Nonlinear Schrödinger problems: existence, symmetry and multiplicity

Ch. Grumiau

Institut de Mathématique
Université de Mons
Mons, Belgium

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Université de Lille
Introduction: the scientific method

1. To observe: by making experiences
2. To model: by using (differential) equations
3. To solve the modelization: by “studying” solutions of the equations
Introduction: the scientific method

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Straight-line motion:
Equation: \( u''(t) = 0 \)
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Uniformly accelerated motion:
Equation: \( u''(t) - c = 0 \)

Parabola
Introduction: the scientific method

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Pendulum equation: $u'' - \sin u = 0$
To observe: by making experiences

To model: by using (differential) equations

To solve the modelization: by “studying” solutions of the equations

Aim:
particularized to some physical problems:

- study the differential equation related to the nonlinear Schroedinger problem
Let $\Omega \subseteq \mathbb{R}^N$ open bounded and $N \geq 2$.

**Problem:**

\[
\begin{cases}
-\Delta u + V(x)u = f(u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

where $V : \overline{\Omega} \to \mathbb{R}$ and the nonlinear function $f : \mathbb{R} \to \mathbb{R}$ are continuous (and $f(0) = 0$).

We assume that solutions are **critical points** of the **energy functional**

\[
E_p : H^1_0 \to \mathbb{R} : u \mapsto \frac{1}{2} \int_{\Omega} |\nabla u|^2 + V(x)u^2 - \int_{\Omega} F(u),
\]

where $F(u) := \int_0^u f(s) \, ds$ and $H$ is a Sobolev space.

**Aim:** To study the symmetries of non-zero solutions, number of non-zero solutions,...
Introduction: nonlinear Schroedinger Problems

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Aim: To study the symmetries of non-zero solutions, number of non-zero solutions,...
1. Lane–Emden problem ($V = 0$ and $f = u^p$): ground state solutions (non-trivial solution with minimal energy)
2. least energy nodal solutions (l.e.n.s.; sign-changing solution with minimal energy)
3. Generalizations
4. Future
Lane–Emden Problem

We consider, for $2 < p < 2^* := \frac{2N}{N-2}$,

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\begin{cases}
-\Delta u = |u|^{p-2}u, & \text{in } \Omega, \\
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Solutions are critical points of the energy functional.

Energy functional

- $\mathcal{E}_p : H^1_0(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p} \int_\Omega |u|^p$
- $\mathfrak{d}\mathcal{E}_p(u) : H^1_0(\Omega) \to \mathbb{R} : v \mapsto \int_\Omega \nabla u \nabla v - \int_\Omega |u|^{p-2} uv$

$H^1_0(\Omega) := \overline{C^2_0(\Omega)}$ for $\|u\|_{H^1_0}^2 := \int_\Omega |
\n0$ is always solution.

What for others?
Lane–Emden Problem

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\textbf{Energy functional}

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\textbf{What for others?}
Lane–Emden problem ($V = 0$ and $f = u^p$): ground state solutions (non-trivial solution with minimal energy)

1. Existence
2. Examples (computing MPA)
3. Symmetry results

Least energy nodal solutions (l.e.n.s.; sign-changing solution with minimal energy)

Generalizations

Future
Existence of Ground State for\[\begin{cases} -\Delta u = |u|^{p-2} u, & \text{in } \Omega \\ u = 0, & \text{on } \partial \Omega. \end{cases}\]

The energy functional is not bounded from below.

**Mountain-Pass theorem (A. Ambrosetti, P. H. Rabinowitz, ’73)**

There exists a ground state solution. It is a one-signed function.

**Sketch:**
- The energy functional $\mathcal{E}_p$ possesses a Mountain-Pass structure
- $\exists u_0 \neq 0, \mathcal{E}_p(u_0) = \inf_{u \neq 0} \max_{\lambda > 0} \mathcal{E}_p(\lambda u) = \inf_{u \in \mathcal{N}_p} \mathcal{E}_p(u)$
- For any sign-changing solution $u$: $\mathcal{E}_p(u^\pm) < \mathcal{E}_p(u)$, where $u^+ := \max(u, 0)$ and $u^- := \min(u, 0)$

$$\mathcal{E}_p(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p} \int_\Omega u^p$$

$\mathcal{N}_p := \{ u \neq 0 : (u, \mathcal{E}(u)) \in \text{“Top of the Mountain”} \}$ and is called

"Nehari manifold"
Existence of Ground State for \[ -\Delta u = |u|^{p-2}u, \quad \text{in } \Omega \]
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Remarks on $\mathcal{N}_p$

- $\mathcal{N}_p$ is formed by all the non-zero functions $u$ such that $d\mathcal{E}(u)u = 0$... maximization in the direction of $u$
- If $u \in \mathcal{N}_p$: $\mathcal{E}_p(u) = (\frac{1}{2} - \frac{1}{p})\|u\|^2$ (because $\|u\|^2 = \|u\|_{L^p}^p$)
- $\mathcal{N}_p$ is a manifold of codimension 1. So, Morse index of a ground state solution is 1.
Mountain-Pass algorithm (MPA)

Y. S. Choi and P. J. McKenna in '92 and J. Zhou and Y. Li in '01

\[ \mathcal{N}_p \]

Algorithm

1. Let \( u \in H^1_0 \) one-signed function and \( n \leftarrow 0 \)
2. Compute \( u_n \leftarrow P(u) \).
3. Deform the path: compute \( g \leftarrow \nabla \mathcal{E}(u_n) \).
4. If \( \| \nabla \mathcal{E}(u_n) \| \leq \varepsilon \) stop;
   else \( v \leftarrow P(u_n - g) \).
5. If \( \mathcal{E}(v) < \mathcal{E}(u_n) \), \( u_{n+1} \leftarrow v \), \( n \leftarrow n + 1 \) and go to step 3;
   else \( g \leftarrow g^2 \) and go to step 4.

Projection \( P : H^1_0 \rightarrow H^1_0 : u \mapsto \lambda_u u \in \mathcal{N}_p \).
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5. If $\mathcal{E}(v) < \mathcal{E}(u_n)$, $u_{n+1} \leftarrow v$, $n \leftarrow n + 1$ and go to step 3; else $g \leftarrow g/\|g\|$ and go to step 4.

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\[\mathcal{N}_p\]
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\[ \mathcal{N}_p \]

\[ -\nabla \mathcal{E}(u_n) \]

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5. If \( \mathcal{E}(v) < \mathcal{E}(u_n) \), \( u_{n+1} \leftarrow v \), \( n \leftarrow n + 1 \) and go to step 3; else \( g \leftarrow g/2 \) and go to step 4.

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Projection \( P : H^1_0 \rightarrow H^1_0 : u \mapsto \lambda_u u \in \mathcal{N}_p \).
Accumulation points of \((u_n)_{n \in \mathbb{N}}\) are **local minimum** of \(E_p\) in \(\mathcal{N}_p\).

They are one-signed solutions of Problem \((P)\).

We can **not** be sure that they are **global minimum** in \(\mathcal{N}_p\).
Examples: convex domains \((-\Delta u = u^3)\)

Mountain-Pass algo (MPA) of Y. Choi and J. McKenna, ’93:
Symmetry result for ground state

Theorem (B. Gidas, W. M. Ni, L. Nirenberg, '79)

When $\Omega$ is “convex”,

- one and only one ground state (up to a multiplicative factor of value $-1$)
  - it respects reflections that leaves $\Omega$ invariant
  - in particular, on balls, ground state is radial

- one and only one maximum (resp. minimum) at the positive (resp. negative) ground state

- positive (resp. negative) ground state is decreasing (resp. increasing) from this maximum to the boundary

Idea: moving-plane method
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Idea: moving-plane method
Non-convex domains \((-\Delta u = u^3)\)
1. Lane–Emden problem ($V = 0$ and $f = u^p$): ground state solutions (non-trivial solution with minimal energy)

2. least energy nodal solutions (l.e.n.s.; sign-changing solution with minimal energy)
   - Existence
   - Examples (MMPA) and symmetry results
     - Symmetries on rectangle, for small $p$,
     - Symmetries on radial domains, for small $p$,
     - Symmetries on square, for small $p$,
     - Symmetry breaking for some rectangles,
     - Nodal line structure
     - What about $p$ large on general domains?

3. Generalizations

4. Future
Existence of nodal solution

Theorem (A. Castro, J. Cossio, J. M. Neuberger, '97)

There exists a nodal solution with minimal energy.

Projection: \( sH_0^1 \setminus \{0\} \rightarrow \mathcal{M}_p : u \mapsto P_{\mathcal{N}_p}(u^+) + P_{\mathcal{N}_p}(u^-) \).

Remark 1: Maximization in \( \{tu^+ + su^- : t, s \geq 0\} \)

Remark 2: Gidas, Ni and Nirenberg method fails.
Ex: Rectangle ($-\Delta u = u^3$)

MMPA of G. Costa, Z. Ding and J. M. Neuberger ('01)
Symmetries

- $\lambda_i$: $i^{th}$ eigenvalue of $-\Delta$ with DBC s.t. $\lambda_1 < \lambda_2 < \ldots$
- $E_i$: eigenspace related to $\lambda_i$

**Theorem (G., C. Troestler, CRAS)**

*For $p$ close to 2, if $\lambda_2(\Omega)$ is simple,*

- *for any reflection $R$ s.t. $R(\Omega) = \Omega$, l.e.n.s. respects the symmetry or antisymmetry of functions in $E_2$ with respect to $R$.*
- *it is unique up to a multiplicative factor of value $-1$.*
Sketch of the proof: "Simple" case (1/3)

\[\begin{aligned}
-\Delta u &= |u|^{p-2}u, & \text{in } \Omega \\
\quad u &= 0, & \text{on } \partial \Omega
\end{aligned}\]  

(P)

Up to a rescaling by \(\lambda_2^{\frac{-1}{p-2}}\), the study of symmetries for (P) is the same as for problem

\[\begin{aligned}
-\Delta u &= \lambda_2 |u|^{p-2}u, & \text{in } \Omega \\
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(P2)

Let \((u_p)_{p>2}\) a family of least energy nodal solutions of (P2).

Let \((u_p)_{p>2}\) a **family of least energy nodal solutions** of (P2).
To obtain that $(u_p)_{p>2}$ is *bounded* in $H^1_0(\Omega)$ and *away* from 0.
Sketch of the proof (3/3)

1. \( \|u_p\| \leq K \Rightarrow u_p \rightharpoonup u_0 \text{ up to a subsequence} \)
   \( \Rightarrow u_p \rightarrow u_0 \text{ up to a subsequence} \)

2. \( u_0 \in E_2 \)

3. \( u_p \) stays away from 0

4. computing \( u_p, \lambda := \frac{u_p}{\|u_p\|} \text{ s.t.} \)
   \( (u_p, \lambda, \lambda \|u_p\|^{p-2}) \rightarrow (\pm e_2, \lambda) \)

5. By IFT, the curve starting from \( e_2 \) and solution of
   \[
   \begin{cases}
   \Delta u(x) + \lambda |u(x)|^{p-2} u(x) = 0, & x \in \Omega \\
   u(x) = 0, & x \in \partial \Omega \\
   \|u\| = 1
   \end{cases}
   \]
   is unique

6. symmetries of \( e_2 \) are respected
Sketch of the proof (3/3)

1. \( \|u_p\| \leq K \Rightarrow u_p \rightharpoonup u_0 \) up to a subsequence
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6. symmetries of \( e_2 \) are respected
Sketch of the proof (3/3)

1. \[ \|u_p\| \leq K \Rightarrow u_p \rightharpoonup u_0 \text{ up to a subsequence} \]
   \[ \Rightarrow u_p \to u_0 \text{ up to a subsequence} \]

2. \( u_0 \in E_2 \)

3. \( u_p \) stays away from 0

4. computing \( u_{p,\lambda} := \frac{u_p}{\|u_p\|} \text{ s.t.} \)
   \[ (u_{p,\lambda}, \lambda_2 \|u_p\|^{p-2}) \to (\pm e_2, \lambda_2) \]

5. By IFT, the curve starting from \( e_2 \) and solution of
   \[
   \begin{cases}
   \Delta u(x) + \lambda |u(x)|^{p-2} u(x) = 0, & x \in \Omega \\
   u(x) = 0, & x \in \partial \Omega \\
   \|u\| = 1
   \end{cases}
   \]
   is unique

6. symmetries of \( e_2 \) are respected
Sketch of the proof (3/3)

1. $\|u_p\| \leq K \Rightarrow u_p \rightarrow u_0$ up to a subsequence
   $\Rightarrow u_p \rightarrow u_0$ up to a subsequence

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   \]
is unique

6. symmetries of \( e_2 \) are respected
The family \((u_p)_{p>2}\) is bounded.

Let \(e_2 \in E_2 \setminus \{0\}\), we obtain

\[
\mathcal{E}_p(u_p) = \left(\frac{1}{2} - \frac{1}{p}\right) \|u_p\|^2 \leq \left(\frac{1}{2} - \frac{1}{p}\right) \left\{ t_p^+ \|e_2^+\|^2 + t_p^- \|e_2^-\|^2 \right\}
\]
$u_p$ stays away from 0

**Lemma**

The family $(u_p)_{p>2}$ stays away from 0.

- $u_p \in \mathcal{M}_p \rightarrow \exists v_p \in \mathcal{N}_p$ such that $\int_{\Omega} v_pe_1 = 0$ and $\|v_p\| \leq \|u_p\|

- By using the Hölder inequality, $\lambda_2 \int_{\Omega} v_p^2 \leq \int_{\Omega} |\nabla v_p|^2$ and Sobolev's inequalities, $(v_p)_{p>2}$ stays away from 0 ($\|v_p\|_p \geq (S\lambda_2^{-1})^{\frac{2^*}{2^*-2}}$).

$$\int u_p^- e_1 < 0$$

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$\bullet \ u_p$
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\[
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\]

\[
\int u_p^- e_1 < 0
\]

\[
\int u_p^+ e_1 > 0
\]

\[
u_p = u_p^+ + u_p^-
\]
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\[
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$v_p : \int v_p e_1 = 0$ and $\|v_p\| \leq \|u_p\|$
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**Lemma**

The family \((u_p)_{p>2}\) stays away from 0.

- \( u_p \in \mathcal{M}_p \rightarrow \exists v_p \in \mathcal{N}_p \) such that \( \int_{\Omega} v_p e_1 = 0 \) and \( \|v_p\| \leq \|u_p\| \)
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Using IFT

Let the problem \((P_\lambda)\)

\[
\begin{cases}
\Delta u + \lambda |u|^{p-2}u = 0, & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega \\
\|u\| = 1
\end{cases}
\]

\(\varphi : [2, 2^*] \times H^1_0 \times \mathbb{R} \to H^1_0 \times \mathbb{R}
\)

\((p, v, \lambda) \mapsto (-(-\Delta)^{-1}(\lambda|v|^{p-2}v) + v, \|v\|^2 - 1)\)

Roots of \(\varphi\) are the solutions of \((P_\lambda)\).

\(\left( dH^1_{(\varphi(2, \varepsilon, \lambda))} \right)(v, t) =
\)

\((-\Delta)^{-1}(\lambda v) + v - t(-\Delta)^{-1}(\varepsilon), 2\int_\Omega (\nabla v \cdot \nabla \varepsilon) \) is injective (and so bijective)

By IFT: one and only one curve.
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\[\left(\text{d}_{H^1_0(\Omega) \times \mathbb{R}} \varphi(2, e_2, \lambda_2)\right)(\nu, t) =
-(-\Delta)^{-1}(\lambda_2 \nu) + \nu - t(-\Delta)^{-1}(e_2), 2 \int_\Omega (\nabla e_2 \nabla \nu) \text{ is injective (and so bijective)}\]

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- By IFT: one and only one curve.
Ex: radial domains \((-\Delta u = u^3)\)
Theorem (A. Aftalion, F. Pacella, ’04)

On a radial domain, l.e.n.s. can not be radial.
Theorem (T. Bartsch, T. Weth, M. Willem, ’05)

On a radial domain, l.e.n.s. is **Schwarz foliated symmetric**. So, it is **even** with respect to $N - 1$ orthogonal directions.

\[ u(A) \geq u(B) = u(C) \geq u(D) \]
What about the last direction?

**Theorem (G., C. Troestler, EJDE)**

For \( p \) close to 2, on radial domains, l.e.n.s. is **odd** with respect to a direction and is **unique** up to rotations.
Sketch of the proof

- When \( \lambda_2 \) isn’t simple, we **can’t use** the previous IFT (\( d\varphi(2, e_2, \lambda_2) \) may **not** be injective).

Idea

Work with \( V \subseteq H^1_0 \): functions which respect a symmetry rotation around \((0, 0, \ldots, 1)\).

- as l.e.n.s. is **Schwarz foliated symmetric**, we assume that \((u_p)_{p>2} \subseteq V\)
- as \( \dim(E_2 \cap V) = 1 \), we use IFT to prove there exists **one and only one** curve of solutions in \( V \)
- as all functions in \( E_2 \cap V \) are antisymmetric with respect to the orthogonal hyperplane of \((0, \ldots, 0, 1)\), \( u_p \) is **odd** in one direction
- **remark**: unicity of the solution (up to rotations).
When $\lambda_2$ isn’t simple, we **cannot use** the previous IFT ($d\varphi(2, e_2, \lambda_2)$ may **not** be injective).

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*Work with $V \subseteq H^1_0$: functions which respect a **symmetry rotation** around $(0, 0, \ldots, 1)$.***

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- As $\dim(E_2 \cap V) = 1$, we use IFT to prove there exists **one and only one** curve of solutions in $V$.
- As all functions in $E_2 \cap V$ are antisymmetric with respect to the orthogonal hyperplane of $(0, \ldots, 0, 1)$, $u_p$ is **odd** in one direction.
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Ex: Square \((-\Delta u = u^3)\)
Theorem (D. Bonheure, V. Bouchez, G., J. Van Schaftingen, CCM)

For $p$ close to 2, on a square, l.e.n.s. is odd with respect to the center.
Ideas for general domains

The IFT approach does not work.

New approach: general approach (but we lose the unicity!!!).

Let $M > 0$, $\exists \epsilon, \bar{p}_M > 0$.

Lemma (D. Bonheure, V. Bouchez, G., J. Van Schaftingen, CCM)

If $\|a(x) - \lambda_2\|_{L^{N/2}} < \epsilon$ and $u$ solves $-\Delta u = a(x)u$ with DBC, then $P_{E_2}u = 0$ implies $u = 0$.

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$\forall p \in (2, \bar{p}_M)$, if $u_p, v_p \in B_M \setminus B_{\frac{1}{\lambda_2}}$ solve $-\Delta u = \lambda_2 |u|^{p-2}u$ with DBC, then $P_{E_2}u_p = P_{E_2}v_p$ implies $u_p = v_p$. 

Ch. Grumiau (UMons)
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Proof of the lemma

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Sketch: Let us work by contradiction. Fix $w = P_{E_1}u$ and $z = P_{(E_1+E_2)\perp}u$.

$$\|w\|^2 = \lambda_2 \int_\Omega w^2 + \int_\Omega (a(x) - \lambda_2)uw$$

$$\geq \frac{\lambda_2}{\lambda_1} \|w\|^2 - C \|a(x) - \lambda_2\|_{L^{N/2}} \|w\| \|u\|,$$

$$\|z\|^2 \leq \frac{\lambda_2}{\lambda_3} \|z\|^2 + C \|a(x) - \lambda_2\|_{L^{N/2}} \|z\| \|u\|.$$

$$\|w\| \leq \frac{\lambda_1 C}{\lambda_2 - \lambda_1} \|a(x) - \lambda_2\|_{L^{N/2}} \|u\| \text{ and } \|z\| \leq \frac{\lambda_3 C}{\lambda_3 - \lambda_2} \|a(x) - \lambda_2\|_{L^{N/2}} \|u\|.$$

As $P_{E_2}u = 0$, $\|u\|^2 = \|w\|^2 + \|z\|^2 \leq C \|a(x) - \lambda_2\|_{L^{N/2}}^2 \|u\|^2$. 

Proof of the lemma

Lemma (D. Bonheure, V. Bouchez, G., J. Van Schaftingen, CCM)

Let \( M > 0, \exists \varepsilon, \bar{p}_M > 0, \) if \( \|a(x) - \lambda_2\|_{L^{N/2}} < \varepsilon \) and \( u \) solves \(-\Delta u = a(x)u\) with DBC, then \( u = 0 \) or \( P_{E_2} u \neq 0 \).

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\[
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\|z\|^2 \leq \frac{\lambda_2}{\lambda_3} \|z\|^2 + C\|a(x) - \lambda_2\|_{L^{N/2}} \|z\|\|u\|. \\
\|w\| \leq \frac{\lambda_1C}{\lambda_2-\lambda_1} \|a(x) - \lambda_2\|_{L^{N/2}} \|u\| \text{ and } \|z\| \leq \frac{\lambda_3C}{\lambda_3-\lambda_2} \|a(x) - \lambda_2\|_{L^{N/2}} \|u\|. \\
\|P_{E_2} u\| = \|w\|^2 + \|z\|^2 \leq C\|a(x) - \lambda_2\|_{L^{N/2}}^2 \|u\|^2.
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\[
\| w \|^2 = \lambda_2 \int \Omega w^2 + \int \Omega (a(x) - \lambda_2)uw \geq \frac{\lambda_2}{\lambda_1} \| w \|^2 - C \| a(x) - \lambda_2 \|_{L^{N/2}} \| w \| \| u \|, 
\]

\[
\| z \|^2 \leq \frac{\lambda_2}{\lambda_3} \| z \|^2 + C \| a(x) - \lambda_2 \|_{L^{N/2}} \| z \| \| u \|. 
\]

\[
\| w \| \leq \frac{\lambda_1 C}{\lambda_2 - \lambda_1} \| a(x) - \lambda_2 \|_{L^{N/2}} \| u \| \text{ and } \| z \| \leq \frac{\lambda_3 C}{\lambda_3 - \lambda_2} \| a(x) - \lambda_2 \|_{L^{N/2}} \| u \|. 
\]

As \(P_{E_2} u = 0\), \(\| u \|^2 = \| w \|^2 + \| z \|^2 \leq C \| a(x) - \lambda_2 \|^2_{L^{N/2}} \| u \|^2 \).
Unicity at projection fixed in $E_2$

Theorem (D. Bonheure, V. Bouchez, G., J. Van Schaftingen, CCM)

\[ \forall p \in (2, \bar{p}_M), \text{ if } u_p, v_p \in B_M \setminus B_{\frac{1}{M}} \text{ solve } -\Delta u = \lambda_2 |u|^{p-2}u \text{ with DBC, then } u_p = v_p \text{ or } P_{E_2}u_p \neq P_{E_2}v_p. \]

Idea: equation verified by $u_p - v_p$ ($\Delta$ linear) + Lebesgue

Corollary 1: If $(u_p)$ is a family of bounded solutions staying away from 0, for $p$ close to 2, $u_p$ respect symmetries of its projection on $E_2$.

Corollary 2: It is working for l.e.n.s.

Corollary 3: As second eigenfunctions odd with respect to the center, we obtain the expected symmetries on squares.
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The square: diagonal or median?... limit functional and Nehari!

Accumulation points $u_*$ verify
$\mathcal{E}^*_*(u_*) = \inf \{ \mathcal{E}^*_*(u) : u \in E_2 \setminus \{0\}, \; d\mathcal{E}^*_*(u)u = 0 \}$, where
$\mathcal{E}^*_* : E_2 \to \mathbb{R} : u \mapsto \frac{\lambda^2}{2} \int_\Omega u^2 - u^2 \log u^2$.

Idea: $0 = \lim_{p \to 2} \int_\Omega \frac{(|u_p|^{p-2}u_p - u_p)v}{p-2} = \int_\Omega u_* \log |u_*| v.$
The square: diagonal or median?...limit functional and Nehari!

Accumulation points $u_*$ verify

$\mathcal{E}_*(u_*) = \inf\{\mathcal{E}_*(u) : u \in E_2 \setminus \{0\}, d\mathcal{E}_*(u)u = 0\}$, where

$\mathcal{E}_* : E_2 \to \mathbb{R} : u \mapsto \frac{\lambda_2}{2} \int_\Omega u^2 - u^2 \log u^2$.

**Idea:** $0 = \lim_{p \to 2} \int_\Omega \frac{(|u_p|^{p-2}u_p - u_p)v}{p-2} = \int_\Omega u_* \log |u_*| v$. 
Limit functional on the limit Nehari manifold

\( \mathcal{E}_* \) has Mountain-Pass structure \( \rightarrow \) projection on the limit Nehari manifold.

![Graph showing eigenfunctions](image)

Eigenfunctions: \( e_\theta = \cos(\theta)v_1 + \sin(\theta)v_2 \), \( v_1 = \cos(\frac{\pi}{2}x)\sin(\pi y) \) and \( v_2 = \sin(\pi x)\cos(\frac{\pi}{2}y) \).

The diagonal seems to be “the winner”... Is it really true???
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The diagonal \textbf{seems} to be “the winner”... Is it really true???
Proposition (G., P. Hauweele, preprint)

Using interval arithmetic, we obtain that l.e.n.s. must be a diagonal function for $p$ small.

Difficulties:

- Compute guaranteed upper and lower bounds
- Prove that the function is convex around the diagonal... difficult part as we need to control singularities in the integral
- Using a Lyapunov-Schmidt reduction.
Symmetry breaking

\[
\text{wide} = \pi \\
\text{length} = \pi + R
\]

\[
1 - \left(\frac{\pi}{\pi + R}\right)^2 = (p - 2)^2
\]

**Idea:** Results work for 
\[- \div(A_p \nabla u) = \lambda_2 |u|^{p-2}u\]
Graph ($p = 6$)
What about \( p \) large and radial domains?

**Theorem (M. Grossi, G., F. Pacella, submit in AIHP)**

For \( p \to +\infty \), the \( L^\infty \)-norm of l.e.n.s. is going to \( \sqrt{e} \) and the nodal line intersects the boundary for large \( p \).

**Sketch:**

- A first blow-up: the maximum picks \( x_p^\pm \) are not going to the nodal line and the boundary too fast (i.e. \( \frac{d(x_p, NL_p \cap \partial \Omega)}{\varepsilon_p} \) where
  \[
  \varepsilon_p = \frac{1}{\sqrt{pu_p(x_p^{p-1})}}.
  \]
- A second blowup \( u_p : z_p^\pm : \varepsilon_p \Omega_{\pm} \ldots \)
- \( \| u_p^\pm \|_\infty \to \sqrt{e} \)
Nodal line structure
Nodal line structure
Linear problem $-\Delta u = \lambda_2 u$

Theorem (G. Alessandrini, '94)

On a convex domain $\Omega$, in dimension 2, nodal line of $u \in E_2$ intersects $\partial \Omega$ in exactly 2 points.

Theorem (M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, N. Nadirashvili, '95)

There exists a not simply connected domain $\Omega$ such that nodal line of $u \in E_2$ does not intersect $\partial \Omega$. 
Example of domain

nodal line
Lane–Emden problem ($V = 0$ and $f = u^p$): ground state solutions (non-trivial solution with minimal energy)

least energy nodal solutions (l.e.n.s.; sign-changing solution with minimal energy)

Generalizations

Future

1. Interval arithmetic and a posteriori estimates for finite element methods
2. Fourth order equations
3. ...
\[-\Delta u + u = \lambda |u|^{p-2} u\]

- Energy defined on the space $H := H^1$.
- Concerning l.e.n.s, it is working in the same way: family bounded and staying away from zero (for the good $\lambda$).
- Ground state solution respects symmetries of its projection in $E_1$ but maybe not for large $p$ and positive solutions are not unique.
- We have the existence of a symmetry breaking.

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$^1$functions in $L^2$ with weak derivative in $L^2$
This is a work in progress with D. Bonheure and C. Troestler.

**We have:**
- For $p > 1 + \lambda_2$, $\pm 1$ is not ground state.
- Radial bifurcations when $p$ crosses $1 + \lambda_i$.
- last result...

**Tools:** bifurcations theory for ODEs (Krasnoselskii-Boehme-Marino, Ambrosetti-Prodi,...)

**Improvements:** it is working for diffusion equations...

**Open questions:**
- Is bifurcation sequence given for $p = 1 + \lambda_2$ a ground state solutions branch?
Numerical experiments

By using Mountain-Pass algorithm:

Figure: \(-\Delta u + u = |u|^{\lambda_2-1+0.1} u\), with \(\lambda_2 = 2 + \frac{\pi^2}{4}\)

\[
\begin{array}{c|c}
\mathcal{E}(u) & \mathcal{E}(1) \\
0.98 & 1.0 \\
\end{array}
\]
What for non-zero $V$? $(-\Delta u + V(x)u = \lambda |u|^{p-2}u)$

**Proposition (G., noDEA)**

$-\Delta + V(x)$ positive definite $\rightarrow$ okay

Otherwise, no Mountain-Pass structure $\rightarrow$ other Nehari manifold (see A. Szulkin, T. Weth, ’09)

$$\mathcal{N}_p := \{u \in H^1_0 \setminus H^- : \mathcal{E}_p'(u)v = 0, v \in \{u\} \cup H^-\},$$

where $H^-$ is the negative spectrum of $-\Delta + V(x)$.

**Remark:** only existence of ground state solutions is proved.

**Work with C. Troestler:** convergence of an algorithm (type MPA) to approach non-zero solution of this problem.
Symmetries \((-\lambda_2 < V(x) = \lambda < -\lambda_1)\)

**Question**: What for symmetries of ground state solution in this case?

**Conjecture**: It seems to keep symmetries of its projection in the first eigenspace with positive eigenvalue.
Future:

- Computer assisted proofs: McKenna + explanations...
- Fourth order equations:
- Squassina:
le chat + merci...