Partially massless spin-2 fields: twisted duality and interactions in (A)dS\(_n\)

Nicolas Boulanger, Service de Physique de l'Univers, Champs et Gravitation
Université de Mons - UMONS

Work in collaboration with Andrea Campoleoni, Nacho Cortese and Lucas Traina
based on 1804.05588 and 1807.04524 for the original parts on twisted duality
and in collaboration with Sebastian Garcia-Saenz and Lucas Traina for interactions

Imperial College London - 11 June 2019
Plan of the talk

1. Generalities

2. Review of duality for spin-2 in flat spacetime

* 3. Parent action in Stuckelberg formulation

4. Twisted (self-) duality in (A)dS_n: massless and PM cases

5. A theory for multiple PM spin-2 fields
1. Generalities

Electric-magnetic duality, perhaps as fundamental as Lorentz symmetry.

In non-Abelian theory, relates strong and weak coupling regimes. [Long story: Heavyside, Dirac, ...]

For spin-2 (linearized), studied by P. West, Hull (2001). Previous attempts in the massive case by Curtright & Freund in 80's. Further studied in 2002, on-shell, by X. Bekaert & N.B., all these studies in flat spacetime.

Quid of massless spin-2 in (A)dS_n: duality property?

→ same question for partially-massless spin-2, only defined in (A)dS_n.
2. Review of duality for spin-2 in flat spacetime

Use condensed notation \((X. \text{ Bekert} & \text{ N.B., 2002})\)

\(EI: \quad \text{Tr} \ K = 0 \iff K_{\mu \nu \rho \sigma} = 0, \quad \text{where } K = d^{(1)}d^{(2)}h,\)

\(BI: \quad \text{Tr}_{12} K = 0 \iff K_{[\mu \nu \rho \sigma]} = 0,\)

\(E\Pi: \quad d^+ \ K = 0 \iff \partial^\mu K_{\mu \nu \rho \sigma} = 0,\)

\(B\Pi: \quad d \ K = 0 \iff \partial_{[\mu} K_{\nu \rho \sigma]} = 0.\)

\(\bullet \ K = K_{[\varepsilon, \zeta]} = \frac{1}{4!} d^{(1)} \xi^\mu d^{(2)} \xi^\nu d^{(2)} \xi^\gamma d^{(2)} \xi^\beta K_{\mu \nu \rho \sigma} K_{\mu \nu \rho \sigma}\)

\(\bullet \ \text{dual } d^+_{(c)} \chi^\mu \text{ s.t. } \{d^{(c)} \chi^\mu, d^+_{(c)} \chi^\nu\} = \eta^{\mu \nu},\)

\(\bullet \ d^{(c)} := d^{(c)} \chi^\mu \frac{\partial}{\partial \chi^\mu}, \quad d^+_{(c)} := d^+_{(c)} \chi^\mu \frac{\partial}{\partial \chi^\mu},\)

\(\bullet \ \text{Tr}_{\varepsilon, \zeta} = \eta_{\mu \nu} d^+_{(c)} \chi^\mu d^+_{(c)} \chi^\nu\)
As operators: \( | K_{[z, z]} \rangle = \frac{i}{4} d^{(1)} x^u d^{(2)} \omega^\alpha d^{(2)} x^\beta K_{\mu \nu \lambda \beta} | 0 \rangle \)
\[ d^{(1)} \omega | 0 \rangle \doteq 0 \quad \text{destruction} . \]

\[
\left[ d^{(1)} x^u , d^{(2)} \omega^\alpha \right] \equiv 0 , \quad \left[ d^{(1)} x^u , d^{(2)} x^\alpha \right] \equiv \delta^u_\alpha \eta^\mu \nu .
\]

**Twisted-duality relations**

\( K \quad \rightarrow \quad *_1 K \quad , \quad *, K \quad \rightarrow \quad -K \)

\[ \overline{K} := \begin{pmatrix} K \\ *_1 K \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} K \\ *_1 K \end{pmatrix} = J \overline{K} , \quad J = \pi_2 \text{rotation} . \]

\[ \rightarrow \begin{pmatrix} E_{\text{I}} \\ E_{\text{II}} \end{pmatrix} \quad \leftrightarrow \quad \begin{pmatrix} B_{\text{I}} \\ B_{\text{II}} \end{pmatrix} \quad \text{under duality} . \]

**Example, \( n = 5 \):**

\[ B_{\text{I}} : \quad \text{Tr}_2 *_1 K = 0 \quad \}
\[ \implies \quad K_{[z, z]} = d^{(1)} d^{(2)} k_{[\alpha, \beta]} , \quad k \sim \]

\[ B_{\text{II}} : d^{(1)} K = 0 \]

\[ E_{\text{I}} : \quad \text{Tr} K = 0 \quad \rightarrow \quad \text{Tr} * \overline{K} = 0 \quad \implies \quad \overline{K} \sim \]

\[ E_{\text{II}} : d^{(1)} K = 0 \quad \rightarrow \quad d \overline{K} = 0 \quad \leftrightarrow \quad \overline{K} \sim \]

\[ E_{\text{II}} : d^{(1)} K = 0 \quad \leftrightarrow \quad d \overline{K} = 0 \quad \leftrightarrow \quad \overline{K} \sim , \quad c \sim \]

\[ B_{\text{II}} \sim \]
So far: on-shell. One can find an action $S[c]$ s.t. $\text{Tr} \tilde{K} = 0$ is the e.o.m.

$\Rightarrow$ [P. West 2001, 2002], [N.B., S. Cnockaert, M. Hommeaux 2003: contact with Curtright's action for $n = 5$]

**Strategy:** Use a Parent action $S[h, Y^{abc}]$

**Dual actions**

$h \sim [1,1]$ \hspace{1cm} C $\sim [n-3,1]$  

**Therefore:** in flat spacetime

$h \sim [1,1]$ \hspace{1cm} $C \sim [n-3,1]$  

**Question:** What about $(A)dS_n$?

Early investigations by B. Julia, Yu. Zinoviev still no identification of what field is the dual graviton in $(A)dS$.
Idea: [Y. Zinov'ev, Th. Basile - X. Behraert - N.B. ]: Parent action = linearised action from Chamesedline - West / MacDowell-Mansouri

\[ S_m[\mathcal{H}, \omega] = \int_{\text{AdS}_n} \left[ -\frac{\epsilon_{abc}[\pi, n, 3]}{n (n-3)!} \left( \nabla^2 \omega^{bc} - \frac{1}{n-2} \omega^a \omega^b \bar{e}^c \right) \bar{e}^p \cdots \bar{e}^{p_{n-3}} 
\]

- \[ \frac{1}{n-2} \omega^a \omega^b \bar{e}^c \cdots \bar{e}^c \left( \frac{-\epsilon_{abc}[\pi, n, 3]}{n (n-3)!} \mu^2 \frac{n-2}{4} h^{a} h^{b} \bar{e}^c \cdots \bar{e}^c \right) \],

\[ \mu^2 := \frac{2 n (n-1)}{n-2} \left( 2 m^2 + \sigma(n-2) \lambda^2 \right), \quad \nabla^2 \lambda^a = -\sigma \lambda^a \bar{e}^a \bar{e}^b \lambda^b, \quad \sigma = \left\{ \begin{array}{ll} 1 & \text{AdS}_n \\ -1 & \text{dS}_n \end{array} \right. \]

Set \( m = 0 \) and integrate out \( h^a = \text{d}x^\mu h_{\mu}^a \), auxiliary for \( \lambda \neq 0 \):

\[ L \rightarrow S[\omega^a, \omega^{ab}] = \frac{1}{\lambda^2} \left( \begin{array}{l} (i) \\ (ii) \\
\end{array} \right) \int_{\text{AdS}_n} C^{ab} \wedge * C^{cd} \]

\[ R_{\mu \nu} \omega^{ab} := 2 \nabla_{[\mu} \omega_{\nu]} \omega^{ab} \]

for \( C^{ab} := \bar{R}^{ab} - \bar{e}^{[a} \bar{e}^{b]} R \) Schouten-like

\[ \delta \omega_{\mu} \omega^{ab} = \nabla_{\mu} \delta \omega^{ab} + \frac{\mu^2}{n-1} \bar{e}_\mu [a \in b] \]

\[ S[\omega_{\mu} \omega^{ab}] \text{ invariant under } \delta \omega_{\mu} \omega^{ab} = \nabla_{\mu} \delta \omega^{ab} + \frac{\mu^2}{n-1} \bar{e}_\mu [a \in b] \]
Hodge dualise: $T_{[n-2,1]} := *_1 \hat{\omega}_{[2,1]}$ ($\hat{\omega}$ = traceless part of $\omega$)

$\leftrightarrow$ Relation with $C_{[n-3,1]}$ via Stückelberg shift:

$$\hat{\omega}_{a_1 b c} \rightarrow \hat{\omega}_{a_1 b c} + \frac{1}{\lambda} \nabla_d \hat{\omega}^{bcd} a$$

Hodge dualise $T_{[n-2,1]} := *_2 \hat{\omega}_{a} \ldots$ & $C_{[n-3,1]} := *_4 \hat{\omega} \ldots$. and observe that, in the flat limit $\lambda \to 0$ of $L_0(C,T)$, $T_{[n-2,1]}$ becomes a topological field:

$$\frac{\delta L_0}{\delta T} = \text{Tr} K_{[n-1,2]}(T) \sim 0 \Rightarrow K(T) \sim 0$$

$n+1 > n$

Hence, only $L(C,\Theta_C)$ remains for local d.o.f.
3. Parent action in Stiickelberg formulation

The best way is to use a Stiickelberg re-formulation \( \rightarrow [N.B., A. Campuleoni, I. Cortes] \) 2018

Continue the analysis of Yuri Zinoviev [2008] and use frame-like,

1st-order action for \( m \neq 0 \):

\[
S \left[ h^a, \omega^a_{\alpha\beta}, A^a, F^a_{\alpha\beta}, \phi, \pi^a \right] = \int (L^{(1)} + L^{(1)} + L^{(0)} + L^{cross})
\]

Spin: \( \begin{array}{ccc} 2 & 1 & 0 \end{array} \)
\[
\mathcal{L}^{(2)} \sim - \frac{\epsilon_{abc} \epsilon_{[n-3]}}{2(n-3)!} \left( \nabla h^a \wedge w^{bc} - \frac{1}{n-2} \omega^a \wedge \omega^b \varepsilon^c \right) \bar{e}_p \ldots \bar{e}_{p-3}
\]

\[
\mathcal{L}^{(1)} \sim F^{ab} \left( \nabla A - \frac{1}{2} F_{cd} \varepsilon^c \varepsilon^d \right)
\]

\[
\mathcal{L}^{(0)} \sim \bar{\pi}^a \left( \nabla \psi - \frac{1}{2} \bar{\pi}_b \bar{e}^b \right)
\]

\[
\mathcal{L}^{\text{cross}} \sim \left[ m(n-1) w^{ab} A + m F_{ab} \bar{h}^d \bar{e}^b + \mu \bar{\pi}^a A \bar{e}^b - \frac{n-2}{n-1} \mu^2 h^a h^b - m \mu \psi h^a \bar{e}^b - \frac{m^2}{n-2} \psi^2 \bar{e}^c \bar{e}^d \right]
\]

\[
\times \in a b c [n-2] \bar{e}^c \ldots \bar{e}^c.
\]

\[ S[\bar{h}, \omega, A, F, \psi, \pi] \] is invariant under the gauge transformations:

\[
\begin{align*}
\delta h^a &= \nabla \xi^a - A^b \bar{e}^b + \frac{\epsilon m}{n-2} \in \bar{e}^a \\
\delta \omega^{ab} &= \nabla \xi^{[a} + \frac{\mu^2}{n-1} \varepsilon^{[a} \bar{e}^b] \\
\delta \bar{\pi}^a &= -m \mu \xi^a
\end{align*}
\]

\[
\delta A = \nabla \varepsilon - m \xi^a \bar{e}_a
\]

\[
\delta F_{ab} = 2 m \Lambda_{ab}
\]

where

\[
\mu^2 := \frac{2(n-1)}{n-2} \left( 2 m^2 + \sigma (n-2) x \right)
\]
When $m = 0$ spin-1 & 0 sectors decouple $\rightarrow$ recovers 1st order formalism of line. gravity in (A)dS$_n$;

When $\mu = 0$ spin-0 sector decouples $\rightarrow$ 1st order formulation of PM spin-2.

$\rightarrow$ smooth flat limit: $\pm 2$ & $\pm 1$ helicities.

**Rem 1**: $\mu = 0$ only if $\sigma = -1$ apparently, but if $L^{(1)} \rightarrow -\sigma L^{(1)}$, PM limit also in $\text{AdS}_n$

$$\mu^2 \rightarrow \tilde{\mu}^2 = \frac{2(n-2)}{n-2} \left( 2m^2 - (n-2)x^2 \right), \text{ in both } \sigma = \pm 1.$$

**Rem 2**: $\gamma^{ab}_{\ c} = \omega_{c1}^{\ ab} + 2S^{ca}_{\ c} \omega_1^{\ b}.$

**Two things can be done**

**A** Electric reduction: Eliminate auxil. $Y, F, \pi \rightarrow S [h_{ab}, \Pi, \gamma]$ $

\rightarrow$ Stuckelberg formulation for massive spin-2 in (A)dS$_n$ [Yu. Zizunov 2001, 2006, 2008]

. Same limits $m \rightarrow 0$ and $\mu \rightarrow 0$ as above.
B Magnetic reduction

B.1) Massless $m=0$ case: Spin $1\&0$ decouple:

\[ \mathcal{L} \rightarrow \mathcal{L}_0 = \nabla_b h_{c_1}^a \gamma^{b c_1} a - \frac{1}{2} (\gamma_{b c_1}^{a b} \gamma^{a b} + \frac{1}{n-2} \gamma^{a 1} \gamma_{a 1}) - \frac{n-2}{2} \sigma \lambda^2 \left( h_{a b} h_{b c} - h_{12}^2 \right) \]

\[ \delta h^a = \nabla^a \delta - \gamma_{a b} \epsilon_b \quad \delta \gamma^{b c_1} = \nabla_b \gamma^{a c_1} + 2 \epsilon_a \left[ b \gamma_{a c_1} - 2 \sigma (n-2) \epsilon_a \gamma_{b c_1} \right] \]

For $\lambda \neq 0, h$ auxiliary \[ \mathcal{L}(\phi, \gamma) = \frac{\sigma}{2(n-2) \lambda^2} \left[ \nabla_a \gamma_{c d b} - \gamma^{a b d} + \sigma \lambda^2 \gamma_{b c d a} \right] \]

where $\gamma = \text{traceless part of } \gamma_{a b c}$.

- Perform the shift $\gamma_{b c_1} a \rightarrow \gamma_{b c_1} a + \frac{1}{\lambda} \nabla_d \gamma_{b c_1 d a}$ to find

\[ \mathcal{L}_0 (\phi, \gamma, \hat{\gamma}) = \frac{1}{\lambda^2} \left[ \frac{1}{2} \nabla_c \hat{\gamma}_{a b c d} \nabla^d \hat{\gamma}_{a b c d} + \lambda \hat{\gamma}_{a b c} \nabla^e \hat{\gamma}_{a b c e} + \frac{\sigma}{2(n-2)} \nabla_b \gamma_{a b c} \nabla^d \gamma_{c d a} + \frac{\lambda^2}{2} \gamma_{a b c} \gamma_{a b c} \right] \]

- Hodge dualise $\hat{\gamma}_{[2,1]} := \star, \hat{\gamma}_{[2,1]}$ and $\hat{\gamma}_{[n-3,1]} := \star, \hat{\gamma}_{[n-3,1]}$ \[ \mathcal{L}_0 = -\frac{1}{2 \lambda^2 (n-3)!} \left[ \mathcal{L}(\nabla C) + 2 \lambda \" T \cdot \nabla C \" + \frac{\sigma}{(n-2) \lambda^2} \mathcal{I} \right] \]

where $\mathcal{I} := \mathcal{L}(\nabla T) + \sigma (n-2) \lambda^2 \left( T^2 - (n-2) T^1 \right)$
. $L_0(T, c)$ invariant under

. $\delta T^{[n-2,1]} : \gamma \overset{\delta}{\rightarrow} \gamma + \gamma + \sigma \lambda \gamma$

. $\delta C^{[n-3,1]} : \gamma \overset{\delta}{\rightarrow} \gamma + \gamma - \lambda \gamma$

. In the flat limit $\lambda \rightarrow 0$, the cross terms vanish and

\[
\lambda^2 L_0(T, c) \longrightarrow L^{\text{Cut}}(C^{[n-3,1]}) + \frac{\sigma}{(n-2)^2} L^{\text{Cut}}(T^{[n-2,1]})
\]

Note:

$L^{\text{Cut}}(T^{[n-2,1]})$ is topological: curvature $K_T^{[n-1,2]} \approx 0$ vanishes on-shell

where $K_T^{[n-1,2]} := d^{(1)} d^{(2)} T^{[n-2,1]}$ [X. Bekaert & N.B. 2002]

Hence there remains only $C^{[n-3,1]} \equiv \check{h}^{[n-3,1]}$ propagating.
B.2) Partially-Massless $\mu = 0$ case: Spin 0 decouples:

- In order to accommodate both $AdS_n$ and $dS_n$, rescale $L^{(n)} \rightarrow (-\sigma) L^{(n)}$ so that

  $$\mu^2 \rightarrow \tilde{\mu}^2 := \frac{2(n-1)}{m-2} \left( 2m^2 -(n-2)\lambda^2 \right).$$

- Take $\tilde{\mu} \rightarrow 0$, get

  $$L_{pm} = \hat{h}_{ab} C^{ab} + \frac{\sigma}{\tilde{\mu}} A_a \nabla_b C^{ab} - \frac{1}{2} \left( Y^{bc} Y_{ab} - Y_{a\alpha} Y_{b\alpha} \right) - \frac{\sigma}{4} F_{ab} F^{ab}$$

  where $C^{ab} := \nabla_b Y_{a\alpha} - \tilde{\mu} F^{ab}$ and $\tilde{\mu}^2 := \frac{m-2}{2} \lambda^2$.

- As in flat spacetime, $\hat{h}_{ab}$ is Lagrange multiplier. Constraint $C^{ab} = 0$

  solved identically by

  $$Y^{bc} a = \frac{1}{\lambda} \nabla_d \hat{W}^{bcd} a - \frac{\sigma}{\tilde{\mu}} \left( \nabla_a F^{bc} + 2 \varepsilon_{[a} \nabla_b \nabla_c F] \right).$$

- Substituting in $L_{pm}(h,Y,A,F)$ gives

  $$L_{pm}(\hat{W}) = -\frac{1}{2\lambda^2} \nabla_a \hat{W}^{ab} \nabla_c \hat{W}^{abc} + T.D.$$  

  $L_{pm}$ enters through total derivative "T.D."

  $$L_{pm} \text{ invariant under } \delta \hat{W}^{bcd} a = \nabla_a \hat{V}^{bcd} a.$$
\[ \mathcal{L}_{PM}(\nabla W) = -\frac{1}{2\lambda^2} \nabla^a \hat{W}_{bcda} \nabla^a \hat{W}_{bcda} \]

**Stückelberg shift**

\[ \hat{W}_{bcda} \rightarrow \hat{W}_{bcda} + \frac{1}{(n-3)\tilde{m}} (\nabla_a U_{bcd} - \text{Trace}) \]

**Hodge dualise**

\[ \begin{align*}
A_{[n-3]} &:= \ast_1 U_{[3]} \\
C_{[n-3,1]} &:= \ast_1 \hat{W}_{[3,1]} 
\end{align*} \]

\[ \tilde{\mathcal{L}}_{PM}(C, A) = -\frac{1}{2(n-3)!\lambda^2} \left[ \mathcal{L}(C_{[n-3]}) - \frac{2\sigma\lambda}{(n-2)\tilde{m}} \mathcal{L}(A, \nabla A) + \frac{4\sigma\lambda^2}{\tilde{m}} \tilde{\mathcal{L}}^{\text{cross}} \right] \]

where

\[ \mathcal{L}(A, \nabla A) = (\nabla_a A_{b[n-3]})^2 - (n-3)(\nabla A)^2 + 3\sigma\lambda^2 A^2 \]

\[ \tilde{\mathcal{L}}^{\text{cross}} = A_{a[n-3]} \left( \nabla_b C_{a[n-3]} \right) + (-)^{n-1} (n-3) \nabla_a C_{a[n-3]} \]

**Gauge transform**

\[ \delta C_{[n-3,1]} = \tilde{\nabla} + \tilde{\nabla} - \sigma \frac{\lambda}{\tilde{m}} \]

\[ \delta A_{[n-3]} = \tilde{\epsilon} - \tilde{\epsilon} \]

**Smooth flat limit**

\[ \lambda^2 \tilde{\mathcal{L}}_{PM} \rightarrow \mathcal{L}^{\text{Curt}}(\partial C_{[n-3,1]} - \sigma \mathcal{L}(A_{[n-3]})) \quad (\text{unitary in } dS_n) \]

**Remark:** Unitarity at classical level from \( \sigma \) in helicity 1: consistent with rules in BMV (\( \sigma = 1 \)) and [Th. Basile, X. Bekaert, N.B. 2017] \( \sigma = -1 \).
4. Twisted (self-)duality in (A)dS$_n$: massless and PM cases

4.1) Massless case

In the standard formulation from $L^\text{FP}_\lambda (\nabla h, h)$:

$$K^{abcd} := -\frac{1}{2} \left( \nabla^a \nabla^{[c} [d] b - \nabla^b \nabla^{[c} [d] c + \nabla^c \nabla^{[a} [b] d - \nabla^d \nabla^{[a} [b] c) + \sigma \chi^2 \left( \frac{\delta^{c [a} \delta^{d b]} - \delta^{b [a} \delta^{c d]} \right) \right)$$

primary gauge-invariant quantity s.t. \( \text{Tr} K = 0 \) \( (\text{BI}) \), \( \nabla^a K^{bc} \mid_{a} = 0 \) \( (\text{BII}) \)

\( \text{Tr} K = 0 \) \( (\text{EI}) \), \( \nabla^a K^{abcd} \mid_{a} = 0 \) \( (\text{EII}) \)

In the dual formulation $L_0 (\hat{\gamma}^{*}, \hat{W}^{*}) \sim \nabla \hat{w} \nabla \hat{w} + \lambda \hat{\gamma} \nabla \hat{w} + \nabla \hat{y} \nabla \hat{y} + \chi^2 \hat{\gamma}^2$

$$R^{abcd} := \lambda \nabla_a \left( \nabla \hat{w}^{cd} \nabla \right)_{b} + \lambda \hat{\gamma}^{cd} \right) \Rightarrow \text{Tr} R$$

$$K^{abcd} := \lambda \nabla_a \left( \nabla \hat{w}^{cd} \right)_{b} + 2 \sigma (n-2) \lambda \left( \nabla \hat{w}^{cd} \right)_{[a} \right)_{b] + \lambda \hat{\gamma}^{cd} \right) \Rightarrow K_a := K_a \right)$$

s.t. \( \{\cdot \} \nabla^{abcd} := R^{abcd} \) - Traces is gauge-invariant

\( \{\cdot \} \nabla^{abc} := K^{abc} + \frac{2}{n-1} \delta^{c [a} K_{b]} \) is gauge-invariant
Recall Hodge–dual $\hat{w} \cdots = *_{n-2} c_{\langle n-3, n-1 \rangle}$, $\hat{y} \cdots = *_{n-1} T_{\langle n-2, n \rangle}$ and define $K^{c}_{\langle n-1, 2 \rangle} := *_{2} \nabla \cdots (\hat{w}, \hat{y})$, $K^{T}_{\langle n-1, 2 \rangle} := *_{2} \nabla \cdots (\hat{w}, \hat{y})$

where $\hat{w}$ and $\hat{y}$ are expressed in terms of $c$ and $T$.

\[ K^{c}_{\langle n-2, 2 \rangle} = C + \lambda, \quad \text{s.t.} \quad \text{Tr}_{*_{1}} K^{c} = 0 \quad (\tilde{\mathcal{B}}_{4}) \]

\[ K^{T}_{\langle n-1, 2 \rangle} = T + \sigma \lambda + \lambda^{2}, \quad \text{Tr}_{*_{1}} K^{T} = 0 \quad (\tilde{\mathcal{B}}_{2}) \]

\[ \nabla^{(1)} K^{c}_{\langle n-2, 2 \rangle} = \lambda K^{T}_{\langle n-1, 2 \rangle} \quad (\tilde{\mathcal{B}}_{2}), \quad \nabla^{(2)} K^{c}_{\langle n-2, 2 \rangle} = \lambda \sigma^{2} \nabla^{(1)} (K^{T}_{\langle n-1, 2 \rangle}) \quad (\tilde{\mathcal{B}}_{2}) \]

From the action, finds $\text{Tr} K^{c} \simeq 0 \quad (\tilde{\mathcal{E}}_{4})$ and $\text{Tr} K^{T}_{\langle n-1, 2 \rangle} \simeq 0 \quad (\tilde{\mathcal{E}}_{2})$

implying that $K^{T} \simeq 0$, whence $\nabla^{(1)} K^{c} \simeq 0$ & $\nabla^{(2)} K^{c} \simeq 0 \quad (\tilde{\mathcal{E}}_{2})$
There is twisted duality in $\text{AdS}_n$:

$$K^c_{[n-2,2]} \simeq *_1 K_{[2,2]}$$

as in flat spacetime, relates

$$
\begin{pmatrix}
B I \\
B II
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
\tilde{E} I \\
\tilde{E} II
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
E I \\
E II
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
\tilde{B} I \\
\tilde{B} II
\end{pmatrix}
$$

In the flat limit $\Lambda \to 0$, reproduces the twisted duality of Hull.

4.2) Partially-massless case

- Standard (electric) Stückelberg formulation with $L(h_{ab}, A_a)$ invariant under

$$\delta h_{ab} = 2 \nabla_{(a} \tilde{y}_{b)} + \frac{2}{m-2} \gamma_{ab} \epsilon,$$
$$\delta A_a = \nabla_a \epsilon + 2 \sigma \tilde{m} \tilde{\delta}_a$$

- $H_{ab} := h_{ab} - \frac{1}{m} \nabla_{(a} A_{b)}$ invariant under $\tilde{\delta}$, not under $\delta$

$$\Rightarrow \quad \boxed{K_{abc} := -4 \sigma \tilde{m} \nabla_{[a} H_{bc]} \epsilon}$$

- $Q^{abcd} := -\frac{1}{2} (\nabla^a \nabla^{[c} H^{d]b} - \ldots - \nabla^d \nabla^{[a} H_{bc]} \epsilon)$

s.t.

$$\frac{\delta L_{PM}}{\delta h_{ab}} = -2(Q_{a,b} - \frac{1}{2} \tilde{g}_{ab} \tilde{Q}^\epsilon)$$

and

$$\frac{\delta L_{PM}}{\delta A_a} = -\frac{2}{m} \nabla^b G^m_{ab} \equiv \sigma K_a \epsilon.$$

$$\text{Tr} \; K_{abc} \simeq 0 \simeq \text{Tr} \; Q$$
One also derives that

\[ \nabla[a Q^{bc}]_{mn} = -\frac{\tilde{m}}{n-2} \delta^{[a}_{[m} K^{bc]}_{n]} \] (BIII)

- In the dual formulation for PM spin-\( n-2 \), \( L(\hat{\mathcal{W}}^{\cdots}, U^{\cdots}) \) Stückelberg

  \[ R_{ab1}^{\cdots} := 2 \nabla_{a} \left( \nabla_{b} \hat{W}_{cd}^{\cdots} - \frac{\sigma}{\tilde{m}} U_{b}^{cd} \right) \]

  \[ \nabla_{ab1}^{\cdots} = R_{ab1}^{\cdots} - \text{Trace} \rightarrow \text{gauge invariant} \]

- Define dual curvatures

  \[ K_{[n-2, z]}^{c} := *_{z} K_{[n-2, z]} \] (Tr \( K^{u} = 0 \))

- \( \tilde{K}_{a[n-2]}^{b} := \frac{(-)^{n}}{2} \epsilon_{a[\cdots]cde} \left( \delta_{e}^{a} K_{c[1]}^{u} - \frac{n-2}{2} \delta_{b}^{a} K_{c[1]}^{u} \right) \)

  i.e. \( K_{a[\cdots]}^{b} = (-)^{n} \frac{2}{(n-2)!} \epsilon_{d[\cdots][a} \tilde{K} \epsilon_{b]} \)

  i.e. \( \tilde{K}_{[n-2, 1]} \sim \quad \)

  \[ \nabla_{a} \nabla_{b} + \tilde{m} \quad - \frac{\sigma}{\tilde{m}} \]

  \[ \quad \] and \( K_{[n-2, z]}^{c} \left( \nabla \nabla c, \tilde{\bar{g}}, \nabla A^{[n-3]} \right) \)

  similar to the massless case

\[ \text{Tr} \ast_{1} K_{[n-2, z]}^{c} \equiv 0 \equiv \text{Tr} \ast_{1} \tilde{K} \] (BII)

16
\[ \text{Tr } \tilde{K}_{[n-2,1]} \cong 0 \quad \text{Tr } K^C_{[n-2,2]} \cong 0 \quad (\tilde{E}I) \]

\[ \nabla^{(1)} K^C_{[n-2,2]} \equiv \frac{1}{\tilde{m}} T^{12} (\tilde{K}_{[n-2,1]}) \quad (\tilde{B}^n_1) \]

\[ \nabla^{(2)} K^C_{[n-2,2]} \equiv \frac{1}{\tilde{m}} T^{12} \sigma^2 (\tilde{K}_{[n-2,1]}) \quad (\tilde{B}^n_2) \]

\[ \nabla^{(2)} K^C_{[n-2,2]} \cong \sigma \frac{1}{\tilde{m}} \tilde{K}_{[n-2,1]} \quad (\tilde{E}II) \quad \text{Tr}_1 K^C_{[n-2,2]} \cong \frac{1}{\tilde{m}} \sigma^2 \tilde{K}_{[n-2,1]} \]

**PM Twisted-duality:**

\[ K^C_{[n-2,2]} \cong *_1 Q_{[2,2]} \quad (TD_1)_\lambda \]

However, \((TD_1)_\lambda\) not enough for smoothness of flat limit!

\[ \rightarrow \text{act } \left(*_1 \nabla^{(1)} \right)[(TD_1)_{[n-2,2]}], \text{ use } (\tilde{B}^n_2), \text{ Tr}_1 (\tilde{B}^n), (EII) \& (\tilde{E}I) \text{ to get} \]

\[ \tilde{K}_{[n-2,1]} \cong (-)^{n-1} \frac{\sigma \tilde{m}^2}{2 \lambda^2} *_1 K_{[2,1]} \quad (TD_2)_\lambda \quad \text{so that} \]

\[ (TD_1)_\lambda \iff (TD_2)_\lambda \lambda \neq 0 \]

When \(\lambda \rightarrow 0\): \((TD_1)_\lambda \rightarrow \text{Hull's spin-2 TD} \]

\[ (TD_2)_\lambda \rightarrow d^{(2)}(\tilde{F}_{[n-2]} = *_1 F_{[2]}) \Rightarrow \tilde{F}_{[n-2]} \cong *_1 F_{[2]} \]
In flat limit, \( (TD_1)_\lambda \rightarrow \) Pair of usual twisted duality relations for spin 2 (h_{ab} \sim C_{[n-3]}^{[n]}) and spin 1 (A_a \sim A_{[n-3]})

Considering \( (TD_2)_\lambda \) and gauge-fixing \( A_{[n]} \equiv 0 \equiv A_{[n-3]} \) which is allowed for \( \lambda \neq 0 \), gets

\[
(TD_2)_\lambda \rightarrow (n-2) \nabla^a C_a^{[n-3]} b \cdot \left( - \frac{\lambda}{2} \right)^{n-2} c_{[n-2]cd} h_{d} b
\]

while \( (TD_1)_\lambda \rightarrow \nabla^2 (TD_2)^*_\lambda \) and

In the \( n=4 \) case, \( (TD_2)^*_\lambda \) reproduces Hinterbichler's duality relation.

Warning: once the St"uckelberg fields \( A_{[n]} \) and \( A_{[n-3]} \) are fixed to zero \( \equiv 0 \), the flat limit is no longer smooth for the counting of d.o.f.

Instead

\[
\begin{align*}
(TD_1)_\lambda & \rightarrow (K_{[n-3]} \approx \ast_4 K_{[2,2]} ) \\
(TD_2)_\lambda & \rightarrow (F_{[n-3]} \approx \ast_4 F_{[2]} )
\end{align*}
\]

is smooth.
5. A theory for multiple PM spin-2 fields


⇒ no consistent 2-derivative (cubic) vertex for a single PM field.

As for gauge algebra, for a set of PM spin-2 [S. Garcia-Saenz, K. Hinterbichler, A. Joyce, E. McKeon & R.A. Rosen 2015]

no non-abelian deformation
to first order in fields, with assumptions on # derivatives.
Revisiting these analyses in the BV BRS cohomological formulation

Start from

\[ S_0 [h_{\mu \nu}] = - \frac{1}{4} \int d^m x \, \varepsilon^b \, h^{ab} \left[ \mathcal{F}^a_{\mu \nu} \mathcal{F}^b_{\mu \nu} - 2 \mathcal{F}^a_{\mu} \mathcal{F}^b_{\mu} \right] \]

\[ \mathcal{F}^a_{\mu \nu} := 2 \, \nabla_{[\mu} h_{\nu]}^a \]

\[ \delta^a_e S_0 = 0 \quad \text{under} \quad \delta^a_e h_{\mu \nu} = \nabla_\mu \varepsilon^a - \frac{\pi}{\lambda^2} g_{\mu \nu} \varepsilon^a \]

1) Most general deformation of gauge algebra:

\[ [\delta_e, \delta_e] h^a_{\mu \nu} = \delta^a \delta h^a_{\mu \nu} \quad \text{where} \]

\[ \chi = \kappa \left( m^a_{\, b c} \, \varepsilon^b_{\, e} \, \varepsilon^c_{\, f} + n^a_{\, b c} \, \nabla^b \varepsilon^c_{\, e} \, \nabla_\mu \varepsilon^c_{\, f} \right) \rightarrow \text{no field dependence} \]

\[ \rightarrow \text{Consistency requires} \quad m^a_{\, b c} = 0 = n^a_{\, b c} \quad \rightarrow \text{Abelian} \]

\[ \rightarrow \text{no higher-order corrections} \]
2) Deformation of gauge symmetry, if \( \Theta \) 's:

Consistency gives only (out of 6 candidates)

\[
\delta_{\epsilon} h_{\mu\nu} = \alpha \, f^{\epsilon}_{\delta \gamma} \, F_{\epsilon \mu \nu} \quad \forall \epsilon \in \mathbb{C}, \quad \text{only in } n = 4.
\]

3) Cubic vertex with \( \Theta \) 's:

\[
S_1 = \int d^4 x \, \sqrt{-\mathcal{g}} \, h_{\mu\nu} \, J^\mu_{\nu}
\]

\[
J^\mu_{\nu} = f_{\delta c \beta} \, \left[ F^\mu_{\epsilon \sigma} \, F^\nu_{\epsilon \sigma} - \frac{1}{4} \, g_{\mu\nu} \, F^\delta_{\epsilon \sigma} \, F^\epsilon_{\delta \sigma} \right] + \text{improvements}
\]

\( \Rightarrow \) # independent deformation at order \( \alpha \); \( \frac{1}{2} N^2 \) \( (N^2) \)

\[
J_{\delta c \beta} \sim \mathbb{A} \mathbb{B} \mathbb{C}
\]

\( \Rightarrow \) Uniqueness result (since existence not new)
Conservation:

Obviously \( \nabla_\mu J^\mu_a = \frac{e}{\ell^2} g_{\mu\nu} J^\nu_a \approx 0 \)

but also, \( n = 4 \),

\( \nabla_\mu J^\mu_a \approx 0 \implies J^\mu_{ab} = \sqrt{-g} J^\mu_a \nabla_b \epsilon_b \)

Neither current rigid symmetry

\( \delta h^a_{\mu\nu} = F^a_{\mu\nu} \nabla^\mu \epsilon^\nu \)

\( \therefore \) Killing

4) Higher-order consistency:

Provided

\( f_{ae,b} f^e_{c,d} = 0 \) \hspace{1cm} (1)

\( f_{ab,e} f^e_{c,d} = 0 \) \hspace{1cm} (2)

Fully consistent to all orders (!)

But \( (1) \& (2) \) non-trivial solution only if \( k_{ab} \neq 0 \)

i.e. “wrong” relative signs.
THANKS!