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Exponentiations over the universal enveloping algebra of $sl_2(\mathbb{C})$

Sonia L'Innocente [a], Angus Macintyre [b], Françoise Point [c,*,1]

[a] Department of Mathematics and Computer Science, University of Camerino, Via Madonna delle Carceri 9, 62032 Camerino (MC), Italy
[b] School of Mathematics, Queen Mary University of London, Mile End Road London E1 4NS, England, UK
[c] Institut de mathématique, Université de Mons, Le Pentagone, 20, place du Parc, B-7000 Mons, Belgium

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ABSTRACT

We construct, by model-theoretic methods, several exponentiations on the universal enveloping algebra U of the Lie algebra $sl_2(\mathbb{C})$.

1. Introduction

Consider the Lie algebra $M_2(\mathbb{C})$ and the Lie (sub)algebra $sl_2(\mathbb{C})$ of all 2×2 trace zero matrices with complex entries. Recall that a standard basis of $sl_2(\mathbb{C})$ (as $\mathbb{C}$-vectorspace) is given by: $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathrm{diag}(1, -1)$. The generators x, y, h satisfy the relations: $[h, x] = 2x$, $[h, y] = -2y$, $[x, y] = h$, where $[u, v]$ is the usual commutator of u and v .

For each positive integer λ, we consider the finite-dimensional simple $sl_2(\mathbb{C})$-module V_λ of dimension $\lambda + 1$ and the (matrix) Lie algebra $M_{\lambda+1}(\mathbb{C})$ (the endomorphism ring of V_λ, viewed as a $\mathbb{C}$-vectorspace) and take the exponential maps from $M_{\lambda+1}(\mathbb{C})$ into the linear group $GL_{\lambda+1}(\mathbb{C})$. (In Section 5, we recall some properties of these exponential maps.)

We connect these exponential maps to the universal enveloping algebra U of $sl_2(\mathbb{C})$ (whose definition and algebraic properties are described in Section 6). We will use some basic facts on the representation theory of this associative algebra (and its analogue over any algebraically closed field of characteristic 0). It has been studied from a model theoretical point of view by [7] and then by [8,13,14].

Using on the one hand the *concrete* exponential maps defined on the matrix rings $M_{\lambda+1}(\mathbb{C})$ and on the other hand the universal property of U, we define a sequence of *exponential maps* indexed by λ from U to $GL_{\lambda+1}(\mathbb{C})$. We describe some of the properties of these maps, which we have formalized (in Section 4) by defining the notion of a non-commutative exponential

* Corresponding author.
E-mail addresses: sonia.linnocente@unicam.it (S. L'Innocente), angus@dcs.qmul.ac.uk (A. Macintyre), point@logique.jussieu.fr (F. Point).

1 Research Director at the "Fonds de la Recherche Scientifique-F.N.R.S".

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ring (generalizing the commutative case) and we explicitly calculate elements lying in their kernels (respectively images). Then, for $\mathcal{V}$ any non-principal ultrafilter on ω, we show that U embeds into the non-principal ultraproduct $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$ and we define an exponential map EXP from U to the non-principal ultraproduct $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$ of the groups $GL_{\lambda+1}(\mathbb{C})$. We show that (U, EXP) is a non-commutative exponential ring, and we explicitly calculate a part of the kernel of EXP. Note that a *formal* exponential map exp was previously defined in the completion $\hat{U}$ of U [17], on the ideal on $\hat{U}$ generated by the generators of U; in Section 9, we will compare the two approaches.

We go on to endow U with a topology using a *norm* in $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$ which takes its values in a non-standard ultrapower of $\mathbb{R}$, and we show that the exponential map EXP is continuous and that the subgroup generated by $\mathrm{EXP}(U)$ is a topological group.

Finally, by considering another norm on each $M_{\lambda+1}(\mathbb{C})$, and the asymptotic cone relative to that norm and a non-principal ultrafilter $\mathcal{V}$ on ω, we embed U in a complete metric space and show that U has a faithful continuous action on that space.

2. Preliminaries on formalism

Let us set up the languages we need.

Let $\mathcal{L}_g := \{\cdot, 1\}$ be the language of groups. Let $\mathcal{L} := \{+, -, \cdot, 0, 1\}$ be the language of (associative) rings (with 1), and let $\mathcal{L}_{\mathfrak{l}} := \{+, -, [\cdot, \cdot], 0\}$ be the language of Lie rings. For a ring R, let $\mathcal{L}_{m,R} := \{+, -, 0, \cdot r; r \in R\}$ be the language of right R-modules.

For the language of R-algebras, where R is a commutative ring, we will choose the expansion $\mathcal{L}_{Alg}$ of $\mathcal{L}$, a two-sorted language with a sort for a ring R, a sort for an algebra A (associative or not) and a scalar multiplication map from $A \times R$ to A, where A is either a $\mathcal{L}$-structure or a $\mathcal{L}_l$-structure and R is a $\mathcal{L}$-structure.

For the language of Lie K-algebras, where K is a field, we will choose either $\mathcal{L}_{Lie} := \mathcal{L}_l \cup \mathcal{L}_{m,K}$ or the two-sorted language $\mathcal{L}_{Alg}$. Note that for the former we omit reference to K when it is understood. We will assume that K is a field of characteristic 0 and is complete with respect to a nontrivial absolute value. Let T_K be the theory of K-vector spaces in $\mathcal{L}_{m,K}$.

Let $T_{\boldsymbol{L}}$ be the theory of Lie K-algebras L in the language $\mathcal{L}_{Lie}$, namely

(1) T_K,
(2) $[\cdot, \cdot]$ is a K-bilinear map from $L \times L$ to L,
(3) $\forall x_1\, [x_1, x_1] = 0$,
(4) $\forall x_1\, \forall x_2\, \forall x_3\, ([[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] = 0)$.

3. Axioms for semi-simple Lie algebras

We will translate, into model-theoretic terms, the basic results on existence and uniqueness of a semi-simple Lie K-algebra with a given reduced abstract root system Φ ([10], chapter 18.2). This is not essential for the present paper, but may be of interest in future generalizations.

Recall that Φ is a subset of an Euclidean space E endowed with a positive definite symmetric bilinear form $(., .)$. Denote by $< \beta, \alpha >:= 2.\frac{(\beta,\alpha)}{(\alpha,\alpha)}$. For a root system Φ these values are integers.

For $x \in L$, let $ad\ x$ be the linear transformation of L sending $y \in L$ to $[x, y]$.

Proposition 3.1. *The theory of any semi-simple Lie algebra L with given reduced root system Φ (and inner product on it) is axiomatisable in $\mathcal{L}_{Alg}$ by the set T_Φ of axioms below. Moreover each T_Φ is $\aleph_1$-categorical.*

(1) *T_{Alg} the theory of K-algebras in $\mathcal{L}_{Alg}$ over some field K;*
(2) *The scheme of axioms expressing that K is an algebraically closed field of characteristic 0;*
(3) *(the α_j are the elements of the root system) $\exists h_1 \cdots \exists h_\ell\, \exists e_1 \cdots \exists e_\ell \exists e_{-1} \cdots \exists e_{-\ell}$*

$$\Big[\bigwedge_{1\le i,j\le\ell} [h_i, h_j] = 0$$
$$\bigwedge_{1\le i\le\ell} [e_i, e_{-i}] = h_i \ \&\ \bigwedge_{1\le i\ne j\le\ell} [e_i, e_j] = 0$$
$$\bigwedge_{1\le i,j\le\ell} [h_i, e_j] =< \alpha_j, \alpha_i > \cdot e_j \ \&\ \bigwedge_{1\le i,j\le\ell} [h_i, e_{-j}] = - < \alpha_j, \alpha_i > \cdot e_{-j}$$
$$\bigwedge_{1\le i\ne j\le\ell} (ad\ e_i)^{-<\alpha_j,\alpha_i>+1}(e_j) = 0$$
$$\bigwedge_{1\le i\ne j\le\ell} (ad\ e_{-i})^{-<\alpha_j,\alpha_i>+1}(e_{-j}) = 0$$
$$\&\ \forall x\, \exists k_1 \in K \cdots \exists k_{3\ell} \in K\ x = \sum_{1\le i\le\ell} k_i.h_i + \sum_{1\le i\le\ell} k_{\ell+j}.e_j + \sum_{1\le j\le\ell} k_{2\ell+j}.e_{-j} \Big].$$

Proof. Serre's work tells us that given a root system Φ and a field of characteristic 0, there exists a unique Lie algebra L that can be presented by these relations and that it is semi-simple. The second statement follows from the fact that if L is a model of these axioms of cardinality $\aleph_1$, then it is a Lie algebra over an algebraically closed field F of characteristic 0 of cardinality $\aleph_1$. $\square$

Question 3.1. Is the theory of any semi-simple Lie $\mathbb{C}$-algebra L with given root system Φ finitely axiomatisable in $\mathcal{L}_{Lie}$ modulo $T_{\mathbb{C}}$?

Let Axiom (3') be got from Axiom (3) by deleting the last part where we quantify over x. Let L be a model of (1), (2) and (3'), and let L_0 be the Lie subalgebra generated by the elements h_i, e_j, e_{-j} satisfying the above relations. Then, Serre's theorem tells us that L_0 is a semi-simple finite-dimensional Lie algebra with root system Φ and Cartan subalgebra generated by $h_i,\ 1 \le i \le \ell$. Then can we add an axiom that forces L to be equal to L_0?

We could have also worked in the language $\mathcal{L}_l$ since any semi-simple Lie algebra has a basis with integral structure constants (a Chevalley basis). We do not pursue this matter here.

4. Exponential rings

Let $\mathcal{L}_E := \mathcal{L} \cup \{E\}$ (respectively $\mathcal{L}_{Alg,E} := \mathcal{L}_{Alg} \cup \{E\}$) where E is a unary function symbol. We will introduce the notion of (non-commutative) exponential ring generalizing the commutative case (see for instance [18]).

Definition 4.1. Let (R, E, G) be a two-sorted structure with R an $\mathcal{L}$-structure, G a $\mathcal{L}_g$-structure and E a map from R to G. We will say that (R, E, G) is an exponential ring if R is an associative ring with 1, G a (multiplicative) group and if $E : R \to G$ satisfies the following axioms:

(1) $E(0) = 1$,
(2) $\forall x\ \ E(x).\,E(-x) = 1$,
(3) $\forall x\, \forall y\ \ (x.\,y = y.\,x \ \to\ E(x+y) = E(x).\,E(y))$.

If in addition R is a K-algebra, then (R, K, E, G) is an exponential K-algebra if (R, K) is a $\mathcal{L}_{Alg}$-structure such that the reduct (R, E, G) is an exponential ring, the $\mathcal{L}_{Alg}$-reduct (R, K) a K-algebra and

$$\forall k_1,\ k_2 \in K\ \ \forall x \in R\ \ E(k_1.\,x).E(k_2.\,x) = E((k_1 + k_2).\,x).$$

Note that this last axiom together with (1) implies (2) above.

One recovers the classical case by taking G the group of units of R, by assuming that R is a commutative ring and then we revert to the one-sorted $\mathcal{L}_E$-structure (R, E). In the case we deal with an exponential K-algebra, we will get that (K, E) is an exponential field.

5. A natural exponential map over $M_{\lambda+1}(\mathbb{C})$

Consider the field $\mathbb{C}$ of complex numbers and, for a fixed natural number λ, the associative $\mathbb{C}$-algebra $M_{\lambda+1}(\mathbb{C})$ of all $(\lambda+1) \times (\lambda+1)$ matrices with coefficients in $\mathbb{C}$ (with the matrix multiplication $\cdot$ as the underlying operation). It is also a Lie $\mathbb{C}$-algebra with the bracket $[A, B] := A \cdot B - B \cdot A$ (see [5,10]). For $A \in M_{\lambda+1}(\mathbb{C})$, denote by A^* the conjugate of the transpose of A, by $tr(A)$ the trace of A, and finally by $\det(A)$ its determinant.

We will denote by $Diag_{\lambda+1}(\mathbb{C})$ (respectively $UT_{\lambda+1}(\mathbb{C})$) the subset of all diagonal matrices (respectively upper triangular matrices) in $M_{\lambda+1}(\mathbb{C})$.

Recall that on the Lie algebra $M_{\lambda+1}(\mathbb{C})$, we have a Hermitian sesquilinear form $(\cdot,\cdot)_{\lambda+1}$ defined by $(A, B)_{\lambda+1} := \mathrm{tr}(B^* \cdot A) = \sum_{i,j} A_{ij} \cdot \bar{B}_{ij}$, where $A,\ B \in M_{\lambda+1}(\mathbb{C})$, its values are in $\mathbb{C}$ ([16] page 9). The Frobenius norm (denoted by F-norm) associated with it, is defined as follows: $\|A\|^2_{F,\lambda+1} := (A, A)_{\lambda+1}$. We use this norm systematically later. In addition to the triangle inequality and submultiplicativity (from which multiplication is continuous for the norm topology) the F-norm satisfies the Cauchy–Schwarz inequality ([16] page 10). Note that for diagonalizable matrices, the F-norm is the square root of the sum of the squares of the norms of the eigenvalues of the matrix.

There are many norms on $\mathbb{C}^{\lambda+1}$, all giving the same topology. For example, on $\mathbb{C}$ we have the usual norm $|\cdot|$, inducing on $\mathbb{C}^{\lambda+1}$ the norm ("the 2-norm") whose value is the square root of the sum of the squares of the absolute values of the entries. This norm, and the Frobenius norm, are both instances of Schatten 2-norms. When we refer later to norms, it will be to such norms, unless we explicitly deal with operator norms. We consider the elements of $M_{\lambda+1}(\mathbb{C})$ as linear operators ϕ from $(\mathbb{C}^{\lambda+1}, \|\cdot\|_1)$ to $(\mathbb{C}^{\lambda+1}, \|\cdot\|_2)$. Then, for any ordered pair of norms on $\mathbb{C}^{\lambda+1}$ there is a corresponding operator norm on $M_{\lambda+1}(\mathbb{C})$. Later, when we consider ultraproducts of the $M_{\lambda+1}(\mathbb{C})$ we will return to discussion of such norms. We will use operator norms only with reference to Schatten 2-norms.

From now on, we will assume that $M_{\lambda+1}(\mathbb{C})$ is equipped with a fixed norm $\|\cdot\|$ satisfying the Cauchy–Schwarz inequality. The topology on $M_{\lambda+1}(\mathbb{C})$ is independent of the norm, but in discussing convergence of series we will appeal to the fixed norm.

If A is any matrix in $M_{\lambda+1}(\mathbb{C})$, one defines ([16] 1.1) the *matrix exponential* of A, denoted by $\exp(A)$, as the power series:

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I_{\lambda+1} + A + \frac{A^2}{2} + \frac{A^3}{3!} + \cdots \tag{1}$$

where $I_{\lambda+1}$ denotes the $(\lambda+1)\times(\lambda+1)$ identity matrix. This exponential series converges in norm for all matrices, so the exponential of A is well defined. If A is a 1×1 matrix, that is, a scalar a of the field $\mathbb{C}$, then $\exp(A) = \mathrm{e}^a$ where e^a denotes the ordinary exponential of the element $a \in \mathbb{C}$.

Recall that the matrix exponential satisfies the following properties:

Proposition 5.1. *Let $A, B \in M_{\lambda+1}(\mathbb{C})$ and $a, b \in \mathbb{C}$ we have:*

(i) $\exp(0_\lambda) = I_{\lambda+1}$, *where $0_{\lambda+1}$ denotes the zero matrix in $M_{\lambda+1}(\mathbb{C})$;*
(ii) $\exp(aA)\cdot\exp(bA) = \exp((a+b)A)$;
(iii) $\exp(A)\cdot\exp(-A) = I_{\lambda+1}$;
(iv) *for A and B commuting,* $\exp(A+B) = \exp(A)\cdot\exp(B)$;
(v) *for an invertible matrix B,* $\exp(BAB^{-1}) = B\,\exp(A)B^{-1}$;
(vi) $\det(\exp(A)) = \exp(tr(A))$.
(vii) *If no two eigenvalues of A have a difference belonging to $2\pi.i.\mathbb{Z}$, then there exists a neighbourhood of A on which* exp *is injective. The exponential is injective on the ball of radius* $\log(2)$ *around the origin, for the Frobenius norm.*

Proof. See [16] Proposition 1 (b), (c), (d), Proposition 3 in section 1.1 and Proposition 7 in section 1.2. See also [1, Chapter 3]. It remains to prove (vi).

By (v), in order to prove (vi), one may assume without loss of generality that A is in Jordan normal form. So A can be written as a sum of a diagonal matrix D and a nilpotent matrix N with N and D commuting. So, by (iv), $\exp(D+N) = \exp(D)\cdot\exp(N)$. Therefore, $\det(\exp(A)) = \det(\exp(D))\cdot\det(\exp(N)) = \det(\exp(D))$. □

For non-commuting matrices A and B, the equality $\exp(A+B) = \exp(A)\cdot\exp(B)$ need not to hold. In that case, the Baker–Campbell–Hausdorff formula can be used to express $\exp(A)\cdot\exp(B)$ (see [16, Section 1.3]).

Now, using the matrix exponential, one defines the *exponential map*

$$\exp : M_{\lambda+1}(\mathbb{C}) \to GL_{\lambda+1}(\mathbb{C}) : A \mapsto \exp(A),$$

and rephrasing, (part of) Proposition 5.1, we get that $(M_{\lambda+1}(\mathbb{C}), \exp, GL_{\lambda+1}(\mathbb{C}))$ is an exponential $\mathbb{C}$-algebra. Moreover, the map exp is surjective from $M_{\lambda+1}(\mathbb{C})$ to $GL_{\lambda+1}(\mathbb{C})$. (Every invertible matrix can be written as the exponential of some other matrix ([16] page 21).)

For future use, we recall some methods for explicitly calculating matrix exponentials.

Diagonalizable case. If a matrix $A \in M_{\lambda+1}(\mathbb{C})$ is diagonal $A = \mathrm{diag}(a_1, a_2, \ldots, a_{\lambda+1})$, then its exponential can be obtained by just exponentiating every entry on the diagonal: $\exp(A) = \mathrm{diag}(\mathrm{e}^{a_1}, \mathrm{e}^{a_2}, \ldots, \mathrm{e}^{a_{\lambda+1}})$.

This also allows one to exponentiate any diagonalizable (so-called *semi-simple*) matrix $S \in M_{\lambda+1}(\mathbb{C})$. If $S = BDB^{-1}$ where B is invertible and D is diagonal, then, according to the property (v) in Proposition 5.1, we have that $\exp(S) = B\exp(D)B^{-1}$ and the exponential of the matrix D is calculated as above.

Nilpotent case. Recall that a matrix $N \in M_{\lambda+1}(\mathbb{C})$ is *nilpotent* if $N^q = 0$ for some positive integer q (without loss of generality $\leq \lambda+1$).

In this case, the matrix exponential $\exp(N)$ can be computed directly from the series expansion (expressed by (1)), as the series terminates after a finite number of terms: $\exp(N) = I_{\lambda+1} + N + \frac{N^2}{2} + \cdots + \frac{N^{q-1}}{(q-1)!}$.

General Case. Since any matrix $A \in M_{\lambda+1}(\mathbb{C})$ can be expressed uniquely as a sum $A = S + N$ where S is diagonalizable, N is nilpotent and $S\cdot N = N\cdot S$, then the exponential of A can be computed by using the property (iv) of Proposition 5.1 and by reducing to the previous two cases, so:

$$\exp(A) = \exp(S+N) = \exp(S)\cdot\exp(N).$$

Note that this uniqueness easily translates, via quantifier elimination for algebraically closed fields, into a constructible version in the sense of algebraic geometry.

We will need a more thorough description of Ker(exp). It is easy to see that the map exp is not injective. For instance, consider a non-zero diagonal matrix $I_{\lambda+1} \neq D \in M_{\lambda+1}(\mathbb{C})$, $D = \mathrm{diag}(d_1, d_2, \ldots, d_{\lambda+1})$, with its matrix exponential, $\exp(D) = \mathrm{diag}(\mathrm{e}^{d_1}, \mathrm{e}^{d_2}, \ldots, \mathrm{e}^{d_{\lambda+1}})$. Then, $\exp(D) \in \mathrm{Ker}(\exp)$ if and only if the entries of D belong to the kernel of the standard complex exponential map , so if and only if $d_1, d_2, \ldots, d_{\lambda+1} \in 2\pi i\cdot\mathbb{Z}$.

Lemma 5.2. *If the matrix $A \in M_{\lambda+1}(\mathbb{C})$ belongs to the kernel of* exp*, then A is diagonalizable and its eigenvalues lie in the kernel of the exponential function e in $\mathbb{C}$.*

Proof. In order to determine whether $A \in \mathrm{Ker}(\exp)$, by Proposition 5.1(v), since $\exp(B^{-1}AB) = B^{-1}\exp(A)B$, we have that $\exp(A) = I_{\lambda+1}$ if and only if $\exp(B^{-1}AB) = I_{\lambda+1}$, for any invertible matrix $B \in M_{\lambda+1}(\mathbb{C})$. Since $\mathbb{C}$ is algebraically closed, A is conjugated to a matrix in Jordan normal form, which can be written as a sum of a diagonal matrix D and a nilpotent matrix N with N and D commuting. So, by Proposition 5.1(iv), $\exp(D+N) = \exp(D)\cdot\exp(N)$. Now $\exp(N)$ is unipotent. So if $\exp(D+N) = I_{\lambda+1}$, then $\exp(D) = \exp(-N)$, so the diagonal matrix $\exp(D) = I_{\lambda+1}$. Thus the eigenvalues of D are periods of the exponential on the complex numbers. Also, $\exp(N)$ must be $I_{\lambda+1}$. Finally, a simple calculation with the polynomial $\exp(N)$ (in N) gives $N = 0_{\lambda+1}$. We conclude that the kernel of exp consists of the diagonalizable matrices with complex periods as eigenvalues. □

Proposition 5.3. *Each associative Lie algebra* $(M_{\lambda+1}(\mathbb{C}), \exp)$ *viewed as an* $\mathcal{L}_E$*-structure is bi-interpretable with the exponential field* $(\mathbb{C}, \mathrm{e}^x)$.

Proof (*See [15]*)**.** We embed $M_{\lambda+1}(\mathbb{C})$ in the direct product $\mathbb{C}^{(\lambda+1)^2}$. Then since $\mathbb{C}$ is algebraically closed, any matrix is conjugate to a matrix in Jordan normal form, namely $D+N$ where D and N commute, D is a diagonal matrix and N a nilpotent matrix with $N^{\lambda+1} = 0$. By Proposition 5.1, $\exp(D+N) = \exp(D) \cdot \exp(N)$. Furthermore, $\exp(N) = 1 + N + \cdots + N^{\lambda}$. For the other direction of interpretability, see [15]. $\square$

However, note that the class of rings $\{M_{\lambda+1}(\mathbb{C}); \lambda \in \omega\}$ is undecidable since the class of their invertible elements $\{GL_{\lambda+1}(\mathbb{C}); \lambda \in \omega\}$ is undecidable (one interprets uniformly in λ a class of finite models whose theory is undecidable) and this implies that the theory of any non-principal ultraproduct of the $M_{\lambda+1}(\mathbb{C})$ is undecidable. Moreover, note that one may replace $\mathbb{C}$ by an arbitrary field, and the group $GL_{\lambda+1}(\mathbb{C})$ by other algebraic groups like $SL_{\lambda+1}(\mathbb{C})$ (see [6]).

6. The universal enveloping algebra of $sl_2(\mathbb{C})$

Recall that the universal enveloping algebra U of $sl_2(\mathbb{C})$ is an associative $\mathbb{C}$-algebra (hence, equipped by a Lie algebra structure) together with a canonical mapping σ which is a Lie algebra homomorphism $\sigma : sl_2(\mathbb{C}) \to U$ such that, if R is any associative $\mathbb{C}$-algebra and $f : sl_2(\mathbb{C}) \to R$ is a Lie algebra homomorphism, then there exists a unique algebra homomorphism $\Theta : U \to R$ sending 1 to 1 and such $f = \Theta \circ \sigma$ (see [4] chapter 2, sections 1, 2).

Diagram 6.1. *Let us choose as R the Lie algebra* $M_2(\mathbb{C})$ *and as f the Lie algebra homomorphism* $f_1 : sl_2(\mathbb{C}) \to M_2(\mathbb{C})$, *so there exists a unique algebra homomorphism* $\Theta_1 : U \to M_2(\mathbb{C})$ *such that (according to what just said above) the following diagram commutes.*

$$\begin{tikzcd} sl_2(\mathbb{C}) \arrow[r, "\sigma"] \arrow[d, "f_1"'] & U \arrow[dl, "\Theta_1"] \\ M_2(\mathbb{C}) & \end{tikzcd}$$

Since the canonical mapping σ of $sl_2(\mathbb{C})$ into U is injective ([4] Proposition 2.1.9), from now on we will identify every element of $sl_2(\mathbb{C})$ to its canonical image in U.

By using this universal property of U, we can construct an exponential map over U. Let us define the exponential map from U to $GL_2(\mathbb{C})$ as follows:

$$\mathrm{EXP}_1 : \quad U \xrightarrow{\Theta_1} M_2(\mathbb{C}) \xrightarrow{\exp} GL_2(\mathbb{C})$$

$$\mathrm{EXP}_1(\alpha) = \exp(\Theta_1(\alpha)) \qquad \forall \alpha \in U.$$

So, the values of $\mathrm{EXP}_1(U)$ are in $GL_2(\mathbb{C})$ and the restriction of EXP_1 to $sl_2(\mathbb{C})$ coincides with the exponential map $\exp : sl_2(\mathbb{C}) \to GL_2(\mathbb{C})$ (viewing $sl_2(\mathbb{C}) \subset M_2(\mathbb{C})$), previously defined (see (1)). Note that the image of the restriction of exp to $sl_2(\mathbb{C})$ is included in $SL_2(\mathbb{C})$ (see Proposition 5.1(vi)). Clearly $(U, \mathrm{EXP}_1, GL_2(\mathbb{C}))$ is an exponential algebra.

Let $c = 2x \cdot y + 2y \cdot x + h^2$ be the Casimir element of U, where x, y, h are the generators of $sl_2(\mathbb{C})$; c generates the center of U. Let us calculate $\mathrm{EXP}_1(c)$. First, let

$$\begin{aligned} \Theta_1(c) &= \Theta_1(2x \cdot y + 2y \cdot x + h^2) = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (\mathrm{diag}(1, -1))^2 \\ &= 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{diag}(1, 1) = \mathrm{diag}\,(3, 3)\,. \end{aligned}$$

By using the universal property of U, we have that $\mathrm{EXP}_1(c) = \exp(\Theta_1(c)) = \mathrm{diag}(\mathrm{e}^3, \mathrm{e}^3)$.

Now we want to describe the map EXP_1 on U. Recall that U is a $\mathbb{Z}$-graded algebra with grading $gr(x) = 1,\ gr(y) := -1$ and so $gr(h) = 0$; a m-homogeneous element $u \in U$ is an element such that $gr(u) = m, m \in \mathbb{Z}$. So U decomposes as a direct sum of m-homogeneous components U_m consisting of m-homogeneous elements, $m \in \mathbb{Z}$,

$$U = \bigoplus_{m \in \mathbb{Z}} U_m.$$

Furthermore, every m-homogeneous component satisfies the following relation, depending on whether m is positive or negative:

$$U_m = x^m U_0 \;=\; U_0\, x^m, \quad \text{for every positive integer number } m,$$
$$U_m = y^{|m|} U_0 = U_0\, y^{|m|}, \quad \text{for every negative integer number } m,$$

where, as well described in [7], the 0-homogeneous component U_0 coincides with the ring of polynomials $\mathbb{C}[h, c]$ in h and c, with coefficients in $\mathbb{C}$. Let u_m be an element in U_m for m a positive integer. So, $u_m = u_0 x^m = x^m v_0$, for some $u_0,\ v_0 \in U_0$, and $u_m^2 = (u_0 x^m)(x^m v_0) = u_0 x^{2m} v_0$. Applying Θ_1 to u_m, we can see that $\Theta_1(u_m^2) = \Theta_1(u_0 x^{2m} v_0) = \Theta_1(u_0)\Theta_1(x)^{2.m}\Theta_1(v_0) = 0$ (because $\Theta_1(x)^2 \;=\; 0$). By similar calculations, we can see that, $\forall u, v \;\in\; U$ with every degree different from $-1, 0,\ 1$, $\Theta_1(uv) \;=\; 0$. Now, we focus on U_0, so pick an element $p \;=\; p(c, h)$ and calculate the corresponding value of EXP_1. Since $\Theta_1(p(c, h)) = p(\Theta_1(c), \Theta_1(h)) = p(\mathrm{diag}(3, 3), \mathrm{diag}(1, -1))$, we can deduce that $\forall p \in U_0,\ \Theta_1(p(c, h) = 0)$ if and only if $p(3, 1) = 0$ and $p(3, -1) = 0$. Note that the corresponding ideal is not prime. Anyway, $\Theta_1(p(c, h))$ is a diagonal matrix with eigenvalues $p(3, 1)$ and $p(3, -1)$, and the matrix $\mathrm{EXP}_1(p) = \mathrm{diag}(e^{p(3,1)}, e^{p(3,-1)})$ with determinant equal to $e^{p(3,1)+p(3,-1)}$.

By what sketched above, Θ_1 acts as zero on $U_{\pm 2}, U_{\pm 3}, \ldots$. So, we restrict our attention to $U_{-1},\ U_0,\ U_1$. Let us pick up in $U_{-1} \oplus U_0 \oplus U_1$ an element $\gamma = yp_{-1}(c, h) + p_0(c, h) + xp_1(c, h)$ where the polynomials $p(c, h),\ p_0(c, h),\ p_1(c, h)$ belong to U_0. We want to calculate the exponential value of γ, as follows:

$$\begin{aligned}
\mathrm{EXP}_1(\gamma) &= \mathrm{EXP}_1\,(yp_{-1}(c, h) \;+\; p_0(c, h) \;+\; xp_0(c, h))\\
&= \exp\,(\Theta_1(yp_{-1}(h, c)) \;+\; \Theta_1(p_0(c, h)) \;+\; \Theta_1(xp_1(c, h)))\\
&= \exp\,(\Theta_1(y)\Theta_1(p_{-1}(c, h)) \;+\; \Theta_1(p_0(c, h)) + \Theta_1(x)\Theta_1(p_1(c, h)))\,.
\end{aligned}$$

Since the value of Θ_1 calculated on any element in U_0 is represented by a diagonal matrix, so $\Theta_1(yp_{-1}(c, h)),\ \Theta_1(p_0(c, h))$, $\Theta_1(xp_1(c, h))$ can be respectively represented by the diagonal matrices $\mathrm{diag}(a_{-1}, b_{-1}),\ \mathrm{diag}(a_0, b_0),\ \mathrm{diag}(a_1, b_1)$, where $a_i,\ b_i \in \mathbb{C}$, with $i = -1,\ 0,\ 1$. So, we have

$$\begin{aligned}
\mathrm{EXP}_1(\gamma) &= \exp\left(\begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix} \cdot \mathrm{diag}(a_{-1}, b_{-1}) + \mathrm{diag}(a_0, b_0) + \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix} \cdot \mathrm{diag}(a_1, b_1)\right)\\
&= \exp\left(\begin{pmatrix} 0 & 0\\ a_{-1} & 0\end{pmatrix} + \mathrm{diag}(a_0, b_0) + \begin{pmatrix} 0 & b_1\\ 0 & 0\end{pmatrix}\right)\\
&= \exp\begin{pmatrix} a_0 & b_1\\ a_{-1} & b_0\end{pmatrix}.
\end{aligned}$$

Thanks to these calculations, we can easily find the EXP_1 of $xp_1(h, c)$: indeed, $\mathrm{EXP}_1(xp_1(h, c)) \;=\; \exp(\Theta_1(xp_1(h, c))) \;=\; \exp\begin{pmatrix} 0 & b_1\\ 0 & 0\end{pmatrix} \;=\; I_2 + \begin{pmatrix} 0 & b_1\\ 0 & 0\end{pmatrix} \;=\; \begin{pmatrix} 1 & b_1\\ 0 & 1\end{pmatrix}$, (because the square of the matrix $\Theta_1(x)$, so of $\Theta_1(xp_1(c, h))$ is null). So, $\mathrm{EXP}_1(xp_1(h, c)) \;=\; I_2 \;+\; \Theta_1(xp_1(h, c))$. A similar property holds for $yp_{-1}(c, h)$, in fact, $\mathrm{EXP}_1(yp_{-1}(c, h)) \;=\; \exp(\Theta_1(yp_{-1}(c, h))) = \exp\begin{pmatrix} 0 & 0\\ a_{-1} & 0\end{pmatrix} = \begin{pmatrix} 1 & 0\\ a_{-1} & 1\end{pmatrix}$.

7. Other exponential maps

In this section, we define other exponential maps over U by using finite-dimensional representations of $sl_2(\mathbb{C})$, that is, finite-dimensional $sl_2(\mathbb{C})$-modules ([4] 1.2). (All our modules will be left modules.) First, recall that by Weyl's theorem, any finite-dimensional representation of $sl_2(\mathbb{C})$ can be decomposed as a direct sum of simple $sl_2(\mathbb{C})$-modules ([4] 1.8.5). For every positive integer λ, there exists a unique (up to isomorphism) simple $sl_2(\mathbb{C})$-module V_λ of dimension $\lambda + 1$; V_λ can be described as the $\mathbb{C}$-vectorspace of all homogeneous polynomials of degree λ with coefficients in $\mathbb{C}$ and variables X and Y (see [5, Chapter 5]). We decompose V_λ with respect to the basis of monomials $X^\lambda,\ X^{\lambda-1}Y, \ldots, XY^{\lambda-1}, Y^\lambda$, $V_\lambda = \oplus_{i=0}^{\lambda}\mathbb{C}[X^{\lambda-i}Y^i]$. The representation f_λ of $sl_2(\mathbb{C})$ can be described as follows:

$$\begin{aligned}
&x \ \text{ acts as } X\frac{\partial}{\partial Y}\\
&y \ \text{ acts as } Y\frac{\partial}{\partial X}\,,\\
&h \ \text{ acts as } X\frac{\partial}{\partial X} - \; Y\frac{\partial}{\partial Y}\,.
\end{aligned}$$

So, for $0 < i \le \lambda$, the basis element $X^{\lambda-i}Y^i$ is shifted to the left by the action of x, sent to $i \cdot X^{(\lambda-i)+1}Y^{i-1}$ and for $i = 0$, X^λ is sent by x to 0. For $0 \le i < \lambda$, the basis element $X^{\lambda-i}Y^i$ is shifted to the right by the action of y, sent to $(\lambda - i)X^{(\lambda-i)-1}Y^{i+1}$ and for $i = \lambda$, Y^λ is sent by y to 0. Each subspace generated by $X^{\lambda-i}Y^i$ is left invariant by the action of h: $X^{\lambda-i}Y^i$ is mapped to $(\lambda - 2)X^{\lambda-i}Y^i$ (so the corresponding eigenvalue is equal to $\lambda - 2.i$).

The $\mathbb{C}$-vectorspace $\mathrm{End}(V_\lambda)$ coincides with the $\mathbb{C}$-vectorspace $M_{\lambda+1}(\mathbb{C})$ of all $(\lambda+1)\times(\lambda+1)$ matrices written with respect to a basis of eigenvectors for h.

More precisely, through the representation f_λ, the actions of x, y and h are represented respectively the following three $(\lambda+1)\times(\lambda+1)$ matrices $X_{\lambda+1}, Y_{\lambda+1}, H_{\lambda+1}, \lambda \in \omega - \{0\}$:

$$X_{\lambda+1} = \begin{pmatrix} 0 & 1 & 0\dots & 0 \\ 0 & 0 & 2\dots & 0 \\ \vdots & \vdots & & \lambda \\ 0 & 0 & 0\dots & 0 \end{pmatrix}, \qquad Y_{\lambda+1} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \lambda & 0 & \dots & 0 \\ 0 & \lambda-1 & & 0 \\ \vdots & \vdots & & \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2}$$

$$H_{\lambda+1} = \operatorname{diag}(\lambda, \lambda-2, \dots, -\lambda+2, -\lambda).$$

Note that the operator norm of $X_{\lambda+1}$ (respectively $Y_{\lambda+1}$) is equal to λ, as is the operator norm of $H_{\lambda+1}$. The operator norm of $\Theta_\lambda(c)$ is equal to $\lambda^2+2\lambda$. On the other hand, the F-norm of $X_{\lambda+1}$ (respectively $Y_{\lambda+1}$) is equal to $\sqrt{\frac{\lambda(\lambda+1)(2\lambda+1)}{6}}$.

For every positive integer λ, we have the following diagram.

Diagram 7.1. *For any simple representation V_λ of $sl_2(\mathbb{C})$ of dimension $\lambda+1$ (with $\lambda \in \omega-\{0\}$), let us consider the representation map $f_\lambda : sl_2(\mathbb{C}) \to M_{\lambda+1}(\mathbb{C})$, and the following (commutative) diagram determined by the universal property of U:*

$$\begin{tikzcd} sl_2(\mathbb{C}) \arrow[r, "\sigma"] \arrow[d, "f_\lambda"'] & U \arrow[dl, "\Theta_\lambda"] \\ M_{\lambda+1}(\mathbb{C}) & \end{tikzcd}$$

where σ is the canonical mapping (which is a Lie algebra homomorphism) from $sl_2(\mathbb{C})$ to U and Θ_λ is the (unique) algebra homomorphism from U to $M_{\lambda+1}(\mathbb{C})$ sending 1 to 1 making the diagram commutes.

Using the commutativity of the above diagram, we obtain that the images of $x,\ y,\ h$ by the representation map $\Theta_\lambda : U \to M_{\lambda+1}(\mathbb{C})$ coincide with their images by the representation map f_λ, and so are equal to the matrices $X_{\lambda+1},\ Y_{\lambda+1},\ H_{\lambda+1}$, (see (2)).

The image by Θ_λ of the Casimir element c in U is given by the following calculation.

$$\begin{aligned} \Theta_\lambda(c) &= \Theta_\lambda(2xy+2yx+h^2) = 2\Theta_\lambda(x)\Theta_\lambda(y) + 2\Theta_\lambda(y)\Theta_\lambda(x) + (\Theta_\lambda(h))^2 \\ &= 2X_{\lambda+1}Y_{\lambda+1} + 2Y_{\lambda+1}X_{\lambda+1} + H_{\lambda+1}^2 \\ &= \operatorname{diag}\left(\lambda^2+2\lambda, \dots, \lambda^2+2\lambda\right). \end{aligned} \tag{3}$$

By the technique used for defining the exponential map EXP_1 from U to $GL_2(\mathbb{C})$, we can define the exponential map EXP_λ for every positive integer λ, as follows.

Definition 7.1. Let $\lambda \in \omega-\{0\}$. The exponential map EXP_λ over U is obtained by composing Θ_λ with the natural exponential map exp from $M_{\lambda+1}(\mathbb{C})$ to $GL_{\lambda+1}(\mathbb{C})$ (see Section 5):

$$\mathrm{EXP}_\lambda(u) = \exp(\Theta_\lambda(u)) \qquad \forall u \in U.$$

Proposition 7.2. $\forall \lambda \in \mathbb{N}-\{0\}$, *the map* EXP_λ *is surjective.*

Proof. Since exp is surjective from $M_{\lambda+1}(\mathbb{C})$ to $GL_{\lambda+1}(\mathbb{C})$, it suffices to prove that $\Theta_\lambda : U \to M_{\lambda+1}(\mathbb{C})$ is surjective. The latter is deduced directly by Jacobson density theorem [11, Section 2.2]. For convenience of the reader, we indicate below the proof.

Let V_λ be the irreducible representation of $sl_2(\mathbb{C})$ of dimension $\lambda+1$. As representation of U, we know by Schur's lemma, that $End_U(V_\lambda) \cong \mathbb{C}$. Consider $\phi \in End_{\mathbb{C}}(V_\lambda)\ (= M_{\lambda+1}(\mathbb{C}))$. Then by Jacobson density theorem we get that, for each finite subset of elements $v_1, \dots, v_{\lambda+1} \in V_\lambda$, that there exists $u \in U$ such that $\bigwedge_{i=1}^m (\ \phi(v_i) = \Theta_\lambda(u).v_i)$. □

We can easily calculate (as matrices in $GL_{\lambda+1}(\mathbb{C})$) the values by EXP_λ of x, y, h, c, using on the one hand that $\Theta_\lambda(x), \Theta_\lambda(y)$ are nilpotent matrices (in $M_{\lambda+1}(\mathbb{C})$), and on the other hand that $\Theta_\lambda(h), \Theta_\lambda(c)$ are diagonal matrices.

$$\mathrm{EXP}_\lambda(x) = \exp(\Theta_\lambda(x)) = \exp(X_{\lambda+1}) = I_{\lambda+1} + X_{\lambda+1} + \frac{X_{\lambda+1}^2}{2} + \dots + \frac{X_{\lambda+1}^\lambda}{\lambda!};$$

$$\mathrm{EXP}_\lambda(y) = \exp(\Theta_\lambda(y)) = \exp(Y_{\lambda+1}) = I_{\lambda+1} + Y_{\lambda+1} + \frac{Y_{\lambda+1}^2}{2} + \cdots + \frac{Y_{\lambda+1}^\lambda}{\lambda!},$$
$$\mathrm{EXP}_\lambda(h) = \exp(\Theta_\lambda(h)) = \exp(H_{\lambda+1}) = \mathrm{diag}(\mathrm{e}^\lambda, \mathrm{e}^{\lambda-2}, \ldots, \mathrm{e}^{-\lambda+2}, \mathrm{e}^{-\lambda});$$
$$\mathrm{EXP}_\lambda(c) = \exp(\Theta_\lambda(c)) = \exp(\mathrm{diag}(\lambda^2+2\lambda, \ldots, \lambda^2+2\lambda)) = \mathrm{diag}(\mathrm{e}^{\lambda^2+2\lambda}, \ldots, \mathrm{e}^{\lambda^2+2\lambda}).$$

Furthermore, we easily see that EXP_λ satisfies the properties of the matrix exponential exp described by Proposition 5.1.

Proposition 7.3. *Let $\lambda \in \mathbb{N} - \{0\}$. Then $(U, \mathrm{EXP}_\lambda, GL_{\lambda+1}(\mathbb{C}))$ is an exponential $\mathbb{C}$-algebra. More precisely, we have the following properties. Let $u, v \in U$ and let $a, b \in \mathbb{C}$, then:*

(i) $\mathrm{EXP}_\lambda\,(0_U) = I_{\lambda+1}$, *where 0_U denotes the identity element (with respect to the addition) in U.*
(ii) $\mathrm{EXP}_\lambda\,(a\cdot u)\cdot \mathrm{EXP}_\lambda\,(b\cdot u) = \mathrm{EXP}_\lambda\,((a+b)\cdot u)$;
(iii) $\mathrm{EXP}_\lambda\,(u)\cdot \mathrm{EXP}_\lambda\,(-u) = I_{\lambda+1}$;
(iv) *for u and v commuting,* $\mathrm{EXP}_\lambda\,(u+v) = \mathrm{EXP}_\lambda\,(u)\cdot \exp(v)$;
(v) *for an invertible element v in U,* $\mathrm{EXP}_\lambda\,(vuv^{-1}) = \Theta_\lambda(v)\mathrm{EXP}_\lambda\,(u)\Theta_\lambda(v)^{-1}$.

Proof. (i) By definition of EXP_λ, $\mathrm{EXP}_\lambda\,(0_U) = \exp(\Theta_\lambda(0_U)) = \exp(0_\lambda) = I_{\lambda+1}$ (see Proposition 5.1(i)).

(ii) $\mathrm{EXP}_\lambda\,(au)\cdot \mathrm{EXP}_\lambda\,(bu) = \exp(\Theta_\lambda\,(au))\cdot\exp(\Theta_\lambda\,(bu)) = \exp(a\,\Theta_\lambda\,(u))\cdot\exp(b\,\Theta_\lambda\,(u))$. Since $\Theta_\lambda(u) \in M_{\lambda+1}(\mathbb{C})$ and Proposition 5.1(ii) can be applied, then $\exp(a\Theta_\lambda\,(u))\cdot\exp(b\Theta_\lambda(u)) = \exp((a+b)\Theta_\lambda(u)) = \exp(\Theta_\lambda((a+b)\,u)) = \mathrm{EXP}_\lambda((a+b)u)$.

(iii) This follows immediately from the corresponding property for the matrix exponential.

(iv) First, note that if u and v commute in U, then $\Theta_\lambda(u)$ and $\Theta_\lambda(v)$ commute also (for Θ_λ is a homomorphism from U to $M_{\lambda+1}(\mathbb{C})$ for every λ). Thus, by using Proposition 5.1(iv) and the fact that Θ_λ is a homomorphism, we have: $\mathrm{EXP}_\lambda(u)\cdot\mathrm{EXP}_\lambda(v) = \exp(\Theta_\lambda(u))\cdot\exp(\Theta_\lambda(v)) = \exp(\Theta_\lambda(u)+\Theta_\lambda(v)) = \exp(\Theta_\lambda(u+v)) = \mathrm{EXP}_\lambda\,(u+v)$.

(v) The map Θ_λ is a morphism of associative rings, so if an element $v \in U$ is invertible, then so is $\Theta_\lambda(v)$. The result follows immediately by the corresponding property for the matrix exponential.

Note that since the Casimir element is central in U, its image $\Theta_\lambda(c)$ is central in $\Theta_\lambda(U) \subseteq M_{\lambda+1}(\mathbb{C})$, so for any $u \in U$, we get by Proposition 5.1 that $\exp(\Theta_\lambda(c)+\Theta_\lambda(u)) = \exp(\Theta_\lambda(c))\cdot\exp(\Theta_\lambda(u))$. So, $\mathrm{EXP}_\lambda(c+u) = \mathrm{EXP}_\lambda(c)\cdot\mathrm{EXP}_\lambda(u)$.

As a direct consequence of the definition of the map EXP_λ, we observe that $u \in \mathrm{Ker}(\mathrm{EXP}_\lambda)$ if and only if $\Theta_\lambda(u) \in \mathrm{Ker}(\exp)$. So in order to describe $\mathrm{Ker}(\mathrm{EXP}_\lambda)$, we should say as much as possible about $\Theta_\lambda(u)$ for $u \in U$. □

Proposition 7.4. *Decompose $U = \oplus_{m\in\mathbb{Z}} U_m$. The representation map Θ_λ sends:*

(i) *an element u_0 of U_0 onto a diagonal matrix,*
(ii) *an element $u_m \in U_m$, $m > 0$, onto an upper triangular matrix if $0 < m \le \lambda$, otherwise (when $m \ge \lambda+1$) $\Theta_\lambda(u_m) = 0_{\lambda+1}$.*
(iii) *an element $u_m \in U_m$, $m < 0$, is mapped to a lower triangular matrix, if $-\lambda \le m \le -1$ and, otherwise, for $m \le -\lambda-1$, to the zero matrix $0_{\lambda+1}$.*

Proof. (i) Let $u_0 \in U_0 - \{0\}$; so u_0 is of the form $p(c, h)$ with $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$, where x_1 and x_2 are two commuting variables. We know that $\Theta_\lambda(p(c, h)) = p(\Theta_\lambda(c), \Theta_\lambda(h))$, where $\Theta_\lambda(c)$ and $\Theta_\lambda(h)$ are the diagonal matrices described respectively by (3) and (2). Since any algebraic operation on diagonal matrices concerns just their diagonal entries, then for any polynomial $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$, we have that:

$$\Theta_\lambda(p(c,h)) = \mathrm{diag}\left(p\left(\lambda^2+2\lambda, \lambda\right), \ldots, p\left(\lambda^2+2\lambda, -\lambda\right)\right) \quad (\in M_{\lambda+1}(\mathbb{C})) \tag{4}$$

(ii) For the positive integer m, let u_m be an element in U_m of the form $u_m = x^m\cdot u_0$ where the 0-component $u_0 = p(c, h)$ as above. On the one hand, suppose that $m \le \lambda$. By using the fact that Θ_λ is a homomorphism and the values of $\Theta_\lambda(x)$ and $\Theta_\lambda(p(c, h))$ (described by (2) and (4) respectively) we have that $\Theta_\lambda(u_m) = \Theta_\lambda(x^m\cdot u_0) = \Theta_\lambda(x)^m\cdot\Theta_\lambda(u_0) = X_{\lambda+1}^m\cdot\mathrm{diag}\left(p\left(\lambda^2+2\lambda, \lambda\right), \ldots, p\left(\lambda^2+2\lambda, -\lambda\right)\right)$, so $\Theta(u_m)$ is represented by the strictly upper triangular matrix with $\star_l \in \mathbb{C}$, $1 \le l \le (\lambda+1) - m$

$$\begin{pmatrix} 0 & 0 & \star_1 & 0\ldots & 0 \\ 0 & 0 & 0 & \star_2\ldots & 0 \\ \vdots & \vdots & 0 & & \star_m \\ \vdots & \vdots & & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}. \tag{5}$$

On the other hand, assume that $m \ge \lambda+1$. Since $\Theta_\lambda(x)$ is a nilpotent matrix, we can easily see: $\Theta_\lambda(u_m) = \Theta_\lambda(x)^m\cdot\Theta_\lambda(u_0) = 0$.

(iii) Similarly, we can repeat the same argument for any element u_m, with $m < 0$, of the form $y^m \cdot u_0$. So, for $-\lambda \leq m \leq -1$ the image by Θ_λ of u_m, is a lower triangular matrix of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0\dots & 0 \\ 0 & 0 & 0 & 0\dots & 0 \\ \star_1 & 0 & 0 & 0\dots & 0 \\ 0 & \star_2 & 0 & 0 & \dots \\ \vdots & \vdots & & 0 & \\ 0 & \dots & 0 & \star_{-m}\dots & \end{pmatrix}. \tag{6}$$

If $m \leq -(\lambda+1)$. we have $\Theta_\lambda(u_m) = \Theta_\lambda(y^m \cdot \alpha_0) = 0$. □

Remark 1. An element $u_0 \in U_0$ belongs to the kernel of EXP_λ if and only if

$$\bigwedge_{0\leq j\leq\lambda} p\left(\lambda^2 + 2\lambda, \lambda - 2j\right) \in 2\pi i \cdot \mathbb{Z}. \tag{7}$$

In fact, for $u_0 = p(c, h)$ the diagonal matrix $\Theta_\lambda(p(c, h))$ belongs to Ker(exp) if and only if their diagonal entries described by (4) belongs to $\mathrm{Ker}(e) = 2\pi i \cdot \mathbb{Z}$.

Proposition 7.5. EXP_λ *maps an element u of U into* $SL_{\lambda+1}(\mathbb{C})$ *whenever*

$$tr(\Theta_\lambda(u)) \in 2\pi i \cdot \mathbb{Z}. \tag{8}$$

In particular, if $u \in \oplus_{m\neq 0} U_m$, *then its image by* EXP_λ *lies always in* $SL_{\lambda+1}(\mathbb{C})$.

Proof. For the first statement, it is enough to apply property (vi) of Proposition 5.1, so for any $u \in U$, the determinant of $\exp(\Theta_\lambda(u))$ equals 1 if the trace of $\Theta_\lambda(u)$ belongs to $\mathrm{Ker}(e) = 2\pi i \cdot \mathbb{Z}$.

As to the second claim, first we can note that the map EXP_λ maps x, y and their powers into $SL_{\lambda+1}(\mathbb{C})$, because their images by Θ_λ are matrices of trace 0. We get the same results with x^m (respectively y^m). Since the subalgebra U_0 is sent to the subalgebra of diagonal matrices in $M_{\lambda+1}(\mathbb{C})$, the image of an element $\alpha_m = x^m \cdot \alpha_0$ in U_m by Θ_λ is a matrix of trace 0 (as illustrated by (5)) and so its matrix exponential has determinant 1. The same argument holds where $\alpha_m = y^m \cdot \alpha_0$ (for negative m). Since the sum of matrices of trace 0 has trace 0, an element of $\oplus_{m\neq 0} U_m$ is sent by EXP_λ to $SL_{\lambda+1}(\mathbb{C})$. □

When we restrict Proposition 7.5 to any element u_0 of U_0, where $u_0 = p(c, h)$ (for some polynomial $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$) the condition (8) means that the sum of eigenvalues of $\Theta_\lambda(u)$, $\sum_{0\leq j\leq\lambda} p(\lambda^2 + 2\lambda, \lambda - 2j)$, has to belong to $2\pi i \cdot \mathbb{Z}$.

Put $p(x_1, x_2) = \sum_{l=0}^{d} q_l(x_1)x_2^l$, then $\sum_{0\leq j\leq\lambda} \sum_{l=0}^{\lfloor d/2\rfloor} q_l(\lambda^2 + 2\lambda)(\lambda - 2j)^{2l} = \sum_{l=0}^{\lfloor d/2\rfloor} q_j(\lambda^2 + 2\lambda)[\sum_{0\leq j\leq\lambda}(\lambda - 2j)^{2l}]$.

Now, let us assume that u_0 is in the kernel of $\bigcap_{\lambda\in\mathbb{Z};\ \lambda>\lambda_0} \mathrm{EXP}_\lambda$, for some λ_0. Then, $p(\lambda^2 + 2\lambda, \lambda - j) \in 2\pi i \cdot \mathbb{Z}$, for all $|\lambda| > \lambda_0$ and $0 \leq j \leq \lambda$.

In the remainder of this section, we will give a partial answer to the question of which elements u of U are such that $\Theta_\lambda(u) \in su_{\lambda+1}$.

Recall that $su_{\lambda+1} := \{A \in M_{\lambda+1}(\mathbb{C}) : A^* = -A,\ tr(A) = 0\}$, and $SU_{\lambda+1} := \{X \in GL_{\lambda+1}(\mathbb{C}) : X \cdot X^* = I_{\lambda+1}, \det(X) = 1\}$, where X^* denotes the conjugate transpose of X; it is a compact Lie group.

Coming back first to the case $\lambda = 1$, it is well known that the exponential map exp (defined in $M_2(\mathbb{C})$) restricted to $sl_2(\mathbb{C})$ does not map it surjectively to its Lie group $SL_2(\mathbb{C})$ ([16] page 38). However if we restrict to the $\mathbb{R}$-subalgebra su_2, exp is surjective onto the (compact) Lie group $SU_2(\mathbb{C})$ (see Lemma 2.a in section 2 of [16]). We have the following decomposition: $SL_2(\mathbb{C}) = SU_2(\mathbb{C}).B$, where B is the subgroup of triangular matrices with determinant 1 and positive real diagonal entries ([16] page 39).

The surjectivity property of exp still holds if one replaces su_2 with $su_{\lambda+1}$ and SU_2 by $SU_{\lambda+1}$ (see Corollary 2 in [16]).

Let $u \in U_0$, so $u = p(c, h)$. So, $\Theta_\lambda(u) \in su_{\lambda+1}$, if $\sum_j p(\lambda^2 + \lambda, \lambda - 2j) = 0$ and for all $-\lambda \leq j \leq \lambda$, $p(\lambda^2 + \lambda, \lambda - 2j) = -\bar{p}(\lambda^2 + \lambda, \lambda - 2j)$. The last condition occurs, for instance if $p(x_1, x_2)$ is the multiple by the complex number i of a polynomial with real coefficients.

Now consider elements $u \in \oplus_{m\neq 0} U_m$, namely $u = \sum_{\ell>0}(p_\ell(c, h) \cdot x^\ell + y^\ell \cdot q_\ell(c, h))$ with $p_\ell,\ q_\ell \in \mathbb{C}[h, c]$. Then the condition under which $\Theta_\lambda(u) \in su_{\lambda+1}$ is that $(\lambda - j)q_\ell(\lambda^2 + \lambda, \lambda - 2j) = (-\lambda + j)p_\ell(\lambda^2 + \lambda, \lambda - 2j)$, for all $-\lambda \leq j \leq \lambda$. Given a polynomial p_ℓ, we can always find a polynomial q_ℓ (of degree $\leq \lambda - 1$) meeting these λ conditions, using Lagrange interpolation theorem.

So, given $u \in \oplus_{m>0} U_m$, there exists $u' \in \oplus_{m<0} U_m$ such that $\Theta_\lambda(u + u') \in su_{\lambda+1}$.

8. Exponentiations and ultraproducts

We will be considering a non-principal ultraproduct of the Lie algebras $M_{\lambda+1}(\mathbb{C})$, $\lambda \in \omega$. Namely, let $\mathcal{V}$ be a non-principal ultrafilter on ω and consider the corresponding ultraproducts $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$ and $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$.

By Łos's theorem, the structure $(\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C}), +, -, 0, [\cdot,\cdot])$ is a Lie algebra over $\mathbb{C}$ or over $\mathbb{C}^* := \prod_{\mathcal{V}} \mathbb{C}$, which is infinite-dimensional.

We first observe the following.

Proposition 8.1. (i) *If u_0 is any element of $U_0 - \{0\}$, then there exists λ_0 such that for all $\lambda \geq \lambda_0$, we have $\Theta_\lambda(u_0) \neq 0$.*
(ii) *For any $u \in U - \{0\}$, there exists λ_0 such that for all $\lambda \geq \lambda_0$ we have $\Theta_\lambda(u) \neq 0$.*

Proof. (i) Let $u_0 \in U_0 - \{0\}$; so u_0 is of the form $p(c, h)$ with $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$, where x_1 and x_2 are two commuting variables.

The claim can be deduced directly from [7, Lemma 19]. For convenience of the reader, we repeat the argument here. We argue by contradiction.

Assume that $\bigwedge_{0\leq j\leq\lambda} p(\lambda^2 + 2\lambda, \lambda - 2j) = 0$. First, we choose λ such that $p(\lambda^2 + 2\lambda, x_2) \neq 0$, so as a polynomial in j, $p(\lambda^2 + 2\lambda, \lambda - 2j)$ is nontrivial of degree k and so the number of roots is bounded by k. So, if we choose λ big enough, we will always find j such that $p(\lambda^2 + 2\lambda, \lambda - 2j) \neq 0$. Therefore, $\Theta_\lambda(p(c, h)) \neq 0$ for some λ.

(ii) Let $u \in U-\{0\}$, then there exists $m \in \mathbb{Z}$ such that its mth component $u_m \neq 0$. Assume that $m \geq 0$ and that m is minimal such. Let $u_m = x^m u_0$, where $u_0 \in U_0$. Let $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ be such that $u_0 = p(c, h)$. Write $p(x_1, x_2) = \sum_{l=0}^{d} q_l(x_1)x_2^l$. We can find (explicitly) an interval $[-r;\ r]$ in $\mathbb{R}$ such that all the roots of the polynomial $q_d(x_1)$ are in that interval. Let $r' = \max\{r,\ d\}$. Then if $\lambda > r'$, then $q_d(\lambda^2 + 2\lambda) \neq 0$ and so the polynomial $\sum_{l=0}^{d} q_l(\lambda^2 + 2\lambda)x_2^l$ has less than d roots and among the $\lambda+1$ elements of the form $(\lambda-2j)$ where $0 \leq j \leq \lambda$, we have such j with the property that $p(\lambda^2+2\lambda, (\lambda-2j)) \neq 0$.

Since the images of any homogeneous components U_m with $-\lambda \leq m \leq \lambda$ are in direct sum and $\Theta_\lambda(u_m) \neq 0$, then we have $\Theta_\lambda(u) \neq 0$. □

Define the obvious $\Theta := [\Theta_\lambda]$ from U to the ultraproduct of the $M_{\lambda+1}(\mathbb{C})$, over any non-principal ultrafilter $\mathcal{V}$ on ω. By Proposition 8.1, the map Θ is an associative ring monomorphism. So, we get the following corollary.

Corollary 8.2. *For any non-principal ultrafilter $\mathcal{V}$ on ω, U embeds in the associative Lie algebra $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$.* □

Recall that U is a left and right Ore domain, so it has a left and right field of fractions which embeds in the ring U' of definable scalars of U. This ring U' has been shown to be von Neumann regular by Herzog [7], equivalently every left (right) principal ideal is generated by an idempotent. Moreover, since any V_λ is also a U'-module, we can send $r \in U'$ in the direct product $\prod_{\lambda\in\omega} M_{\lambda+1}(\mathbb{C})$ by sending it in each factor to the element of $M_{\lambda+1}(\mathbb{C})$, representing its action on each V_λ.

Then, [13] explicitly identifies certain idempotents of U' of the form e_u, $u \in U$, corresponding to the projections on $\ker(\Theta_\lambda(u))$ on V_λ, $\lambda \in \omega$. For instance e_x is the projection on the highest weight space of V_λ. When $u \in U_0$, so of the form $p(c, h)$, with $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$, they call p *standard* if there are only finitely many λ such that $p(\lambda^2 + 2\lambda, \lambda - 2j) = 0$ for some $0 \leq j \leq \lambda$ (and *non-standard* otherwise). Note that if $u = p(c, h)$ with p standard, then $[\Theta_\lambda(u)]_{\mathcal{V}}$ is invertible in $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$. (Note that the converse holds if $[\Theta_\lambda(u)]_{\mathcal{U}}$ is invertible with respect to any non-principal ultrafilter $\mathcal{U}$.)

Let now $u = p(c, h) \in U_0$ be such that p is non-standard, so for some non-principal ultrafilter $\mathcal{V}$ the action of e_u in the ultraproduct $\prod_{\mathcal{V}} V_\lambda$ will be a non-invertible element of the form $[(\mathrm{diag}(0, \ldots, 1, \ldots, 0, 1, \ldots, 0)] \neq 0$, where the number of possible 0's is bounded by the degree of p with respect to the second variable.

We know that Θ is a surjection from $\prod_{\mathcal{V}} U$ to $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$ (see the proof of Proposition 7.2). Then, we will compose with the map

$$\mathrm{Exp} : \prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C}) \to \prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C}) : [\mathrm{A}_\lambda]_{\mathcal{V}} \to [\exp(\mathrm{A}_\lambda)]_{\mathcal{V}}.$$

So, by composing with $[\Theta_\lambda]_{\mathcal{V}}$, we get a map $\mathrm{EXP}^* = Exp[\Theta_\lambda]_{\mathcal{V}}$ from $\prod_{\mathcal{V}} U$ to $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$, which is surjective. The kernel of that map is in bijection with the kernel of *Exp* on $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$.

Definition 8.1. Let EXP from U to $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$ be defined as follows:

$$\mathrm{EXP} : U \to \prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C}) : \mathrm{u} \to [\mathrm{EXP}_\lambda(\mathrm{u})]_{\mathcal{V}}.$$

Proposition 8.3. *Both $(U, \mathrm{EXP}, \prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C}))$ and $(U_0, \mathrm{EXP}, \prod_{\mathcal{V}} Diag_{\lambda+1}(\mathbb{C}))$ are exponential $\mathbb{C}$-algebras. Moreover we have that $\mathrm{EXP}(\oplus_{m\neq 0}U_m) \subset \prod_{\mathcal{V}} SL_{\lambda+1}(\mathbb{C})$, $\mathrm{EXP}(\oplus_{m\geq 0}U_m) \subset \prod_{\mathcal{V}} UT_{\lambda+1}(\mathbb{C})$, and $\mathrm{EXP}(U_0) \subset \prod_{\mathcal{V}} Diag_{\lambda+1}(\mathbb{C})$.*

Proof. A direct application of Łos Theorem shows that EXP satisfies the properties stated for each EXP_λ in Proposition 7.3. □

Note that the above properties are independent of the non-principal ultrafilter $\mathcal{V}$ on ω.

Question 8.1. What is the kernel of EXP?

It is the set of elements u such that for a subset of λ belonging to $\mathcal{V}$, $\exp(\Theta_\lambda(u)) = 1$. So, the eigenvalues of $\Theta_\lambda(u)$ belong to $2\pi i \cdot \mathbb{Z}$; does it translate into an independently interesting property of $u \in U$? For $u_0 \in U_0$, we have the following answer. Let $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ such that $u_0 = p(c, h)$. Then, for almost all λ and all $0 \le j \le \lambda$, we have $p(\lambda^2 + \lambda, \lambda - 2j) \in 2\pi i \cdot \mathbb{Z}$.

Proposition 8.4. *Let $u := p(c, h) \in U_0$, with $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$. Write $p(x_1, x_2)$ in the form $2\pi i \cdot q(x_1, x_2)$. Then, if $u \in \ker(\mathrm{EXP})$, then $q(x_1, x_2) \in \mathbb{Q}[x_1, x_2]$.*

Proof. Let $q(x_1, x_2) = \sum_{k=0}^{d} q_k(x_1) \cdot x_2^k$ and assume that $q(c, h) \in \ker(\mathrm{EXP})$. Then, the set $\{\lambda \in \omega : \bigwedge_{0 \le j \le \lambda} \; q(\lambda^2 + 2\lambda, \lambda - 2j) \in 2\pi i \cdot \mathbb{Z}\} \in \mathcal{V} \quad (\star)$.

Set $c_k := q_k(\lambda^2 + 2\lambda)$ and consider the following system of linear equations, with $z_\ell \in \mathbb{Z}$, $0 \le \ell \le n$:

$$\begin{pmatrix} 1 & y_0 & y_0^2 & \cdots & y_0^d \\ 1 & y_1 & y_1^2 & \cdots & y_1^d \\ & \vdots & & & \\ 1 & y_n & y_n^2 & \cdots & y_n^d \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_n \end{pmatrix}.$$

When $n = d$, the determinant of the (square) matrix $\begin{pmatrix} 1 & y_0 & y_0^2 & \cdots & y_0^d \\ 1 & y_1 & y_1^2 & \cdots & y_1^d \\ & \vdots & & & \\ 1 & y_d & y_d^2 & \cdots & y_d^d \end{pmatrix}$ is equal to $\prod_{0 \le n_1 < n_2 \le d} (y_{n_1} - y_{n_2})$. So it is a non-zero integer whenever the y_i's are d pairwise distinct integers and so in that case, the coefficients c_k are rational numbers.

So, it suffices to express hypothesis $(\star)$ for $\lambda > d$.

Now, write each $q_k(x_1^2 + 2x_1)$ as $q_k'(x_1) = \sum_{h=0}^{d_k} f_h \cdot x_1^h$ and again write the system of equations expressing that each $q_k(\lambda^2 + 2\lambda) \in \mathbb{Q}$, for $\lambda \in \omega$. Let $q_j \in \mathbb{Q}$, $0 \le j \le n$.

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{d_k} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{d_k} \\ & \vdots & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{d_k} \end{pmatrix} \cdot \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_n \end{pmatrix}.$$

Then, again when $n = d_k$, the determinant of the (square) matrix $\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{d_k} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{d_k} \\ & \vdots & & & \\ 1 & x_{d_k} & x_{d_k}^2 & \cdots & x_{d_k}^{d_k} \end{pmatrix}$ is equal to

$\prod_{0 \le n_1 < n_2 \le d_k} (x_{n_1} - x_{n_2})$. So it is a non-zero integer whenever the x_i's are d_k pairwise distinct integers and so in that case, the coefficients f_k are rational numbers. So, it suffices to express hypothesis $(\star)$ for $d_k + 1$ values of λ's, as soon as $\lambda > d$. □

Remark 2. We have a partial converse to the above proposition. Namely, let $q(x_1, x_2) = \sum_{k=0}^{d} q_k(x_1) \cdot x_2^k$, where each $q_k(x_1) \in \mathbb{Q}[x_1]$, so can be written as $1/n_k \cdot \sum_{h=1}^{d_k} z_h \cdot x_1^h + q_{0,k}$, where $n_k \in \mathbb{N} - \{0\}$, $\;z_h \in \mathbb{Z}$ and $q_{0,k} \in \mathbb{Q}$.

If, we assume in addition that each $q_{0,k} \in \mathbb{Z}$, then for some ultrafilter $\mathcal{V}$, $2\pi i \cdot q(c, h) \in \ker(\mathrm{EXP})$. Indeed, let $n = \mathrm{lcm}\{n_k : 0 \le k \le d\}$. Then we choose an ultrafilter $\mathcal{V}$ containing $2n \cdot \omega$.

So, if $\lambda = 2n \cdot m$, for some $m \in \omega$, $q_k(\lambda^2 + 2\lambda) = n/n_k \cdot \sum_{h=1}^{d_k} z_h \cdot (2n \cdot m^2 + 2m)^h + q_{0,k}$, then $q_k(\lambda^2 + 2\lambda) \in \mathbb{Z}$ and so $\{\lambda \in \omega : \bigwedge_{0 \le j \le \lambda} \; q(\lambda^2 + 2\lambda, \lambda - 2j) \in 2\pi i \cdot \mathbb{Z}\} \in \mathcal{V}$.

Corollary 8.5. *Let $u := p(c, h) \in U_0$, with $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$. Write $p(x_1, x_2)$ in the form $2\pi i \cdot q(x_1, x_2)$. Write $q(x_1, x_2) = \sum_{k=0}^{d} q_k(x_1) \cdot x_2^k$, with $q_k(x) \in \mathbb{Q}[x_1]$.*

Then, $u \in \ker(\mathrm{EXP})$ for all non-principal ultrafilters on ω, if and only if $q(x_1, x_2) \in \mathbb{Q}[x_1, x_2]$ and for each $0 \le k \le d$, $q_k(0) \in \mathbb{Z}$. □

Proposition 8.6. *Let $u := p(c, h) \in U_0$, with $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$. Write $p(x_1, x_2)$ in the form $2\pi i \cdot q(x_1, x_2)$. Then, if $\mathrm{EXP}(u) \in \prod SL_{\lambda+1}(\mathbb{C})$, then $q(x_1, x_2) \in \mathbb{Q}[x_1, x_2]$.*

Proof. Let $q(x_1, x_2) = \sum_{k=0}^{d} q_k(x_1) \cdot x_2^k$ and assume that the set $\{\lambda \in \omega : \mathrm{EXP}_\lambda(q(c, h)) \in SL_{\lambda+1}(\mathbb{C})\} \in \mathcal{V}$. Equivalently, $\{\lambda \in \omega : [\sum_{\ell=0}^{\lfloor d/2 \rfloor} q_\ell(\lambda^2 + 2\lambda) \cdot \sum_{0 \le j \le \lambda} (\lambda - 2j)^{2 \cdot \ell}] \in 2\pi i \cdot \mathbb{Z}\} \in \mathcal{V} \quad (\star)$.

Set $c_k := q_k(x_1)$ and consider the following system of linear equations, with $z_\ell \in \mathbb{Z}$, $0 \le \ell \le n$:

$$\begin{pmatrix} 1 & y_0 & y_0^2 & \cdots & y_0^d \\ 1 & y_1 & y_1^2 & \cdots & y_1^d \\ & \vdots & & & \\ 1 & y_n & y_n^2 & \cdots & y_n^d \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_n \end{pmatrix}.$$

When $n = d$, the determinant of the (square) matrix $\begin{pmatrix} 1 & y_0 & y_0^2 & \cdots & y_0^d \\ 1 & y_1 & y_1^2 & \cdots & y_1^d \\ & \vdots & & & \\ 1 & y_d & y_d^2 & \cdots & y_d^d \end{pmatrix}$ is equal to $\prod_{0 \le n_1 < n_2 \le d} (y_{n_1} - y_{n_2})$. So it is a non-zero integer whenever the y_i's are d pairwise distinct integers and so in that case, the coefficients c_k are rational numbers.

So, it suffices to express hypothesis ($\star$) for $\lambda > d$ and show that $\sum_{0 \le j \le \lambda} (\lambda - 2j)^{2 \cdot \ell}$ are pairwise distinct.

The rest of the proof is similar to the previous one. $\square$

9. Comparison with Serre's definition of an exponential map

Recall that the completion $\hat{U}$ of U [17] is defined as the infinite product $\Pi_{n=0}^{\infty} U^n$, where U^n denotes the component of degree n of U (generated by all products of length $\le n$ of generators $x,\ y$ of U); an element $f \in \hat{U}$ can be represented as $\sum_{n=0}^{\infty} f_n$, where $f_n \in U^n$ (see [17] Part 1, chapter 4, paragraph 6). (Note that U^n differs in general from U_n.)

Denote by $\mathcal{M}$ the ideal of U generated by $x,\ y$ and let $\hat{\mathcal{M}}$ be the ideal of $\hat{U}$ generated by $\mathcal{M}$. For $f \in \hat{\mathcal{M}}$, Serre defines $\exp_S$ by the usual formula $\exp_S(f) := \sum_n \frac{f^n}{n!}$. It takes $\hat{\mathcal{M}}$ to $1 + \hat{\mathcal{M}}$ (see [17] Part 1, chapter 4, paragraph 7). (Similarly, one can define $\log_S$ from $1 + \hat{\mathcal{M}}$ to $\hat{\mathcal{M}}$ by $\log_S(1 + x) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, obtaining that $\exp_S \circ \log_S = 1 = \log_S \circ \exp_S$ (see Theorem 7.2, Chapter 4, Part 1 in [17].)

Let $f := \sum_{n=0}^{\infty} f_n \in \hat{U}$ and assume that $\sum_{n=0}^{\infty} \Theta_\lambda(f_n)$ belongs to $M_{\lambda+1}(\mathbb{C})$. Then, define $\hat{\Theta}(f) := [\sum_{n=0}^{\infty} \Theta_\lambda(f_n)]_{\mathcal{V}}$. Since, if $u \in U$, there exists a bound on the number of non-zero components, this map is always well defined on the elements of U.

Proposition 9.1. *For any $u \in \mathcal{M}$, $\hat{\Theta}(\exp_S(u)) = \mathrm{EXP}(u)$.*

Proof. Now, let $u \in \mathcal{M}$ with $u = \sum_{j=1}^{k} u_j$, where $u_j \in U^j$, then $u^n := (\sum_{j=1}^{k} u_j)^n$. So, for each m, the m-component of $\exp_S(u)$ is a finite sum. Therefore $\hat{\Theta}(\exp_S(u))$ is well defined and $\hat{\Theta}(\exp_S(u)) = [\sum_{n=0}^{\infty} \Theta_\lambda(\frac{u^n}{n!})]_{\mathcal{V}} = \mathrm{EXP}(u)$. $\square$

10. A $\star$-norm on the universal enveloping algebra of $sl_2(\mathbb{C})$

Now, we would like to put a natural topology on U in such a way that EXP is continuous. As in the previous section, we fix a non-principal ultrafilter $\mathcal{V}$ on ω; let $\mathbb{C}^* := \prod_{\mathcal{V}} \mathbb{C}$ be a non-principal ultrapower of the field $(\mathbb{C}, +, \cdot, -, 0)$. We equip $\mathbb{C}^*$ with the ultrapower of the standard complex conjugation, and in addition consider the ultraproduct of the various Frobenius norms. This takes values in the corresponding ultrapower of the reals, and satisfies the obvious modification of the norm axioms. By functoriality this norm comes formally from the ultraproduct of the Hermitian sesquilinear forms.

Finally, by taking ultraproducts of normed algebras we get a natural notion of a $\star$-normed algebra, satisfying a natural version of the Cauchy–Schwarz inequality if the component algebras do. Since $\|\cdot\|_{\lambda+1}$ is a norm on each $M_{\lambda+1}(\mathbb{C})$, by the usual properties of an ultraproduct, we get a natural $\star$-norm $\|\cdot\|$ on $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$.

This in turn, by Corollary 8.2 induces a star norm on U.

In the next lemma, we will give an estimate of the norm of $u \in U$ in terms of a polynomial in λ, with coefficients in $\mathbb{R}$.

Lemma 10.1. *For each $u \in U$, there exist non-zero polynomials $q_1(.),\ q_2(.)$ with coefficients in $\mathbb{R}$ such that for λ sufficiently big, we have $q_1(\lambda) \le \|\Theta_\lambda(u)\|_F^2 \le q_2(\lambda)$ and so $q_1([\lambda]_{\mathcal{V}}) \le \|u\| \le q([\lambda]_{\mathcal{V}})$.*

Proof. Let us examine the norm of $\Theta_\lambda(u)$ for any element of U. Let $u = \sum_{m \in \mathbb{Z}} u_m$ (where $u_m \in U_m$ and $m \in \mathbb{Z}$). Moreover, for each $m \ge 0$, each $u_m = x^m \cdot p_m(c, h)$, and $u_{-m} = y^m \cdot p_{-m}(c, h)$, where $p_m(x_1, x_2),\ p_{-m}(x_1, x_2) \in \mathbb{C}[x_1, x_2]$. Assume that for some $k \in \mathbb{N}$, we have $u = \sum_{-k \le m \le k} u_m$, then we estimate $\|\Theta_\lambda(u)\|$ as follows. Assume $\lambda \ge k$, then

$$\begin{aligned} ||\Theta_\lambda(u)||_F^2 &= ||\Theta_\lambda\left(\sum_{m \in \mathbb{Z}} u_m\right)||_F^2 = ||\sum_{m \in \mathbb{Z}} \Theta_\lambda(u_m)||_F^2 \\ &= ||\sum_{m=-k}^{-1} \Theta_\lambda(u_m) + \Theta_\lambda(u_0) + \sum_{m=1}^{k} \Theta_\lambda(u_m)||_F^2 \end{aligned}$$

$$= \sum_{m=-k}^{-1} ||\Theta_\lambda(u_m)||_F^2 + ||\Theta_\lambda(u_0)||_F^2 + \sum_{m=1}^{k} ||\Theta_\lambda(u_m)||_F^2$$
$$= \sum_{m=-k}^{-1} ||\Theta_\lambda(y^{|m|}p_{-m}(c,h))||_F^2 + ||\Theta_\lambda(p_0(c,h))||_F^2 + \sum_{m=1}^{k} ||\Theta_\lambda(x^m p_m(c,h))||_F^2$$
$$= \sum_{m=-k}^{-1} ||\Theta_\lambda(y)^{|m|} \cdot p_{-m}(\Theta_\lambda(c), \Theta_\lambda(h))||_F^2 + ||p_0(\Theta_\lambda(c), \Theta_\lambda(h))||_F^2$$
$$+ \sum_{m=1}^{k} ||\Theta_\lambda(x)^m \cdot p_m(\Theta_\lambda(c), \Theta_\lambda(h))||_F^2.$$

Then, we make the following estimate. Write $p_m(x_1, x_2) = \sum_{j=0}^{d_m} q_j(x_1) \cdot x_2^j$. Let $f_j(x_1) = \frac{q_j(x_1)}{q_{d_m}(x_1)}$ and write the roots of $\sum_{j=0}^{d_m} f_j(x_1) \cdot x_2^j$ as $\alpha_1(x_1), \cdots, \alpha_{d_m}(x_1)$. Note that these roots are all in a ball of radius $M_m(\lambda) := 1 + \sum_{j=0}^{d_m-1} |f_j(\lambda^2 + 2\lambda)|$; let $R_m(\lambda) := \sum_{j=0}^{d_m} |q_j(\lambda^2 + 2\lambda)|$. Then $p_m(x_1, x_2) = q_{d_m}(x_1). \prod_{j=1}^{d_m}(x_2 - \alpha_j(x_1))$.

We have $|p_m(\lambda^2+2.\lambda, \lambda-2i)| = |q_{d_m}(\lambda^2+2\lambda)|. \prod_j |((\lambda-2i)-\alpha_j(\lambda^2+2.\lambda)|$. Since the number of roots of $p_m(\lambda^2+2\lambda, x_2)$ is at most d_m, there is at least one integer in the interval $[-\lambda; \lambda]$ at distance bigger than $\lfloor \frac{\lambda}{d_m} \rfloor$ of all of these roots. So, $|q_{d_m}(\lambda^2 + 2\lambda)|^2 . \lfloor \frac{\lambda}{d_m} \rfloor^{2d_m} \le \sum_{-\lambda \le i \le \lambda} |p_m(\lambda^2 + 2.\lambda, \lambda - 2i)|^2 \le R_m(\lambda)^2.(2.\lambda^{2d_m+1} + 1) \le R_m(\lambda)^2.(3\lambda^{2d_m+1})$. So we get on the one hand,

$$||\Theta_\lambda(u)||_F^2 \le \sum_{m=-k}^{-1} \lambda^{2.|m|} \cdot \sum_{-\lambda \le i \le \lambda} |p_{-m}(\lambda^2 + 2\lambda, \lambda - 2i)|^2 + \sum_{-\lambda \le i \le \lambda} |p_0(\lambda^2 + 2\lambda, \lambda - 2i)|^2$$
$$+ \sum_{m=1}^{k} \lambda^{2m} \cdot \sum_{-\lambda \le i \le \lambda} |p_m(\lambda^2 + 2\lambda, \lambda - 2i)|^2$$
$$\le \sum_{m=-k}^{-1} \lambda^{2.|m|} \cdot R_m(\lambda)^2.(3\lambda^{2d_m+1}) + R_0(\lambda)^2.(3\lambda^{2d_0+1}) + \sum_{m=1}^{k} \lambda^{2m} \cdot R_m(\lambda)^2.(3\lambda^{2d_m+1})$$

and on the other hand,

$$||\Theta_\lambda(u)||_F^2 \ge \sum_{m=-k}^{-1} (\lambda - k)^{2.|m|}. \sum_{-\lambda \le i \le \lambda} |p_{-m}(\lambda^2 + 2\lambda, \lambda - 2i)|^2 + \sum_{-\lambda \le i \le \lambda} |p_{d_0}(\lambda^2 + 2\lambda, \lambda - 2i)|^2$$
$$+ \sum_{m=1}^{k} (\lambda - k)^{2m}. \sum_{-\lambda \le i \le \lambda} |p_m(\lambda^2 + 2\lambda, \lambda - 2i)|^2$$
$$\ge \sum_{m=-k}^{-1} (\lambda - k)^{2.|m|}.|q_{d_m}(\lambda^2 + 2\lambda)|^2. \left\lfloor \frac{\lambda}{d_m} \right\rfloor^{2d_m} + |q_{d_0}(\lambda^2 + 2\lambda)|^2. \left\lfloor \frac{\lambda}{d_0} \right\rfloor^{2d_0}$$
$$+ \sum_{m=1}^{k} (\lambda - k)^{2m}.|q_{d_m}(\lambda^2 + 2\lambda)|^2. \left\lfloor \frac{\lambda}{d_m} \right\rfloor^{2d_m}.$$

We can give an estimate of the degrees of q_1 and q_2. Namely, the degree of q_2 is equal to $\max_{-k \le m \le k}\{2.deg(R_m) + 2|m| + 2.d_m + 1\}$ and the degree q_1 is equal to $\max_{-k \le m \le k}\{4.deg(q_{d_m}) + 2|m| + 2.d_m\}$. (Note that $2.deg(q_{d_m}) \le deg(R_m)$.) $\square$

The ultraproduct of the norms induces a topology both on $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$ (under which $+$ and . are continuous) and on U. A basis of neighbourhoods O_ϵ of 0 (in U) is given by $O_\epsilon := \{u \in U : \|u\| \le \epsilon\}$, where $\epsilon \in \mathbb{R}^{*,+} - \{0\}$. When we just consider them as topological spaces, we will call them $\star$-normed spaces.

Then, we will consider the following topological subspaces $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$ (dense in $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$) and $\prod_{\mathcal{V}} SL_{\lambda+1}(\mathbb{C})$ which is a closed subspace of $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$.

Lemma 10.2 (*See [9] Corollary 6.2.32*). *Let $A, B \in M_\lambda(\mathbb{C})$, then $|| \exp(A + B) - \exp(A)||_\lambda \le ||B||_\lambda \exp(||B||_\lambda) \exp(||A||_\lambda)$. So, the exponential map is continuous on $M_\lambda(\mathbb{C})$ and Lipschitz continuous on each compact subset of $M_\lambda(\mathbb{C})$.* $\square$

Proposition 10.3. *Consider the $\star$-normed spaces $(U, \|\cdot\|)$ and $(\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C}), \|\cdot\|_{\lambda+1})$. The map $\mathrm{EXP} : U \to \prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$ is continuous and maps bounded sets to bounded sets. The image $\mathrm{EXP}(U_0)$ is an abelian subgroup of $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$ and $\mathrm{EXP}(\oplus_{m\neq 0}U_m)$ is included in $\prod_{\mathcal{V}} SL_{\lambda+1}(\mathbb{C})$.*

Proof. The continuity is clear from Łos theorem and the preceding lemma.

Note that if the sequence $A_{\lambda+1} \in M_{\lambda+1}(\mathbb{C})$ is bounded, namely the sequence $\|A_{\lambda+1}\|_{\lambda+1}$ is bounded, then the corresponding sequence $\|\exp(A_{\lambda+1})\|_{\lambda+1}$ is bounded. Indeed, by definition, $\exp(A_{\lambda+1}) = \sum_{k=0}^{\infty} \frac{A_{\lambda+1}^k}{(k)!}$, so the norm $\|\exp(A_{\lambda+1})\|_{\lambda+1} \leq e^{\|A_{\lambda+1}\|_{\lambda+1}}$.

The last statement follows from Proposition 7.5. □

Note that a priori, $\mathrm{EXP}(U)$ is not a subgroup of $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$; we will denote by $\langle \mathrm{EXP}(U)\rangle$ the subgroup generated by $\mathrm{EXP}(U)$ in $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$. The Campbell–Baker–Hausdorff formula which expresses for two matrices A, B, $\exp(A) \cdot \exp(B)$ as $\exp(C)$ where C is expressed as an infinite series in commutators in A and B, can be translated back with u and v in place of A and B to express $\mathrm{EXP}(u) \cdot \mathrm{EXP}(v)$ in terms of an infinite series in u, v ([16] section 1.3).

Does $\langle \mathrm{EXP}(U)\rangle$ have finite width with respect to $\mathrm{EXP}(U)$, namely does there exist a finite number k such that every element of $\langle \mathrm{EXP}(U)\rangle$ can be written as a product of k elements of $\mathrm{EXP}(U)$?

We consider the field $\mathcal{R} := (\mathbb{R}, +, ., 0, 1, e^x)$, and we denote by $\mathcal{R}^*$ a non-principal ultrapower of $\mathcal{R}$ with respect to the ultrafilter $\mathcal{V}$ on ω. We will extend the exponential map EXP to $U \otimes \mathbb{R}^*$ as follows. Let $u \in U$ and $s := [r_\lambda]_{\mathcal{V}} \in \mathbb{R}^*$ with $r_\lambda \in \mathbb{R}$, then $\mathrm{EXP}(u \otimes s) := Exp[r_\lambda.\theta_\lambda(u)]_{\mathcal{V}} = Exp[\theta_\lambda(r_\lambda.u)]_{\mathcal{V}}$ and $\mathrm{EXP}(\sum_i u_i \otimes s_i) := Exp[\theta_\lambda(\sum_i u_i.r_{i,\lambda})]_{\mathcal{V}}$, where $s_i := [r_{i,\lambda}]_{\mathcal{V}}$. Note that $\sum_i u_i.r_{i,\lambda} \in U$. This is well defined.

We will say that a topological group G is $\star$-path connected if given any two elements $h_0, h_1 \in G$, there is a continuous map g from $[0;1]^* := \mathbb{R}^* \cap [0;1]$ to G with $g(0) = h_0$ and $g(1) = h_1$.

Proposition 10.4. *The subgroups $\langle \mathrm{EXP}(U)\rangle$ and $\mathrm{EXP}(U_0)$ of $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$ (respectively $\langle \mathrm{EXP}(U \otimes \mathbb{R}^*)\rangle$ and $\mathrm{EXP}(U_0 \otimes \mathbb{R}^*)$ are topological groups. Moreover, $\langle \mathrm{EXP}(U \otimes \mathbb{R}^*)\rangle$ and $\mathrm{EXP}(U_0 \otimes \mathbb{R}^*)$ are $\star$-path connected.*

Proof. First note that $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$ is a topological group as an ultraproduct of topological groups. So, the subgroups $\langle \mathrm{EXP}(U)\rangle$, $\mathrm{EXP}(U_0)$, $\langle \mathrm{EXP}(U \otimes \mathbb{R}^*)\rangle$ and $\mathrm{EXP}(U_0 \otimes \mathbb{R}^*)$ are topological subgroups.

The groups $\langle \mathrm{EXP}(U \otimes \mathbb{R}^*)\rangle$ and $\mathrm{EXP}(U_0 \otimes \mathbb{R}^*)$ are $\star$-path connected. We only prove that $\langle \mathrm{EXP}(U \otimes \mathbb{R}^*)\rangle$ is $\star$-path connected. Let $g_0, \; g_1 \in \langle \mathrm{EXP}(U \otimes \mathbb{R}^*)\rangle$. Then we can write $g_1 = \mathrm{EXP}(u_1) \cdot \ldots \cdot \mathrm{EXP}(u_n)$ and $g_0 = \mathrm{EXP}(v_1) \cdot \ldots \cdot \mathrm{EXP}(v_m)$, where $u_1, \cdots, u_n, v_1, \cdots, v_m \in U \otimes \mathbb{R}^*$. So, $g_1 = g_0 \cdot \mathrm{EXP}(y_1) \cdot \ldots \cdot \mathrm{EXP}(y_k)$, for some $y_1, \cdots, y_k \in U \otimes \mathbb{R}^*$. Let $t \in [0;1]^*$ and set $g(t) = g_0 \cdot \mathrm{EXP}(t \cdot y_1) \cdot \ldots \cdot \mathrm{EXP}(t \cdot y_k)$, so $g(0) = g_0$ and $g(1) = g_1$. Let us denote the set $\{g \in \langle \mathrm{EXP}(U)\rangle : \exists t \in [0;1]^* \;\; g = \mathrm{EXP}(t \cdot y_1) \cdot \ldots \cdot \mathrm{EXP}(t \cdot y_k)\}$ by C_{g_0,g_1}.

First, let us check that the map from $[0;1]^*$ to $\mathrm{EXP}(U)$, sending t to $\mathrm{EXP}(tu)$ is continuous at $t_1 \in [0;1]^*$.

Let $\epsilon \in [0;1]^*$, then we have to find η such that if $|t_0 - t_1| < \eta$, then $\|\mathrm{EXP}(t_0 \cdot u) - \mathrm{EXP}(t_1 \cdot u)\| \leq \epsilon$. We have $\mathrm{EXP}(t_0 \cdot u) - \mathrm{EXP}(t_1 \cdot u) = \mathrm{EXP}(t_1 \cdot u) \cdot [\mathrm{EXP}((t_0 - t_1) \cdot u) - 1]$. So, $\|\mathrm{EXP}(t_0 \cdot u) - \mathrm{EXP}(t_1 \cdot u)\| \leq \|\mathrm{EXP}(t_1 \cdot u)\| \cdot \|\mathrm{EXP}((t_0 - t_1) \cdot u) - 1]\|$. Now, $\|\mathrm{EXP}((t_0 - t_1) \cdot u) - 1]\| \leq |(t_0 - t_1)| \cdot \|u\| \cdot e^{\|((t_0 - t_1) \cdot u)\|}$.

Then we use the fact that the product (possibly non-commutative) of two continuous functions is continuous $(*)$. So, by induction on n, we may deduce that the map sending t to $\mathrm{EXP}(t \cdot y_1) \cdot \mathrm{EXP}(t \cdot y_2) \cdot \ldots \cdot \mathrm{EXP}(t \cdot y_k)$ is also continuous.

Now suppose $\langle \mathrm{EXP}(U)\rangle$ is the disjoint union of two open sets U_1 and U_2. Denote the intersection of U_1 (respectively U_2) with C_{g_0,g_1} by O_1 (respectively O_2). The inverse image of O_1 and O_2 gives rise to a partition of $[0;1]^*$, which is a contradiction.

For convenience of the reader, let us prove $(*)$. Let $f(t), g(t)$ be two continuous maps on the interval $[0;1]^*$ and assume one of them is bounded. Then consider the map sending t to the product $f(t) \cdot g(t)$; let us show it is continuous at t_1, assuming that f is bounded. Estimate the difference: $f(t) \cdot g(t) - f(t_1) \cdot g(t_1) = (f(t) - f(t_1)) \cdot g(t_1) + f(t) \cdot (g(t) - g(t_1))$. So, $\|f(t) \cdot g(t) - f(t_1) \cdot g(t_1)\| \leq \|(f(t) - f(t_1))\| \cdot \|g(t_1)\| + \|f(t)\|.\|(g(t) - g(t_1))\|$. Note that the map sending t to $\mathrm{EXP}(tu)$ is bounded. Indeed, $\|\mathrm{EXP}(tu)\| \leq e^{\|t \cdot u\|} \leq e^{|t| \cdot \|u\|} \leq e^{\|u\|}$. □

11. The asymptotic cone

In the previous section, we embedded U in a $\star$-normed space, namely $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$. Here, we will embed U into a complete metric space (with an $\mathbb{R}$-valued metric) which will be the asymptotic cone associated with the family of normed algebras $M_{\lambda+1}(\mathbb{C})$, $\lambda \in \omega$, and a non-principal ultrafilter $\mathcal{V}$ on ω. We will first endow each $M_{\lambda+1}(\mathbb{C})$ with a new norm scaled down by λ; this norm differs from the norms we previously introduced in the fact that the norms of $\Theta_\lambda(x)$, $\Theta_\lambda(y)$, $\Theta_\lambda(c)$, $\Theta_\lambda(h)$ will be a multiple of λ (see Proposition 11.2).

Even though they did not name it *asymptotic cone*, it was introduced by van den Dries and Wilkie when they revisited Gromov's proof that a finitely generated group of polynomial growth is nilpotent-by-finite. Given a group of polynomial growth, Gromov associated a converging sequence of discrete metric spaces scaled down by a sequence of well-chosen natural numbers. Then, van den Dries and Wilkie associated with any finitely generated group G a limited ultraproduct of discrete metric spaces quotient out by infinitesimals. This space is usually denoted by $Cone(X, \mathcal{V})$, where X is a metric space associated with G and $\mathcal{V}$ a non-principal ultrafilter on ω, note that $Cone(X, \mathcal{V})$ may depend on $\mathcal{V}$ (see for instance [12,3]). The advantage of using an ultraproduct construction is that one can easily transfer certain properties from the factors.

First, we introduce the map ϕ from $M_{\lambda+1}(\mathbb{C})$ to $\mathbb{N}$, sending $A \in M_{\lambda+1}(\mathbb{C})$ to the number of non-zero coefficients of A. Of course, $\phi(A) = 0$ iff $A = 0$.

Let us check that

(1) $\phi(A + B) \leq \phi(A) + \phi(B)$,

(2) $\phi(A \cdot B) \le \phi(A) \cdot \phi(B)$.

We denote the ij coefficient of $A + B$ by $(A + B)_{ij}$. We have that if $(A + B)_{ij} \neq 0$, then either $A_{ij} \neq 0$ or $B_{ij} \neq 0$.

Let $C := A \cdot B$, then $C_{ij} = \sum_k A_{ik} \cdot B_{kj}$ and so $C_{ij} \neq 0$ implies that for some k, $A_{ik} \neq 0$ and $B_{kj} \neq 0$. We prove the second claim by induction on the number $\phi(C)$. For $\phi(C) = 1$, it is clear. By induction suppose that for any $1 \le n \le m$, if $\phi(C) = n$, then for some 2-tuple (k_1, k_2) with $k_1 \ge 1$, $k_2 \ge 1$, we have $\phi(A) \ge k_1$ and $\phi(B) \ge k_2$ and $n \le k_1 \cdot k_2$.

Assume now that $\phi(C) = m+1$, so there are $m+1$ tuples (i, j) with $C_{ij} \neq 0$. For each of these tuples, there are two tuples (i, k), (k, j) such that $A_{ik} \neq 0$ and $B_{kj} \neq 0$. By induction corresponding to the first m non-zero tuples, we know that there are k_1 (respectively k_2) non-zero coefficients of the matrix A (respectively of the matrix B) which are non-zero and such that $m \le k_1.k_2$. Corresponding to the $m + 1$ non-zero coefficient of C, there exists another non-zero coefficient of either A or B and so either $\phi(A) \ge k_1 + 1$, or $\phi(B) \ge k_2 + 1$, so $m + 1 \le \min\{(k_1 + 1) \cdot k_2, k_1 \cdot (k_2 + 1)\}$.

So, this map ϕ defines a norm on $M_{\lambda+1}(\mathbb{C})$, that we will denote by $\| \cdot \|_{c,\lambda+1}$.

In the ultraproduct $\prod_{\mathcal{V}}(M_{\lambda+1}(\mathbb{C}), \frac{\|\cdot\|_{c,\lambda+1}}{\lambda})$, we consider the set $\prod^*_{\mathcal{V}}(M_{\lambda+1}(\mathbb{C}), \frac{\|\cdot\|_{c,\lambda+1}}{\lambda})$ of elements $[a_\lambda]$ such that for some natural number N, we have $\{\lambda \in \omega : \|a_\lambda\|_{c,\lambda} \le N \cdot \lambda\} \in \mathcal{V}$. Then we quotient out this set by the equivalence relation $\sim$ defined by $[a_\lambda]_{\mathcal{V}} \sim [b_\lambda]_{\mathcal{V}}$ if $(\frac{\|a_\lambda - b_\lambda\|_{c,\lambda}}{\lambda}) \to_{\mathcal{V}} 0$. Let us denote the equivalence class of an element by $[a_\lambda]_\sim$ and by st the standard part of an element of $\prod_{\mathcal{V}} \mathbb{R}$ whose absolute value is bounded by some natural number.

On $X_{\mathcal{V}} := \prod^*_{\mathcal{V}}(M_{\lambda+1}(\mathbb{C}), \frac{\|\cdot\|_{c,\lambda+1}}{\lambda})/\sim$, we define the following distance with values in $\mathbb{R}^{\ge 0}$.

Let $a := [a_\lambda]_\sim$ and $b := [b_\lambda]_\sim$, then $d(a, b) := st([\frac{\|a_\lambda - b_\lambda\|_{c,\lambda}}{\lambda}])$.

Lemma 11.1. *The space $(X_{\mathcal{V}}(\mathbb{C}), d)$ is an infinite-dimensional complete metric space.*

Proof. The only thing which remains to be checked is the completeness of the space, but this follows from the countable saturation of the ultraproduct (see [2] Theorem 6.1.1). □

We will say that $(X_{\mathcal{V}}(\mathbb{C}), d)$ is the asymptotic cone associated with $\{(M_{\lambda+1}(\mathbb{C}), \frac{\|\cdot\|_{c,\lambda+1}}{\lambda}); \lambda \in \mathbb{N}\}$ and $\mathcal{V}$.

Proposition 11.2. *The universal enveloping Lie algebra U of $sl_2(\mathbb{C})$ embeds in $(X_{\mathcal{V}}(\mathbb{C}), d)$ via its embedding in the ultraproduct of the matrix rings.*

Proof. We proceed in two steps.

Firstly, we show that for any $u \in U$, $[\Theta_\lambda(u)]$ belongs to $\prod^*_{\mathcal{V}}(M_{\lambda+1}(\mathbb{C}), \frac{\|\cdot\|_{c,\lambda+1}}{\lambda})$. This is direct by inspection of the proof of Proposition 7.4.

Secondly, let $u \in \oplus_{|j| \le m} U_j$, then there exist $d_0, d_1, \ldots, d_m, d_{-1}, \ldots, d_{-m}$ such that for all $\lambda \in \mathbb{N}$, $\phi(\Theta_\lambda(u)) = (\lambda - d_0) + \sum_{j=1}^m ((\lambda - i) - d_i) = \lambda.m - (m(m + 1))/2 - \sum_{i=1}^m d_i - \sum_{i=-1}^{-m} d_{-i}$. Again, this is seen by inspection of the proof of Proposition 7.4.

So if $[\Theta_\lambda(u)] \sim 0$, then $[\Theta_\lambda(u)] = 0$. □

We will denote the image of U in $(X_{\mathcal{V}}(\mathbb{C}), d)$ by $U_\sim$.

Definition 11.1. A matrix (a_{ij}) in $M_{\lambda+1}(\mathbb{C})$ is called a m-band matrix if there exists m such that for any $1 \le i, j \le \lambda + 1$, we have $a_{ij} \neq 0$ implies that $|i - j| \le m$. (Namely, the non-zero entries of a m-band matrix are confined to a diagonal band comprising the main diagonal and the adjacent m *diagonals* on either side.) The band-width is equal to $2.m + 1$.

Proposition 11.3. *Every element of U acts by left multiplication on the image $U_\sim$ of U in $(X_{\mathcal{V}}(\mathbb{C}), d)$, in a continuous way. More generally, any element $[a_\lambda]_{\mathcal{V}} \in \prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$ acts by left multiplication on $(X_{\mathcal{V}}(\mathbb{C}), d)$, whenever there exists m independently of λ such that a_λ is a m-band matrix.*

Proof. Let $u, v \in U$. Let us show that $[\Theta_\lambda(u)].[\Theta_\lambda(v)]_\sim$ is well defined. Namely, if $[\epsilon_\lambda] \in \prod_{\mathcal{V}}(M_{\lambda+1}(\mathbb{C})$ with $[\epsilon_\lambda] \sim 0$, then $\Theta_\lambda(u) \cdot \epsilon_\lambda \sim 0$. Assume that $u \in \oplus_{|j| \le m} U_j$, so $\Theta_\lambda(u)$ is a band matrix of width $\le m$. Namely $\Theta_\lambda(u)_{ij} = 0$ unless $|i - j| \le m$. So, if we denote by c the matrix in $M_{\lambda+1}(\mathbb{C})$ which is the product $\Theta_\lambda(u) \cdot \epsilon_\lambda$, then $c_{ij} = \sum_{k=1}^{\lambda+1} \Theta_\lambda(u)_{ik} \cdot \epsilon_{kj}$. If we fix the matrix element ϵ_{kj}, then there are at most $2m$ indices i such that $c_{ij} \neq 0$. Now since $[\epsilon_\lambda] \sim 0$, $lim \frac{\phi(\epsilon_\lambda)}{\lambda} = 0$. We have that $\phi(c) \le \phi(\epsilon_\lambda) \cdot 2m$, so $lim \frac{\phi(c)}{\lambda} \le lim \frac{\phi(\epsilon_\lambda)}{\lambda} \cdot 2m = 0$.

This action is continuous. Let $\epsilon > 0$, choose $\eta := \frac{\epsilon.\lambda}{\phi(\Theta_\lambda(u))}$. Then for $v_1, v_2 \in U_\sim$, if $d(v_1, v_2) \le \eta$, then $d(u.v_1, u.v_2) \le \epsilon$. Indeed, we have $\|\Theta_\lambda(u).\Theta_\lambda(v_1) - \Theta_\lambda(u).\Theta_\lambda(v_2)\|_{c,\lambda} = \phi(\Theta_\lambda(u).\Theta_\lambda(v_1) - \Theta_\lambda(u).\Theta_\lambda(v_2)) = \phi(\Theta_\lambda(u).(\Theta_\lambda(v_1) - \Theta_\lambda(v_2))) \le \phi(\Theta_\lambda(u)).\phi(\Theta_\lambda(v_1) - \Theta_\lambda(v_2))$. □

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