a is invertible (i.e. $a \in U$) and if yes produce a^{-1}. Therefore given any coset $a.U$, there is an algorithm which chooses a representative (for instance the element with the smallest index in the given enumeration). There are also algorithm which given $a,\ b \in B^\star$, produces:

$gcd(a,b)$ (the algorithm enumerates the elements $a.r_n + b.r_m$ and checks whether it divides a and b),

$(a:b)$ (the algorithm checks whether $a = gcd(a,b).r_n.u$ for some $u \in U$).

4.1. **Abelian structures revisited.** Recall that the language $\mathcal{L}_V$ has been defined as the expansion of the language $\mathcal{L}$ of B-modules together with a set of unary predicates that we index by the submonoid of the positive elements of the group of divisibility Γ of B, namely $\mathcal{L}_V := \mathcal{L} \cup \{V_\gamma; \gamma \in \Gamma^+\}$.

Definition 4.2. Let $T_{B,V}$ be the $\mathcal{L}_V$-theory, consisting of the $\mathcal{L}$-theory of B-modules together with:

$(1)_V$ $V_\delta(0)$, for each $\delta \in \Gamma^+$,
$(2)_V$ $\forall m_1\, \forall m_2\, ((V_{\delta_1}(m_1)\, \&\, V_{\delta_2}(m_2)) \to V_{\delta_1 \wedge \delta_2}(m_1 + m_2))$, for every $\delta_1,\ \delta_2 \in \Gamma^+$,
$(3)_V$ $\forall m\, (V_\delta(m) \to V_{\delta \wedge \mu}(m))$, for any $\delta,\ \mu \in \Gamma^+$,
$(4)_V$ $\forall m\, (V_\delta(m) \to V_{\delta \cdot v(a)}(m.a))$, for each $a \in B^\star$, $\delta \in \Gamma^+$.

When the context is clear, we will simply use the notation T_V (instead of $T_{B,V}$). Given a B-module M, we denote by M_V its expansion as an $\mathcal{L}_V$-structure, namely M together with a family a submodules $V_\delta(M)$, $\delta \in \Gamma^+$.

Remark 4.3. From the above axioms, we deduce easily the following properties, letting $M_V \models T_{B,V}$:

1. each $V_\delta(M)$ is a subgroup and it is a B-submodule of M,
2. if $u \in B$ is an invertible element i.e. $u \in U$, then for any $\delta \in \Gamma$, $V_\delta(m.u) \leftrightarrow V_\delta(m)$ (this is due to the fact that $v(u) = 1$),
3. suppose $v(a) = v(b)$ with $a,\ b \in B^\star$ and that $V_{\delta.v(b)}(m.b)$, then $V_{\delta.v(b)}(m.a)$.

We will consider a subclass of the class of abelian structures we just defined, namely those which satisfy in addition:

$$(5)_V \qquad \forall m\, V_1(m),$$

together with the following divisibility scheme:

$$(6)_{V,div} \qquad \forall m\, \exists n\, (V_{v(a)}(m) \ \to n.a = m), \text{for each } a \in B^\star.$$

Definition 4.4. We will denote by $T_{B,V,div}$ the theory $T_{B,V}$ together with $(5)_V$ together with the divisibility axioms scheme $(6)_{V,div}$.

Remark 4.5.

1. The theory $T_{B,V,div}$ is consistent.
   Indeed, the ring B itself can be expanded to a model of $T_{B,V,div}$. Define $V_\gamma(B) = \{b \in B : v(b) \geq \gamma\}$, $\gamma \in \Gamma^+$. If $\gamma := v(a)$, $a \in B^\star$, $V_\gamma(B) = B.a$. Then $(B, (Ba)_{a \in B^\star})$ is a model of $T_{B,V,div}$. By definition of the ℓ-valuation on B, we have that for $a,\ b \in B^\star$, $v(b) \geq v(a)$ iff $b.a^{-1} \in B$ and so B will satisfy axiom $(5)_V$ together with axiom schemes $(4)_V$ and $(6)_{V,div}$. In fact, B satisfies a stronger form of axiom scheme $(4)_V$, as we will see below.

(2) In fact, $T_{B,V,div}$ is what is called *an expansion by definitions* of the theory of B-modules. Explicitly, it means that given any B-module M, we can expand it to a model of $T_{B,V,div}$ and in any model of $T_{B,V,div}$, the new predicates V_γ are definable in the language of B-modules.
Indeed, given any B-module M, we set $V_\delta(M) := \{m \in M : \exists n\ m = n.a\}$ with $v(a) = \delta$, $a \in B^\star$. To check that this expansion satisfies all the axioms $(1)_V$ up to $(6)_{V,div}$ is rather straightforward. Let us check for instance $(2)_V$. Let $\delta_1 = v(a_1)$ and $\delta_2 = v(a_2)$ and write $a_1 = a'.gcd(a_1, a_2)$, $a_2 = a''.gcd(a_1, a_2)$. Assume $V_{\delta_1}(m_1)\,\&\,V_{\delta_2}(m_2)$ holds. So for some n_1, n_2 we have $m_1 = n_1.a_1$ and $m_2 = n_2.a_2$ and $m_1 + m_2 = (n_1.a' + n_2.a'').gcd(a_1, a_2)$. Therefore $V_{\delta_1\wedge\delta_2}(m_1 + m_2)$.
By definition of the ℓ-valuation on B, we have that for a, $b \in B^\star$, $v(b) \geq v(a)$ iff $b.a^{-1} \in B$. Using this, one can easily deduce axioms $(3)_V$, $(5)_V$ and $(6)_{V,div}$.
Conversely, if $M_V \models T_{B,V,div}$, then the subgroup $V_{v(a)}(M)$, $a \in B^\star$ is definable by the $\mathcal{L}_B$-formula: $\exists x\ y = x.a$. It follows from $(5)_V$ and $(6)_{V,div}$.

(3) Let $M_V \models T_{B,V,div}$ and assume M is a torsion-free B-module. Let $m \in M$ and suppose that $V_{\delta.v(b)}(m.b)$ holds with $b \in B^\star$, $\delta \in \Gamma^+$. Then let us show that $V_\delta(m)$ holds. Let $a \in B^\star$ be such that $v(a) = \delta$. By axiom $(6)_{V,div}$, for some $n \in M$, $m.b = n.a.b$. So $(m - n.a).b = 0$. Since M is torsion-free, $m = n.a$ and so $V_\delta(m)$ holds.

4.2. **The case of valuation domains.** As recalled in the introduction, there are general results on decidability of the theories of modules over valuation domains [22], [14]. For instance, G. Puninskiĭ, V. Puninskaya and C. Toffalori proved that if A is a valuation domain satisfying (EF) with infinite residue field and archimedean densely ordered value group, then the theory of all A-modules is decidable [22, Theorem 6.2]. Then L. Gregory removed these two assumptions and proved that for an effectively given valuation domain A, if the prime radical relation is recursive, then the theory of A-modules is decidable [10, Theorem 7.1]. In addition she proved for any effectively given commutative ring R, if the theory of the R-modules is decidable then the prime radical relation is recursive.

In order to be self-contained, we will present here a direct proof of their decidability result, under the hypothesis of infinite residue field. As said in the introduction, this proof is also more algebraic than the one presented in [22]. It will also be a key step in the decidability result for Bézout domains in the next section. The hypothesis on the residue field is satisfied in all good Rumely domains containing the prime field $\mathbb{F}_p$ (see section 6).

In this subsection, A will denote a valuation domain, $Q(A)$ its fraction field and $\Gamma := \Gamma(A)$ its value group. Recall that a *fractional ideal* of A is an additive subgroup of $Q(A)$ which is closed under multiplication by A. We will use that a pure-injective indecomposable module over a valuation domain is the pure hull $\overline{I/J}$ of a module of the form I/J, where I, J are two fractional ideals of A and that if a type is realized in $\overline{I/J}$, it is already realized in I/J [34, section 5].

Note that $v(J)$ is a subset of $\overline{\Gamma}$ which is upward closed (in particular if $x \in I \setminus J$, then $v(x) < v(J)$). We define the predicates V_δ on I/J using axiom $(6)_{V,div}$. Namely let $u \in I \setminus J$, then $V_\delta(u+J)$ holds if there exists $s \in A$ with $v(s) \geq \delta$ and $y \in I$ such that $u - y.s \in J$. We have that $v(u) \geq min\{y.s, u-ys\}$, since $u \notin J$, $y.s \notin J$ and so $v(ys) < v(J)$ and $v(u) = v(y.s)$, so $v(u) \geq \delta$ and $s|u$ (in I).

Notation 4.6. Let E be a subset of Γ which is upward closed. Let $^*\Gamma$ be an elementary extension of Γ which is $|\Gamma|^+$-saturated. We consider the partial type with parameters in Γ, $p(x) := \{\delta < x \leq \gamma : \ \delta \in \Gamma \setminus E, \ \gamma \in E\}$ and we denote by $\inf E$ a realisation of $p(x)$ in $^*\Gamma$.

Let E be a subset of Γ which is upward closed and assume that E has no minimum in Γ. Then any non empty interval in $^*\Gamma$ of the form $]\inf E \ \gamma]$, where $\gamma \in \Gamma$ has infinite intersection with Γ.

Corollary 4.7. *Assume that A satisfies (EF), and that the quotient $A/\mathcal{M}$ of A by its maximal ideal $\mathcal{M}$, is infinite, then T_A is decidable.*

Proof: Since the theory $T_{A,V,div}$ is recursively enumerable, in order to prove its decidability we need to show that we can enumerate the set of sentences false in some element of $T_{A,V,div}$ (or taking the negation, true in some element of $T_{A,V,div}$).

By the Baur–Monk pp elimination theorem, it suffices to consider boolean combinations of invariant sentences: $(\phi/\psi) \geq n$, $n \in \mathbb{N}$, where $\phi(x), \psi(x)$ are pp $\mathcal{L}_V$-formulas (see subsection 2.2). Note that since the quotient of A by its maximal ideal $\mathcal{M}$ is infinite, then $(\phi/\psi) > 1$ implies that (ϕ/ψ) is infinite.

Since a disjunction of formulas is true whenever one of them is true, we may only consider conjunctions of formulas of the form $\sigma := \bigwedge_{i\in I}(\phi_i/\psi_i) > 1 \ \& \ \bigwedge_{j\in J}(\chi_j/\xi_j) = 1$. Moreover suppose we find for each $i \in I$, a model M_i of $T_{A,V,div}$ satisfying $(\phi_i/\psi_i) > 1 \ \& \ \bigwedge_{j\in J}(\chi_j/\xi_j) = 1$, then we can form the direct sum of the $M_i's$, $i \in I$ and get a model M of $T_{A,V,div}$ satisfying σ. So the sentences we need to consider are of the following form: $(\phi_i/\psi_i) > 1 \ \& \ \bigwedge_{j\in J}(\chi_j/\xi_j) = 1$. Furthermore note that if we cannot find a model M_V of $T_{A,V,div}$ where σ holds, it exactly means that in the Ziegler spectrum the basic open set $[\phi_i/\psi_i]$ is included in the union: $\bigcup_{j\in J}[\chi_j/\xi_j]$.

By Theorem 3.2, we may reduce ourselves to only consider pp $\mathcal{L}_V$-formulas of the form: $x.a = 0 \ \& \ \bigwedge_{i\in I} V_{\delta_i}(x.b_i)$ with $a, \ b_i \in A^*$, $\delta_i \in \Gamma(A)$, I finite, and or equivalently, $x.a = 0 \ \& \ \bigwedge_{i\in I}((x.b_i = 0) + V_{\delta_i.v(b_i)^{-1}}(x))$, assuming that $v(b_i) < \delta_i$ (otherwise we may delete the corresponding congruence condition). In order to write that formula in the same way when I is empty, we will express $x.a = 0$ by $V_{+\infty}(x.a)$.

Furthermore, note that the quantifier elimination procedure described in Theorem 3.2 is effective. In the course of the proof, we had to decide whether $v(a_1) < v(a_2)$, $a_1, \ a_2 \in A^\star$. This is equivalent to decide whether $a_1|a_2$ and $a_2 \nmid a_1$. By hypothesis (EF), we can do that in an effective way in A.

As in [22], we use the duality functor introduced by M. Prest on the lattice of pp formulas (see subsection 2.2) in order to simplify the form of the pairs of pp formulas we need to consider. Let $\phi(x)$ be a pp formula, then $D(D(\phi)) \leftrightarrow \phi$. Assuming that $D(\phi(x)) \leftrightarrow \bigwedge_i(c_i|x + x.d_i = 0)$, we get $\phi(x) \leftrightarrow \sum_i D(c_i|x + x.d_i = 0) \leftrightarrow$

$\sum_i (d_i|x \ \& \ x.c_i = 0)$. Finally we note that if $\phi \leftrightarrow \sum_i \phi_i$ and $\psi \leftrightarrow \bigwedge_j \psi_j$, then $[\phi/\psi] = [\bigcup_{i,j}(\phi_i/\psi_j]$. Since the formula $a|x + x.d = 0$ is equivalently $V_{v(a.d)}(x.d)$, by making the same abuse of notation as above by allowing the possibility to have $V_{+\infty}(x.d)$, we get the following Claim.

Claim 4.8. [22, Section 5] We only need to consider basic open sets in the Ziegler spectrum of the form $[b|x \ \& \ x.c = 0/V_{v(a.d)}(x.d)]$.

Claim 4.9. [22, Corollary 4.3]

$$[x.b = 0 \ \& \ V_{\delta_1}(x)/V_{\delta_2}(x.c)] \neq \emptyset \text{ if and only if } \quad (v(b) > v(c) \text{ and } \delta_1.v(c) < \delta_2).$$

Proof: ($\leftarrow$) Let $s \in A$ be such that $v(s) = \delta_1$. Consider the A-module $M := A/A.bs$; the element $x := s + A.bs$ belongs to $ann(b) \cap V_{\delta_1}(M)$. By way of contradiction, assume that $V_{\delta_2}(s.c + A.bs)$. We have that $v(s.c) = \delta_1.v(c) < min\{\delta_1.v(b), \delta_2\}$, a contradiction.

($\rightarrow$) Let N be an A-module such that there exists $m \in N$ such that $m.b = 0$, $V_{\delta_1}(m)$ and $\neg V_{\delta_2}(m.c)$. If $v(b) \leq v(c)$, then $m.c = 0$ and so $V_{\delta_2}(m.c)$. Now assume for a contradiction that $\delta_1.v(c) \geq \delta_2$. Since $V_{\delta_1}(m)$, we have that $V_{\delta_2}(m.c)$, contradicting the assumption on m. □

Now, let us consider the two open sets:

$$[x.b = 0 \ \& \ V_{\delta_1}(x)/V_{\delta_2}(x.c)],$$
$$[x.b' = 0 \ \& \ V_{\delta'_1}(x)/V_{\delta'_2}(x.c')],$$

with $b, c, b', c' \in A^*$, $\delta_1, \delta_2, \delta'_1, \delta'_2 \in \Gamma^+$ and $v(b) > v(c)$, $v(b') > v(c')$, $\delta_1.v(c) < \delta_2$, $\delta'_1.v(c') < \delta'_2$.

Claim 4.10. (See also [22, Proposition 4.5]) Under the above assumptions, we have

$$[x.b = 0 \ \& \ V_{\delta_1}(x)/V_{\delta_2}(x.c)] \subset [x.b' = 0 \ \& \ V_{\delta'_1}(x)/V_{\delta'_2}(x.c')] \text{ if and only if}$$
$$\delta_2.v(c)^{-1} \leq \delta'_2.v(c')^{-1} \text{ and } \delta_2.v(c)^{-1}.v(b) \leq \delta'_2.v(c')^{-1}.v(b').$$

Proof: ($\rightarrow$)

(i) By way of contradiction, assume that $\delta'_2.v(c')^{-1} < \delta_2.v(c)^{-1}$.
Choose $I, \ J$ two fractional ideals of $Q(A)$, $J \subset I$ such that $\min(v(I)) = \delta'_2.v(c')^{-1}$ and $\min(v(J)) = \delta_2.v(c)^{-1}$.
Let us show that $\overline{I/J}$ belongs to the first open set but not to the second one.
Let $u \in I$ with $v(u) \geq \delta_1$ and $v(u).v(b) \in v(J)$; by choice of $I, \ J$ we have that $v(u) \geq \delta_1$ and for any $m \in J$, $v(u.c + m) < \delta_2$. However all elements of I have valuation $\geq \delta'_2.v(c')^{-1}$.

(ii) Now, let us show that if $\delta_2.v(c)^{-1}.v(b) > \delta'_2.v(c')^{-1}.v(b')$, then we get a contradiction.
We can choose a fractional ideal J such that $\max\{\delta'_2.v(c')^{-1}.v(b'), \delta_2\} \leq \min(v(J)) < \delta_2.v(c)^{-1}.v(b)$. This is feasible since $\delta_2 < \delta_2.v(c)^{-1}.v(b)$.
Let $u \in A$ be such that $\delta_1 \leq v(u) < \delta_2.v(c)^{-1}$ and $v(u.b) \in v(J)$, equivalently $\min(v(J)).v(b)^{-1} \leq v(u)$. Moreover, since $\delta_2 \leq v(J)$, we have that $v(u.c + m) < \delta_2$, for any $m \in J$.

However, any element $\tilde{u} \in I \setminus J$ such that $v(\tilde{u}.c') < \delta'_2$ has the property that $\tilde{u}.b' \notin J$, a contradiction.

($\leftarrow$) Now take any pure-injective indecomposable module belonging to the first pair. As already recalled, this module is of the form $\overline{I/J}$, where I, J are two fractional ideals. Since $\overline{I/J}$ belongs to the first pair, then there exists $u \in I \setminus J$ with $v(u) \in [\delta_1 \ \delta_2.v(c)^{-1}[$, $\neg V_{\delta_2}(u.c + J)$ and such that $v(u.b) \in v(J)$. Note that this implies that $u.c \notin J$ and since $v(J)$ is upward closed, it implies that $v(u.c) < v(J)$, and so $\delta_1.v(c) \le v(u.c) < v(J)$.
Now we look for an element $u' \in I$, such that $v(u') \ge \delta'_1$ and belonging to the interval $[\inf v(J)v(b')^{-1} \ \delta'_2.v(c')^{-1}[$. By hypothesis the interval $[\inf v(J)v(b)^{-1} \ \delta_2.v(c)^{-1}[$ is non trivial, equivalently $[\inf v(J) \ \delta_2.v(c)^{-1}v(b)[\neq \emptyset$.
Since $\delta'_2.v(c')^{-1}.v(b') \ge \delta_2.v(c)^{-1}.v(b)$, $[\inf v(J)v(b')^{-1} \ \delta'_2.v(c')^{-1}[\neq \emptyset$. The interval $[\delta'_1 \ \delta'_2.v(c')^{-1}[$ is non trivial, as well as $[\inf v(I) \ \delta'_2.v(c')^{-1}[$ (since $[\inf v(I) \ \delta_2.v(c)^{-1}[\neq \emptyset$ and $\delta_2.v(c)^{-1} \le \delta'_2.v(c')^{-1}$).
So the intersection of these three intervals is non empty. It remains to check that $\neg V_{\delta'_2}(u.c' + J)$, namely for all $m \in J$, $v(u'.c' + m) < \delta'_2$.
We always have that $\inf v(J).v(b')^{-1} < \inf v(J).v(c')^{-1}$.
Either, $\inf v(J).v(c')^{-1} < \delta'_2.v(c')^{-1}$, in which case we replace the interval $[\inf v(J)v(b')^{-1} \ \delta'_2.v(c')^{-1}[$ by $[\inf v(J)v(c')^{-1} \ \delta'_2.v(c')^{-1}[$,
or $\inf v(J) \ge \delta'_2$. In that last case, any element $m \in J$ will have the property that $v(u'.c' + m) < \delta'_2$ provided that $v(u'.c') < \delta'_2$. □

Before considering the general case, let us consider the case when an open set in the Ziegler spectrum is included in the union of two open subsets. We assume that each of the open sets is non trivial (see Claim 4.9).

Claim 4.11. $[x.b = 0 \ \& \ V_{\delta_1}(x)/V_{\delta_2}(x.c)] \subset$
$[x.b' = 0 \ \& \ V_{\delta_{1'}}(x)/V_{\delta_{2'}}(x.c')] \cup [x.b'' = 0 \ \& \ V_{\delta_{1''}}(x)/V_{\delta_{2''}}(x.c'')]$ if and only if

$$\text{either } (\delta_2.v(c)^{-1} \le \delta'_2.v(c')^{-1} \text{ and } \delta_2.v(c)^{-1}.v(b) \le \delta'_2.v(c')^{-1}.v(b')),$$
$$\text{or } (\delta_2.v(c)^{-1} \le \delta''_2.v(c'')^{-1} \text{ and } \delta_2.v(c)^{-1}.v(b) \le \delta''_2.v(c'')^{-1}.v(b'')).$$

Proof: ($\rightarrow$) Suppose otherwise. By symmetry, it suffices to consider the following cases:
(i) $\delta''_2.v(c'')^{-1} < \delta_2.v(c)^{-1}$ and $\delta'_2.v(c')^{-1}.v(b') < \delta_2.v(c)^{-1}.v(b)$.
Choose a fractional ideal J such that $\max\{\delta'_2.v(c')^{-1}.v(b'), \delta_2\} \le \min(v(J)) < \delta_2.v(c)^{-1}.v(b)$ and a fractional ideal I such that $\min v(I) = \max\{\delta_1, \delta''_2.v(c'')^{-1}\}$.
First let us check that $\overline{I/J}$ belongs to $[x.b = 0 \ \& \ V_{\delta_1}(x)/V_{\delta_2}(x.c)]$. Since $\min(v(J)) < \delta_2.v(c)^{-1}.v(b)$, there is an element $u \in I$ such that $v(u.c) < \delta_2$ but $u.b \in J$ (and by choice of I, $v(u) \ge \delta_1$).
But any element $u \in I$ will have the property that $v(u.c'') \ge \delta''_2$ and any element $u' \in I \setminus J$ with $v(u'.c') < \delta'_2$ will have the property that $u'.b' \notin J$.
(ii) $\delta_2.v(c)^{-1} > \max\{\delta'_2.v(c')^{-1}, \delta''_2.v(c'')^{-1}\}$.
Choose a fractional ideal I such that $\min v(I) = \max\{\delta_1, \delta'_2.v(c')^{-1}, \delta''_2.v(c'')^{-1}\}$ and a fractional ideal J with $\delta_2 \le \min(J) < \delta_2.v(c)^{-1}.v(b)$. We similarly check that $\overline{I/J}$

belongs to $[x.b = 0 \ \& \ V_{\delta_1}(x)/V_{\delta_2}(x.c)]$. But, any $u \in I \setminus J$ will have the property that $V_{\delta'_2}(u.c')$ and $V_{\delta''_2}(u.c'')$, but there is $u \in I$ such that $v(u) < \delta_2.v(c)^{-1}$ and $u.b \in J$.

(iii) $\delta_2.v(c)^{-1}.v(b) > \max\{\delta'_2.v(c')^{-1}.v(b'), \delta''_2.v(c'')^{-1}.v(b'')\}$.
Choose a fractional ideal J such that

$$\max\{\delta'_2.v(c')^{-1}.v(b'), \delta''_2.v(c'')^{-1}.v(b'')\} \leq \min v(J) < \delta_2.v(c)^{-1}.v(b).$$

Choose a fractional ideal I with the property that $\min(I) = \delta_1$. Again, it is easily checked that $\overline{I/J}$ belongs to $[x.b = 0 \ \& \ V_{\delta_1}(x)/V_{\delta_2}(x.c)]$.
But no $u \in I \setminus J$ such that $\neg V_{\delta'_2}(u.c')$ has the property that $u.b' \in J$. Similarly $\neg V_{\delta''_2}(u.c'')$ implies that $u.b'' \notin J$.

($\leftarrow$) This direction is clear using the previous claim. □

Finally we will show that to decide whether a basic open set in the Ziegler spectrum is included in a finite union of basic open subsets reduces to divisibility conditions on the elements of the ring, which we can decide by assumption (EF). As before, we assume that each of the basic open sets is non trivial (see Claim 4.9).

Claim 4.12.

$$[x.b = 0 \ \& \ V_{\delta_1}(x)/V_{\delta_2}(x.c)] \subset \bigcup_{\ell \in L} [x.b_\ell = 0 \ \& \ V_{\delta_{1\ell}}(x)/V_{\delta_{2\ell}}(x.c_\ell)] \ \text{ if and only if}$$

$$\bigvee_{\ell \in L} (\delta_2.v(c)^{-1} \leq \delta_{2\ell}.v(c_\ell)^{-1} \ \text{ and } \ \delta_2.v(c)^{-1}.v(b) \leq \delta_{2\ell}.v(c_\ell)^{-1}.v(b_\ell)).$$

Proof: ($\rightarrow$) Suppose otherwise, namely that either

(i) we can partition L into two non-empty subsets L', L'' with $\max_{\ell \in L''} \delta_{2\ell}.v(c_\ell)^{-1} < \delta_2.v(c)^{-1} \leq \min_{\ell \in L'} \delta_{2\ell}.v(c_\ell)^{-1}$ and $\max_{\ell \in L'} \delta_{2\ell}.v(c_\ell)^{-1}.v(b_\ell) < \delta_2.v(c)^{-1}.v(b)$. In this case, we choose a fractional ideal J such that $\max\{\max_{\ell \in L'} \delta_{2\ell}.v(c_\ell)^{-1}.v(b_\ell), \delta_2\} \leq \min(v(J)) < \delta_2.v(c)^{-1}.v(b)$ and a fractional ideal I such that
$\min v(I) = \max\{\delta_1, \max_{\ell \in L''} \delta_{2\ell}.v(c_\ell)^{-1}\}$. First let us check that $\overline{I/J}$ belongs to $[x.b = 0 \ \& \ V_{\delta_1}(x)/V_{\delta_2}(x.c)]$. Since $\min(v(J)) < \delta_2.v(c)^{-1}.v(b)$, there is an element $u \in I$ such that$v(u.c) < \delta_2$ but $u.b \in J$ (and by choice of I, $v(u) \geq \delta_1$).

But any element $u \in I$ will have the property that $v(u.c_\ell) \geq \delta_{2\ell}$, for $\ell \in L''$ and any element $u' \in I \setminus J$ with $v(u'.c_\ell) < \delta_{2\ell}$ will have the property that $u'.b_\ell \notin J$, $\ell \in L'$.

(ii) $\max_{\ell \in L} \delta_{2\ell}.v(c_\ell)^{-1} < \delta_2.v(c)^{-1}$, then we proceed as in Claim 4.11 (ii).

(iii) $\max_{\ell \in L} \delta_{2\ell}.v(c_\ell)^{-1}.v(b_\ell) < \delta_2.v(c)^{-1}.v(b)$, then we proceed as in Claim 4.11 (iii).

($\leftarrow$) This direction is clear using Claim 4.10.

This ends the proof of the Claim and the proof of the Corollary.

□

Remark 4.13. Let T be a theory of R-modules, where the invariant sentences are of the form $(\phi/\psi) > 1$. The discussion above showed that an equivalent formulation of whether a sentence holds in some R-module is asking whether a basic open set is included in a given finite union of other basic open sets in the closed subset of Ziegler spectrum of R, consisting of models of T (see for instance [22, section 6]).

When B is a Bézout domain, each point of the Ziegler spectrum is an indecomposable pure-injective B-module (and so a $B_{\mathcal{M}}$-module, where $B_{\mathcal{M}}$ denotes the localization of B at $\mathcal{M}$, for some $\mathcal{M} \in MSpec(B)$) and a basic open set is the set of points in the Ziegler spectrum where the index the two pp definable subgroups is nontrivial. By the discussion above, in case $B_{\mathcal{M}}$ is a model of $T_{B_{\mathcal{M}},V,div}$ we reduce ourselves to consider open sets in the Ziegler spectrum of the form $[x.b = 0 \ \& \ V_{\delta_1}(x)/V_{\delta_2}(x.c)]$.

5. Feferman-Vaught theorem for Bézout domains

Let B be a Bézout domain and let $\Gamma(B)$ be its group of divisibility.

S. Garavaglia [10] showed that any B-module M, can be embedded in a direct sum of modules over the localizations $B_{\mathcal{M}}$, $\mathcal{M}$ varying in the space $MSpec(B)$ of maximal ideals of B, and this embedding is elementary (i.e. respects pp formulas). Even though we could have directly applied his result, we will present here a slight generalization for abelian structures (see Proposition 5.2 below)).

5.1. **Localizations.** Let us review basic definitions on localizations by maximal ideals of both the ring and the module [13, Chapter 9].

Let $\mathcal{M} \in MSpec(B)$. Let M be a B-module and let $M_{\mathcal{M}}$ be the localization of M by $\mathcal{M}$. There is an embedding of M into the direct product $\prod_{\mathcal{M} \in MSpec(B)} M_{\mathcal{M}}$ (as a B-module) and S. Garavaglia showed that this embedding is elementary [10, Theorem 3] (namely respects pp formulas). We want to extend this result when M is viewed as an $\mathcal{L}_V$-structure, namely not only as a B-module but endowed with a distinguished lattice of submodules; to that end, we will use the following description of the localizations $M_{\mathcal{M}}$ and of the embedding of M into the direct product $\prod_{\mathcal{M} \in MSpec(B)} M_{\mathcal{M}}$.

Recall that $M_{\mathcal{M}}$ is also a $B_{\mathcal{M}}$-module and that one can view $M_{\mathcal{M}}$ as the module of fractions $M \otimes B_{\mathcal{M}}$ of M with respect to the multiplicative set: $B \backslash \mathcal{M}$ ([13, Proposition 9.14]).

Let $E \subset B$, denote by E-tors$(M) := \{m \in M : \ \exists r \in E \ m.r = 0\}$. For $m \in M^{\star}$, let $Ann(m) := \{r \in B : \ m.r = 0\}$; it is a proper ideal of B and so it is included in a maximal ideal $\mathcal{M}_0$ of B. Therefore $m \notin (B \setminus \mathcal{M}_0)$-tors$(M)$.

We can embed M into $\prod_{\mathcal{M}} M_{\mathcal{M}}$ as follows. Set $m_{\mathcal{M}} := m + (B \setminus \mathcal{M})$-tors$(M)$, with $m \in M$. The map sending m to $(m_{\mathcal{M}})_{\mathcal{M} \in MSpec(B)}$ is injective by the above and clearly a morphism of B-modules.

Now we consider the expansion of M to the abelian structure M_V as defined in Definition 4.2. It induces the following abelian structure on $(M_{\mathcal{M}})_V$ by setting $V_\delta(m_{\mathcal{M}})$, whenever there exists $n \in m + (B \setminus \mathcal{M})$-tors$(M)$ such that $V_\delta(n)$, where $\delta \in \Gamma$.

Lemma 5.1. *Let M be a B-module and let M_V be its expansion as an abelian $\mathcal{L}_V$-structure. Let $\mathcal{M} \in MSpec(B)$. Then there is a morphism of $\mathcal{L}_V$-structures sending $m \in M_V$ to $m_{\mathcal{M}} \in (M_{\mathcal{M}})_V$.*

Proof: Let $S := (B \setminus \mathcal{M})$, we have to check that M_V is a model of T_V and that the map sending $m \in M$ to $m + S$-tors(M) is a morphism of $\mathcal{L}_V$-structures. □

Proposition 5.2. *Let M_V be a model of $T_{B,V}$. Then we have an elementary embedding $M_V \hookrightarrow \prod_{\mathcal{M}}(M_{\mathcal{M}})_V$ as $\mathcal{L}_V$-structures.*

Proof: By the pp elimination result for abelian structures, it is enough to show that given any pp $\mathcal{L}_V$-formula $\phi(\bar{x})$ and $\bar{a} \in M$ such that $M_V \not\models \phi(\bar{a})$, then for some maximal ideal $\mathcal{M}$ we have $(M_{\mathcal{M}})_V \not\models \phi(\bar{a}_{\mathcal{M}})$, where $\bar{a}_{\mathcal{M}} = \bar{a} + (B \setminus \mathcal{M})$-tors$(M)$.

Let $I := \{r \in B : \ M_V \models \phi(\bar{a}.r)\}$. Then I is a proper ideal of B; let $\mathcal{M}$ be a maximal ideal containing I and $S := (B \setminus \mathcal{M})$.

By way of contradiction, suppose that $(M_{\mathcal{M}})_V \models \phi(\bar{a}_{\mathcal{M}})$. The formula $\phi(\bar{x})$ is of the form $\exists \bar{y}\ \theta(\bar{x}, \bar{y})$ where $\theta(\bar{a}, \bar{y}) := (\bar{a}.A_1 + \bar{y}.A_2 = 0\ \&\ \bigwedge_i V_{\delta_i}(t_i(\bar{a}) + t'_i(\bar{y})))$, with $\delta_i \in \Gamma^+$, and A_1, A_2 are two matrices with coefficients in B. Let $\bar{b} = (d_1.s_1^{-1}, \cdots, d_n.s_n^{-1})$, with $d_i \in M$, $s_i \in S$, $1 \leq i \leq n$, be such that $(M_{\mathcal{M}})_V \models \theta(\bar{a}_{\mathcal{M}}, \bar{b}_{\mathcal{M}})$. Equivalently, $\bar{a}.A_1 + \bar{b}.A_2 \in S$-tors$(M)$ and $\bigwedge_i V_{\delta_i}(t_i(\bar{a}_{\mathcal{M}}) + t'_i(\bar{b}_{\mathcal{M}}))$. We multiply both expressions by $s = \prod_i s_i \in S$ and we get $\bar{a}.s.A_1 + \bar{b}.s.A_2 \in S$-tors$(M)$ and $\bigwedge_i V_{\delta_i.v(s)}(t_i(\bar{a}_{\mathcal{M}}.s) + t'_i(\bar{b}_{\mathcal{M}}.s))$ (using $(4)_V$). Since $\bar{a}.s.A_1 + \bar{b}.s.A_2 \in S$-tors$(M)$, there exists $\tilde{s} \in S$ such that $\bar{a}.s.\tilde{s}.A_1 + \bar{b}.s.\tilde{s}.A_2 = 0$. Finally we get: $\bar{a}.s.\tilde{s}.A_1 + \bar{b}.s.\tilde{s}.A_2 = 0$ and $\bigwedge_i V_{\delta_i.v(s.\tilde{s})}(t_i(\bar{a}_{\mathcal{M}}.s.\tilde{s}) + t'_i(\bar{b}_{\mathcal{M}}.s.\tilde{s}))$. Since $\delta_i.v(s.\tilde{s}) \geq \delta_i$, we get $V_{\delta_i}(t_i(\bar{a}.s.\tilde{s}) + t'_i(\bar{b}.s.\tilde{s}))$. Therefore, noting that $\bar{b}.s \in M$ and $M_V \models \theta(\bar{a}.s.\tilde{s}, \bar{b}.s.\tilde{s})$, we obtain that $M_V \models \phi(\bar{a}.s.\tilde{s})$. This shows that $s.\tilde{s} \in I \cap S$, a contradiction. □

5.2. **Feferman-Vaught theorem.** Below, we introduce a property of the ring B that implies the existence of relative complement for the basic closed sets in the Zariski spectrum of B.

Recall that for $c, d \in B$, we denoted $c \in rad(d)$ the Jacobson radical relation, where $rad(d)$ is the intersection of all maximal ideals that contains d.

Definition 5.3. Recall that B has *good factorisation* [6], if given any pair a, b of non zero elements of B, there exist c, $d \in B$ such that $a = c.d$ with $gcd(c, b) = 1$ and $b \in rad(d)$.

First, let us link that last property with other possibly better known properties.

If B has good factorization, then given any two basic closed subsets $V(a), V(b)$ of $MSpec(B)$, there is an element c such that $V(c) = V(a) \setminus V(b)$. From that property it follows that the constructible subsets of $MSpec(B)$ are either basic open or basic closed subsets [6, Lemma 2.12]. It also follows that if B has good factorization, then B is an elementary divisor ring [9, Chapter III, Exercice 6.2].

Definition 5.4. A ring R is *adequate* [9, Exercice 6.4, page 118], if for all $a \neq 0, b$, there exist c, d such that $a = c.d$, $bR + cR = R$ and for all d' $[(dR \subseteq d'R \subsetneq R) \rightarrow (bR + d'R \subsetneq R)]$.

It is easy to see that a Bézout ring B with good factorization is adequate. Take $c, d \in B$ such that $a = c.d$ with $gcd(c, b) = 1$ and $b \in rad(d)$ and let d' be such that $dR \subset d'R$ and $d'R$ a proper ideal. Let $\mathcal{M}$ be a maximal ideal of B containing $d'R$, so it contains d and since $b \in rad(d)$, $b \in \mathcal{M}$. Therefore, $bR + d'R$ is a proper ideal of B.

Therefore, a Bézout ring with good factorization has the property that any prime ideal is contained in a unique maximal ideal [9, Exercice 6.4, page 118].

Finally let us make the connection between B *has good factorisation* and $\Gamma(B)$ *is a projectable* ℓ*-group*. Let us first recall that last notion.

Given an ℓ-group Γ, an ideal is a convex ℓ-subgroup [12, Section 3.2] and a prime ideal is an ideal P with the property that for any $f,\ g \in \Gamma$ such that $f \wedge g = 1$, we have either $f \in P$ or $g \in P$ [12, Section 3.3]. By Zorn's Lemma, there exist minimal prime ideals. Let $Min(\Gamma)$ denotes the space of minimal prime ideals of Γ endowed with the co-Zariski topology, namely the basis of open sets consists of $\{V(\delta) :\ \delta \in \Gamma\}$ where $V(\delta) := \{P \in Min(\Gamma) :\ \delta \in P\}$.

Define for any $g \in \Gamma$, $g^{\perp} := \{f \in \Gamma :\ |f| \wedge |g| = 1\}$, where $|f| := f \vee f^{-1}$ belongs to Γ^{+}. Recall that a *cardinal* sum of ℓ-groups is a sum of ℓ-groups endowed with the partial order defined componentwise [12, Example 1.3.13]. Then Γ is *projectable* if for any $g \in \Gamma$, $g^{\perp}$ is a cardinal summand [12, Section 3.5]. Since $MSpec(B)$ is homeomorphic to $Min(\Gamma(B))$ [32, Proposition 8], the property for B to have good factorization, translates into the property for $\Gamma(B)$ to be a projectable ℓ-group.

Remark 5.5. Using Remark 3.1 and the fact that B is Bézout, one can further show [5, Lemma 1.3 and its proof] for $a_i, b_i, c_j, d_j \in B$, I, J finite, the following equivalences:

$$\forall \mathcal{M} \in MSpec(B)\ B_{\mathcal{M}} \models\ (\bigvee_{i\in I} a_i \nmid b_i \vee \bigvee_{j\in J} c_j | d_j),$$

$$\forall \mathcal{M} \in MSpec(B)\ (\bigwedge_{i\in I} (a_i : b_i) \in \mathcal{M} \rightarrow \prod_{j\in J} (c_j : d_j) \in \mathcal{M}),$$

$$B \models\ \prod_{j\in J} (c_j : d_j) \in rad(gcd((a_i : b_i)_{i\in I})),$$

where $gcd((a_i : b_i)_{i\in I})$ denotes a generator of the ideal generated by the elements $(a_i : b_i)$, $i \in I$.

Theorem 5.6. *Let M_V be a model of $T_{B,V,div}$. Let $\phi(\mathbf{y})$ be a pp $\mathcal{L}_{V,B}$-formula. Then there are finitely many conjunctions of atomic (c.a.) $\mathcal{L}_{V,B_{\mathcal{M}}}$-formulas $\psi_k(\mathbf{y})$, $k \in K$, and a finite covering of $MSpec(B)$ into constructible subsets $C_{\phi,k}$ such that for any $\mathbf{u} \in M$, we have:*

$$M_V \models \phi(\mathbf{u}) \leftrightarrow (\bigwedge_{k\in K} \text{ for all } \mathcal{M} \in C_{\phi,k}\ (\mathcal{M}_{\mathcal{M}})_V \models \psi_k(\mathbf{u}_{\mathcal{M}})),$$

where $\mathbf{u}_{\mathcal{M}}$ denotes the image of the tuple $\mathbf{u}$ in $M_{\mathcal{M}}$.

Proof: First, by Proposition 5.2, we have $M_V \models \phi(\mathbf{u})$ iff for all $\mathcal{M} \in MSpec(B)$, $(M_{\mathcal{M}})_V \models \phi(\mathbf{u}_{\mathcal{M}})$. Each $M_{\mathcal{M}}$ is a $\mathcal{L}_{B_{\mathcal{M}},V}$-structure and since $B_{\mathcal{M}}$ is a valuation domain, we may apply Theorem 3.2 to these classes of $\mathcal{L}_{B_{\mathcal{M}},V}$-structures $M_{\mathcal{M}}$. So there exist finitely many constructible subsets $C_{\phi,k}$, $k \in K$, with K finite such that for any $\mathcal{M} \in C_{\phi,k}$, $Mod_{B_{\mathcal{M}}} \models \forall \mathbf{y}(\phi(\mathbf{y}) \leftrightarrow \psi_k(\mathbf{y}))$, where ψ_k is a c.a. $\mathcal{L}_{V,B_{\mathcal{M}}}$-formula. □

In case B has good factorization, we obtain a neater statement.

Corollary 5.7. *Assume that B has good factorization. Let M_V be a model of $T_{B,V,div}$. Let $\phi(\mathbf{y})$ be a pp $\mathcal{L}_{V,B}$-formula. Then there are finitely many conjunctions of atomic $\mathcal{L}_{V,B_{\mathcal{M}}}$-formulas $\theta_i(\mathbf{y})$, $i \in I$, and a finite partition of $MSpec(B)$ into basic open or basic closed subsets O_i, such that for any $\mathbf{u} \in M$, we have:*

$$M_V \models \phi(\mathbf{u}) \leftrightarrow (\bigwedge_{i\in I} \text{ for all } \mathcal{M} \in O_i \ (\mathcal{M}_{\mathcal{M}})_V \models \theta_i(\mathbf{u}_{\mathcal{M}})),$$

where $\mathbf{u}_{\mathcal{M}}$ denotes the image of the tuple $\mathbf{u}$ in $M_{\mathcal{M}}$.

Proof: Since B has good factorisation, any constructible subset of $MSpec(B)$ is either a basic open or basic closed subset of $MSpec(B)$ [6, Lemma 2.12]. □

Proposition 5.8. *Suppose that a B is a countable Bézout domain and assume that for each $\mathcal{M} \in MSpec(B)$, the quotient $B/\mathcal{M}$ is infinite. Further, assume that B satisfy hypothesis (EF) and that the Jacobson radical relation rad is recursive. Then T_B is decidable.*

Proof: First recall that the theory $T_{B,V,div}$ is a definable expansion by definition of the theory T_B. The key ingredient is Theorem 5.6 which, given a pp $\mathcal{L}_V$-formula $\phi(x)$, enables us to obtain (in an effective way) a finite covering of $MSpec(B)$ into constructible subsets: $C_{\phi,k}$, $k \in K$ and finitely many c.a. $\mathcal{L}_{V,B_{\mathcal{M}}}$-formulas $\psi_k(x)$ such that over each $C_{\phi,k}$, $\phi(x)$ is equivalent to ψ_k which can be assumed to be of the form: $x.a_k = 0$ & $\bigwedge_{i\in I} V_{\delta_{k_i}}(x.b_{k_i})$, a_k, $b_{k_i} \in B_{\mathcal{M}}, \delta_{k_i} \in \Gamma^+_{\mathcal{M}}$ by Theorem 3.2 $(\star)$.

Then we use a standard procedure to obtain decidability of the theory $T_{B,V,div}$ (see for instance [22, Theorem 6.2]), that we detail below.

The hypothesis on the ring B implies that the theory $T_{B,V,div}$ is recursively enumerable. As recalled in Remark 4.13, since for every maximal ideal $\mathcal{M}$, $B/\mathcal{M}$, is infinite, proving that $T_{B,V,div}$ is decidable is equivalent to being able to answer the question whether in Zg_B, a basic open set is included in a given finite union of other basic open sets, namely $[\phi_0/\psi_0] \subset \bigcup_{i=1}^n [\phi_i/\psi_i]$, with ϕ_i, ψ_i pp $\mathcal{L}_V$-formulas, and $\psi_i \to \phi_i$, $0 \le i \le n$ $(\star\star)$. A point in the Ziegler spectrum is (the isomorphism class of) an indecomposable pure-injective B-module and so a $B_{\mathcal{M}}$-module for some maximal ideal $\mathcal{M}$ of B [34, Theorem 5.4].

Given the above finite set of pp $\mathcal{L}_{V,B}$-formulas ϕ_i, ψ_i, $0 \le i \le n$, we apply to each of these formulas procedure $(\star)$ and we obtain (in an effective way) a finite covering of $MSpec(B)$ into constructible subsets: C_ℓ, $\ell \in L$, over which each of these pp $\mathcal{L}_{V,B}$-formulas is equivalent to a c.a. formula of the form: $x.a_\ell = 0$ & $\bigwedge_{i\in I} V_{\delta_{\ell_i}}(x.b_{\ell_i})$, a_ℓ, $b_{\ell_i} \in B_{\mathcal{M}}, \delta_{\ell_i} \in \Gamma^+_{\mathcal{M}}$. Moreover using duality, by Corollary 4.7 (see Claim 4.8), we may only consider open sets of the form $[x.b = 0 \,\&\, V_{\delta_1}(x)/V_{\delta_2}(x.c)]$, $b, c \in B_{\mathcal{M}}$, $\delta_1, \delta_2 \in \Gamma^+_{\mathcal{M}}$.

In order to check whether $(\star\star)$ holds, we proceed then as in the proof of Corollary 4.7 (see Claim 4.12), and it reduces on each element C_ℓ, $\ell \in L$, of the covering of $MSpec(B)$, to divisibility conditions on elements of B in the localizations and order relations between the δ's, which reduce to divisibility conditions in the corresponding $B_{\mathcal{M}}$.

Finally, using Remark 5.5, this can be expressed using the Jacobson radical relation in B. We have to answer statements which are finite conjunctions of the following form: $\prod_{i\in I'}(r'_i : s'_i) \in rad(gcd(a'_j : c'_j)_{j\in J'})$, where the (finite) index sets I', J' and the elements a'_j, c'_j, r'_i, s'_i can be effectively determined from the previous data. Then we use the hypothesis (EF) on our ring to effectively obtain the elements $(r'_i : s'_i)$, $(a'_j : c'_j)$, $gcd(a'_j : c'_j)_{j\in J'}$ from the previous ones. Finally we use the hypothesis that the Jacobson radical relation is recursive in order to decide whether $\prod_{i\in I'}(r'_i : s'_i)$ belongs to $rad(gcd(a'_j : c'_j)_{j\in J'})$. □

6. Applications

In this section we will revisit in detail the examples mentioned in the introduction: on one hand the ring of algebraic integers and on the other hand the ring of holomorphic functions over $\mathbb{C}$ and we also examine the cases of real and p-adic algebraic integers.

6.1. **Good Rumely domains.** In order to axiomatize the elementary theory of the ring $\widetilde{\mathbb{Z}}$ of algebraic integers, the following subclasses of Bézout domains were introduced in [6]. A domain B with fraction field K is a *Rumely domain* if it has the following properties:

(Ru 1.) The field K is algebraically closed.
(Ru 2.) Every finitely generated ideal of B is principal.
(Ru 3.) (Local-global principle) If $C \subseteq \mathbb{A}^m(K)$ is a smooth irreducible closed curve, $f \in K[X_1, \ldots, X_m]$ and $C_f = \{x \in C : f(x) \neq 0\}$ has points in $(1/a)\mathbb{A}^m(B)$ and in $(1/b)\mathbb{A}^m(B)$, where $a, b \in B \setminus \{0\}$ are relatively prime, then C_f has a point in $\mathbb{A}^m(B)$.

B is a *good Rumely domain* if it satisfies, moreover, the following properties.

(Ru 4.) (Good factorization) For all $a,\ b \in B^\star$, there are $c,\ a_1 \in B$ such that $a = c.a_1$ with $gcd(c, b) = 1$ and $b \in rad(a_1)$.
(Ru 5.) Every nonzero nonunit is the product of two relatively prime nonunits.
(Ru 6.) Its Jacobson radical, namely the intersection of all maximal ideals of B, is equal to $\{0\}$ and $B \neq K$.

All these properties are first-order expressible in the language of rings ([6, 1.6]). Consider the Boolean algebra $\mathcal{B}(B)$ generated by all basic closed subsets $V(a)$, $a \in B$, of the maximal spectrum $MSpec(B)$; if a Bézout domain $B \neq K$ satisfies (Ru 4), then one can check that (Ru 5) holds in B if and only if $\mathcal{B}(B)$ is an atomless boolean algebra.

In [27], A. Prestel and J. Schmid axiomatize a class of (commutative) domains endowed with *a radical relation* $\preceq$ [27, Introduction]. Alternatively they show that for each such relation, one can associate a non-empty subset $P_\preceq$ of the prime spectrum in such a way $a \preceq b$ if and only if for every prime ideal $I \in P_\preceq$, $a \in I \ \rightarrow b \in I$ [27, Theorem 2.5]. A domain R is called a r_0*-domain* if it is endowed with a radical relation $\preceq$ with the additional property that $(0 \preceq a \rightarrow a = 0)$. They show that the class of good Rumely domains is exactly the class of existentially closed (e.c.) r_0-domains [27, Theorem 3.3]. They also note that in an e.c. r_0-domain R, the relation

$\preceq$ is induced by the maximal spectrum $MSpec(R)$ of R, namely $a \preceq b$ if and only if $V(a) \subseteq V(b)$.

Examples of good Rumely domains are: the ring $\widetilde{\mathbb{Z}}$ of algebraic integers, the integral closure of $\mathbb{F}_p[t]$.

All localizations of Rumely domains are again Rumely domains ([6, Corollary 3.5]). Localizations of Bézout domains with good factorisation are again Bézout domains with good factorisation ([6, 2.10]).

Let $\mathcal{O}$ be either the ring of algebraic integers in a number field or the integral closure of $\mathbb{F}_p[t]$ in a finite degree field extension of $\mathbb{F}_p(t)$ and let S denote a multiplicative subset of $\mathcal{O}$. Then assume that the ring $S^{-1}.\mathcal{O}$ is not a field and that $S^{-1}.\mathcal{O}$ has infinitely many maximal ideals, then the Jacobson radical of $(\widetilde{S^{-1}.\mathcal{O}})$ is $\{0\}$ [6, Lemma 2.15].

Note that when B is a good Rumely domain, the prime radical relation and the Jacobson radical relation coincide [27, Theorem 3.3] and L. Gregory showed that the decidability of the theory of B-modules implies that the prime radical relation is recursive [14, Lemma 3.2]. Therefore we get the following Corollary to Proposition 5.8.

Corollary 6.1. *Let B be a countable good Rumely domain, assume that B satisfies (EF). Then T_B is decidable if and only if the prime radical relation is recursive.* □

Proof: The only thing to note is that for each $\mathcal{M} \in MSpec(B)$, the quotient $B/\mathcal{M}$ is infinite. If the characteristic of B is zero, this is immediate and if the characteristic of B is a prime p, all good Rumely domains containing the prime field $\mathbb{F}_p$ also contains its algebraic closure [6, Theorem 4.2]. □

The field $\widetilde{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, can be equipped with a recursive structure [31, page 131] and from that presentation one can deduce that the ring $\widetilde{\mathbb{Z}}$ can also be equipped with a recursive structure. Earlier, M. Rabin showed that if F is a computable field, then so is its algebraic closure [29, Theorem 7].

Remark 6.2. [5, Fact 2] Suppose the ring R satisfies hypothesis (EF) and that the Jacobson radical relation is equal to the prime radical relation, then the Jacobson radical relation rad on R is recursive.

Proof: For the reader's convenience, we give the proof below [5, page 192] (note that van den Dries uses that the ring is equipped with a recursive structure, but in our context, we may replace this by assumption (EF)). Van den Dries uses that the relation $x \in rad(y_1, \cdots, y_\ell)$ is recursively enumerable (r.e.) as well as its complement. To show it is r.e., one writes: $x \in rad(y_1, \cdots, y_\ell) \leftrightarrow \exists n \in \mathbb{N}\ x^n \in (y_1, \cdots, y_\ell)$ and that its complement is r.e. $x \notin rad(y_1, \cdots, y_\ell) \leftrightarrow \exists z\ (1 \in (z, x)\ \&\ 1 \notin (z, y_1, \cdots, y_\ell))$. Since our ring is Bézout, we have that $1 \in (z, x) \leftrightarrow gcd(z, x) = 1$. □

Therefore, one can deduce the following Corollary.

Corollary 6.3. *The theory $T_{\widetilde{\mathbb{Z}}}$ is decidable.* □

The decidability of the theory of modules over the ring of algebraic integers $\widetilde{\mathbb{Z}}$ has also been obtained by S. L'Innocente, G. Puninskiĭ and C. Toffalori, using different methods [19].

6.2. **Real algebraic integers and p-adic integers.** A. Prestel and J. Schmid used the same analysis as described above (for $\widetilde{\mathbb{Z}}$) in order to study the rings $\widetilde{\mathbb{Z}} \cap \mathbb{R}$ and $\widetilde{\mathbb{Z}} \cap \mathbb{Q}_p$ ([28]). They showed that in the case of $\widetilde{\mathbb{Z}} \cap \mathbb{R}$ and $\widetilde{\mathbb{Z}} \cap \mathbb{Q}_p$, the axiomatizability depends on a certain local-global principle (as in the case of $\widetilde{\mathbb{Z}}$). Furthermore in these two rings, any prime ideal is maximal since it holds in $\mathbb{Z}$ [1, Corollary 5.8, page 61]. By working in the setting of rings $(R, \preceq)$ with a radical relation $\preceq$, they proved that the related theories of $\widetilde{\mathbb{Z}} \cap \mathbb{R}$ and $\widetilde{\mathbb{Z}} \cap \mathbb{Q}_p$ (in the language of rings) are decidable [28, Corollary 2.5 and Corollary 3.5]. So, Proposition 5.8 leads us to the following corollary.

Corollary 6.4. *Let B be one of the two rings $\widetilde{\mathbb{Z}} \cap \mathbb{R}$ and $\widetilde{\mathbb{Z}} \cap \mathbb{Q}_p$. Then the corresponding theory T_B is decidable.* □

Proof: It suffices to show that each of these rings satisfy hypothesis (EF), by Remark 6.2. Moreover since $\widetilde{\mathbb{Z}}$ satisfies hypothesis (EF) [5, Fact 2], it remains to check that the intersections $\widetilde{\mathbb{Z}} \cap \mathbb{R}$ and $\widetilde{\mathbb{Z}} \cap \mathbb{Q}_p$ are recursive. As in [5], we will use the recursive structure on $\widetilde{\mathbb{Z}}$ defined by Rumely [31, III, page 131].

As in [31, III, page 131], we fix an embedding of $\widetilde{\mathbb{Q}}$ in $\mathbb{C}$ and represent each element α of $\widetilde{\mathbb{Z}}$ as a pair $(f_\alpha(x), a + bi)$ where $f_\alpha(x) \in \mathbb{Z}[x]$ is the minimal monic polynomial of α and $a + b.i \in \mathbb{C}$ is a *sufficiently good* decimal approximation of α to distinguish it from its conjugates. (There is a discussion in [31, page 132] to how *good* is an approximation good enough.)

First consider $\widetilde{\mathbb{Z}} \cap \mathbb{R}$. One can give an estimate of the minimal distance $B(f_\alpha)$ of each of the roots of $f_\alpha(x)$ (in terms of the coefficients of $f_\alpha(x)$) and in order to check that $\alpha \in \widetilde{\mathbb{Z}} \cap \mathbb{R}$, we express that the complex conjugate $a - bi$ is also a root of $f_\alpha(x)$ at distance strictly smaller than $B(f_\alpha)$.

In case of $\widetilde{\mathbb{Z}} \cap \mathbb{Q}_p$, we can proceed as follows. By the result of A. Prestel and J. Schmid recalled above, the theory of the ring $\widetilde{\mathbb{Z}} \cap \mathbb{Q}_p$ is decidable. So, we can check whether the sentence $\exists x\, (f(x) = 0 \;\&\; |x - (a + bi)| < B(f_\alpha))$ holds in $\widetilde{\mathbb{Z}} \cap \mathbb{Q}_p$. If the answer is *yes*, we keep such α. □

6.3. **The ring of entire functions and its integral closure.** Let B be the ring $\mathcal{H}(\mathbb{C})$ of entire functions in $\mathbb{C}$. As we already recalled (see Example 2.3), $\mathcal{H}(\mathbb{C})$ is a Bézout domain, as is its integral closure $\widetilde{\mathcal{H}}(\mathbb{C})$. Moreover $\widetilde{\mathcal{H}}(\mathbb{C})$ satisfy the algebraic properties (Ru1.) up to (Ru6.) listed above, except the property (Ru3.) of the local-global principle [6, 5.6]. Here we will restrict ourselves to $\mathcal{H}(\mathbb{C})$.

Lemma 6.5. *The Bézout domain $\mathcal{H}(\mathbb{C})$ has good factorization.*

Proof: Let $f,\ g \in \mathcal{H}(\mathcal{C})^\star$. By the Weierstrass factorization theorem [30, Theorem 15.10], one can write f as $e^h.z^d.\prod_{n=1}^{\infty} E_{n-1}(\frac{z}{z_n})$, where $Z(f)\setminus\{0\} = \{z_n :\ n \in \omega\}$ and

d is the multiplicity of 0 as a zero of f. Then let $Z_1 = \{n \in \omega : z_n \in Z(f) \cap Z(g)\}$ and set, if $g(0) \neq 0$, $f_1 := e^h.\prod_{n \in Z_1} E_{n-1}(\frac{z}{z_n})$, $f_2 := z^d.\prod_{z_n \notin Z(g)} E_{n-1}(\frac{z}{z_n})$ and if $g(0) = 0$, set $f_1 := e^h.z^d.\prod_{n \in Z_1} E_{n-1}(\frac{z}{z_n})$, $f_2 := \prod_{z_n \notin Z(g)} E_{n-1}(\frac{z}{z_n})$. We have $f = f_1.f_2$ and $gcd(f_1, f_2) = 1$. Furthermore, if $\mathcal{M}$ is any maximal ideal containing f_1, since $Z(f_1) \subset Z(g)$, we get $g \in \mathcal{M}$ (†), so $g \in rad(f_1)$. ((†) In order to see that if $f_1 \in \mathcal{M}$ and $Z(f_1) \subset Z(g)$, then $g \in \mathcal{M}$, one applies the Weierstrass factorization theorem, in order to construct an element $\tilde{f_1}$ with $Z(\tilde{f_1}) = Z(f_1)$ and $\mu_{\tilde{f_1}} \upharpoonright Z(f_1) = 1$. So, $\mu_{\tilde{f_1}} \leq \{\mu_g, \mu_f\}$ which implies that $\tilde{f_1} \in \mathcal{M}$ and $\tilde{f_1}|g$.) □

I. Kaplansky noted that there are prime non-maximal ideals in $\mathcal{H}(\mathbb{C})$ [15, Theorem 1]. A necessary and sufficient condition for a prime ideal P to be non-maximal is that for all $f \in P$, the multiplicity function μ_f is unbounded (μ_f as in Example 2.3).

Since $\mathcal{H}(\mathbb{C})$ is uncountable, there is the usual problem of defining a suitable notion of decidability of a theory of modules in that case. One could take R a countable elementary substructure of $\mathcal{H}(\mathbb{C})$ (in the language of rings) (respectively of $\widetilde{\mathcal{H}}(\mathbb{C})$) and assume that R is effectively given and that the Jacobson radical relation is recursive. From Proposition 5.8, we get that the corresponding theory $T_{R,V}$ is decidable. Of course it would be more informative to exhibit such a subring. In a forthcoming paper with G. Puninskiĭ and C. Toffalori, we describe the Ziegler spectrum over $\mathcal{H}(\mathbb{C})$ [20].

Finally let us note that, contrary to the other examples of rings we considered, it is an open question whether the positive existential theory of $\mathcal{H}(\mathbb{C})$ in the language of rings expanded with a new constant symbol interpreted by the identity function of $\mathcal{H}(\mathbb{C})$ is decidable [11, Problem 1.1].

7. Further work

Now, we introduce the notion of ℓ-valued B-modules which extends the notion of valued modules occurring in, for instance [3], [8] or [18](§2) and also in [4], [2], for a model-theoretic point of view. Let M be a B-module and set $M^\star := M \setminus \{0\}$. Let $\Gamma := \Gamma(B)$ be the ℓ-group of divisibility of B (with the group law $\cdot$, lattice operation $\wedge$ and neutral element 1) (see section 2.1).

Definition 7.1. A *ℓ-valued B-module* is a two-sorted structure $(M, \bar{\Gamma}(B)^+, w)$, where M is a B-module and $w : M \to \bar{\Gamma}(B)^+$ such that

(1) for all $m_1, m_2 \in M$, $w(m_1 + m_2) \geq w(m_1) \wedge w(m_2)$, and $w(0) = \infty$;
(2) for all $m \in M^\star$, $w(m.a) = w(m) \cdot v(a)$, for each $a \in B^\star$.

Example 7.2. *Considering B as a module over itself and letting v the ℓ-valuation on its group of divisibility, we get that B is a ℓ-valued B-module.*

Remark 7.3. We could have taken Γ any ℓ-group, or even we could have considered a distributive lattice Δ, assuming that for each of them we have an action of $\Gamma(B)$, as in, for instance, [2].

From the axioms above, we easily deduce the following properties:
$(P.1)$ $w(m) = w(-m)$ and for all $m \in M$, $w(m) \leq w(m) \cdot s$, for each $s \in \Gamma(B)^+$.
$(P.2)$ Let $m,\ n \in M$ and assume that $w(n) \geq w(m)$. We have $w(m+n) \geq w(m) \wedge w(n) = w(m)$ and $w(m) \geq w(m+n) \wedge w(-n) = w(m+n) \wedge w(n) = w(m+n)$. So, $w(m+n) = w(m)$.

Given $(M, \bar{\Gamma}, w)$ an ℓ-valued B-module; we may consider it in the weaker formalism of abelian structures. Namely we associated with it the $\mathcal{L}_V$-structure M_V where $M_\gamma := \{m \in M : w(m) \geq \gamma\}$, $\gamma \in \Gamma^+$. It is easily checked that M_V is a model of $T_{B,V}$.

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