On the asymptotics of solutions of the Lane-Emden problem for the p-Laplacian

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Abstract. In this paper we consider the Lane–Emden problem adapted for the p-Laplacian

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $\lambda > 0$ and $p < q < p^*$ (with $p^* = \frac{np}{n-p}$ if $p < n$, and $p^* = \infty$ otherwise). After some recalls about the existence of ground state and least energy nodal solutions, we prove that, when $q \to p$, accumulation points of ground state solutions or of least energy nodal solutions are, up to a "good" scaling, respectively first or second eigenfunctions of $-\Delta_p$.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. We consider the Lane-Emden equation for the p-Laplacian, that is

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here is $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\lambda > 0$, $1 < p < q < p^*$ (with $p^* = \frac{np}{n-p}$ if $p < n$, and $p^* = +\infty$ otherwise). In this paper we are interested in the existence and

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the asymptotic behaviour, as $q \to p$, of ground state solutions and of least energy nodal solutions. The existence question for $p = 2$ and $1 < q < 2^*$ was already studied in 1973 by Ambrosetti and Rabinowitz in [2], where it was shown that the problem admits a positive ground state solution. The existence of a nodal solution with least energy among nodal solutions was proved in [7] by Castro, Cossio and Neuberger in 1997.

Recently Bonheure, Bouchez, Grumiau and Van Schaftingen ([6]) have shown that for $p = 2$ any accumulation point of sequences of (renormalized) least energy nodal solutions is a second eigenfunction that minimizes a reduced functional on a reduced Nehari manifold. The approch of [6] works and could provide an alternative proof (for $p = 2$) to Theorem 3.1 (ii), which states that accumulation points of ground state solutions belong to the first eigenspace of $-\Delta$, when $q \to 2$. [6] contains also a study of the possible symmetries of least energy nodal solutions.

For $p \neq 2$ things seem to become more complicated, due to the lack of linearity of the p-Laplacian and the fact that the associated energy functional is defined on a Banach space which is not a Hilbert space. Nevertheless, it is logical to suppose that ground state solutions and least energy nodal solutions of our problem converge, after being suitably scaled, to solutions of the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p u &= \lambda |u|^{p-2}u, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega. \end{cases} \tag{2}$$

In contrast to the linear case, this problem has not yet completely been solved. It is known (see e.g. [9]) that there exists a sequence of eigenvalues (defined in (3) below)

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_n \leq ...$$

with $\lambda_n \to +\infty$, but it is still an open question whether other eigenvalues can exist. The first eigenvalue can be characterized as

$$\lambda_1 := \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}$$

and there exists only one eigenfunction e_1 (up to a multiplicative factor) which has constant sign (for a new and direct proof, see [5]). The higher eigenvalues can be obtained through the following minimax principle: let us define the *Krasnoselskii genus* of a set $A \subseteq W_0^{1,p}(\Omega)$:

$$\gamma(A) := \min\left\{k \in \mathbb{N} \,|\, \exists f : A \mapsto \mathbb{R}^k \setminus \{0\},\ f \text{ continuous and odd}\right\}.$$

Define

$$\Gamma_k := \left\{A \subseteq W_0^{1,p}(\Omega) \,|\, A \text{ symmetric, } A \cap \{||v||_p = 1\} \text{ compact, } \gamma(A) \geq k\right\}.$$

Then,

$$\lambda_k := \inf_{A \in \Gamma_k} \sup_{u \in A} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}. \tag{3}$$

It turns out that there do not exist other eigenvalues between λ_1 and λ_2 (see [3]). Eigenfunctions associated with higher eigenvalues must be sign-changing. Notice that if u and v are eigenfunctions with identical eigenvalue, $u+v$ is in general not an eigenfunction, due to the nonlinearity of the problem. Due to elliptic regularity theory eigenfunctions belong to $\mathcal{C}^{1,\alpha}_{loc}(\Omega)$ for $0<\alpha<1$ (see [8]).

By default we denote by $\|.\|$ the norm in $W^{1,p}_0(\Omega)$ defined as

$$||u|| = \left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}}.$$

The norm $\|.\|_X$ will denote the traditional norm in the Banach space X, and $\|.\|_p$ will stand for the L^p-norm.

In Section 2, we recall some known results on our problem. To be complete, we give a sketch of the proof of the existence of ground state solution and least energy nodal solution for the problem (1). We use a idea presented in [7]. In [4] the authors prove the existence of a nodal solution using a different method.

In Section 3 we prove our main result:

Proposition 1.1. *Let $(u_q)_{q>p}$ be a family of ground state solutions (resp. least energy nodal solutions) of the Lane-Emden problem (1). Then, if $\lambda<\lambda_1$ (resp. $\lambda<\lambda_2$), u_q diverges to infinity as $q\to p$. If $\lambda=\lambda_1$ (resp. $\lambda=\lambda_2$), $u_q\to u_*\neq 0$ in $L^p(\Omega)$, as $q\to p$, where the function u_* solves the equation*

$$\begin{cases} -\Delta_p u_* &= \lambda |u_*|^{p-2}u_*, \qquad \text{in } \Omega,\\ \quad u_* &= 0, \qquad\qquad\quad \text{on } \partial\Omega.\end{cases}$$

Finally, if $\lambda>\lambda_1$ (resp. $\lambda>\lambda_2$), u_q converges to 0, as $q\to p$.

2. Existence of solutions

Let us fix $1<p<+\infty$ and $p<q<p^*$. For the sake of completeness we will prove the existence of at least two non-trivial solutions to the Lane-Emden problem (1). In particular we prove the existence of a ground state solution (non-trivial solution with minimum energy) and a least energy nodal solution (sign-changing solution with minimum energy). In order to do this, we introduce the energy functional

$$\varphi_q(u) := \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{q}\int_\Omega |u|^q$$

defined on $W^{1,p}_0(\Omega)$. A function u is a solution of (1) if and only if it is a critical point of φ_q. Remark that φ_q is a $\mathcal{C}^2$ functional for $p\geq 2$ and $\mathcal{C}^1$ functional for $1<p<2$.

Let us define the first variation of φ_q at u in direction v

$$\mathrm{d}\varphi_q(u)(v) := \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v - \lambda\int_\Omega |u|^{q-2}uv$$

and the *Nehari manifold*

$$\mathcal{N}_q := \{u \in W_0^{1,p}(\Omega) \setminus \{0\} \,|\, \mathrm{d}\varphi_q(u)(u) = 0\}.$$

Clearly, all the non-trivial solutions belong to $\mathcal{N}_q$. We will also make use of the *positive Nehari manifold*

$$\mathcal{N}_q^+ := \{u \in \mathcal{N}_q \,|\, u \geq 0\},$$

of the *negative Nehari manifold*

$$\mathcal{N}_q^- := \{u \in \mathcal{N}_q \,|\, u \leq 0\}$$

and of the *nodal Nehari set*

$$\mathcal{M}_q := \{u \in \mathcal{N}_q \,|\, u^+ \in \mathcal{N}_q^+,\, u^- \in \mathcal{N}_q^-\},$$

where we defined the positive part $u^+ := \max(0, u)$ and the negative part $u^- := \min(0, u)$.

Notice that by definition the functions belonging to $\mathcal{M}_q$ are sign-changing. Moreover, all sign-changing solutions of the problem belong to $\mathcal{M}_q$. The following results prove that ground state solutions are characterized by functions minimizing the energy functional in $\mathcal{N}_q$ and least energy nodal solutions are characterized by functions minimizing the energy functional in $\mathcal{M}_q$.

Proposition 2.1. *For every $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, there exists one and only one $t_q^* > 0$ such that $t_q^* u \in \mathcal{N}_q$. Moreover,*

$$\varphi_q(t_q^* u) = \max_{t>0} \varphi_q(tu).$$

Proof. For $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, we have

$$tu \in \mathcal{N}_q \Leftrightarrow \int_\Omega |\nabla(tu)|^p - \lambda \int_\Omega |tu|^q = 0 \Leftrightarrow t^p \int_\Omega |\nabla u|^p - \lambda t^q \int_\Omega |u|^q = 0.$$

The last equation admits

$$t_q^* := \left(\frac{\int_\Omega |\nabla u|^p}{\lambda \int_\Omega |u|^q} \right)^{\frac{1}{q-p}} \tag{4}$$

as unique positive solution. For $t \geq 0$ we define

$$\psi(t) := \varphi_q(tu) = \frac{1}{p} \int_\Omega |\nabla(tu)|^p - \frac{\lambda}{q} \int_\Omega |tu|^q = \frac{t^p}{p} \int_\Omega |\nabla u|^p - \frac{\lambda t^q}{q} \int_\Omega |u|^q.$$

We have

$$\psi'(t) = t^{p-1} \int_\Omega |\nabla u|^p - \lambda t^{q-1} \int_\Omega |u|^q,$$

so that the only positive critical point is $t = t_q^*$. Since $\psi(0) = 0$ and $\psi(t) \to -\infty$ as $t \to +\infty$, t_q^* must be a maximum point, which means

$$\varphi_q(t_q^* u) = \max_{t>0} \varphi_q(tu).$$

□

By the previous result and since the support of u^+ and u^- are disjoint, we obtain

Corollary 2.2. *For every $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, the numbers $t_q^+, t_q^- > 0$ such that $t_q^+ u^+ + t_q^- u^- \in \mathcal{M}_q$ are uniquely defined.*

Proposition 2.3. *The Nehari manifold $\mathcal{N}_q$ is closed in $W_0^{1,p}(\Omega)$.*

Proof. Since φ_q is of class $\mathcal{C}^1$, it is clear that $\mathcal{N}_q \cup \{0\}$ is closed. So we must prove that 0 is not an accumulation point for $\mathcal{N}_q$; this follows from the fact that the $W_0^{1,p}$-norm of every function $u \in \mathcal{N}_q$ is uniformly bounded from below. Indeed, from Sobolev's embedding Theorem we have

$$||\nabla v||_p \geq C||v||_q \qquad \forall\ v \in W_0^{1,p}(\Omega).$$

For $v \in W_0^{1,p}(\Omega) \setminus \{0\}$ the unique positive multiplicative function $t_q^* v \in \mathcal{N}_q$ (with t_q^* as in (4)) satisfies

$$||\nabla(t_q^* v)||_p \geq C||t_q^* v||_q = C\left(\frac{||\nabla v||_p^p}{\lambda||v||_q^q}\right)^{\frac{1}{q-p}} ||v||_q = C\lambda^{-\frac{1}{q-p}}\left(\frac{||\nabla v||_p}{||v||_q}\right)^{\frac{p}{q-p}} \geq C^{\frac{q}{q-p}}\lambda^{-\frac{1}{q-p}}.$$

□

The following result proves that we can compute the minimum of the energy on the positive and negative Nehari manifold, and on the nodal Nehari set. The idea for it is the same as the one used by Castro, Cossio and Neuberger in [7].

Proposition 2.4. *The infima*

$$\inf_{u\in\mathcal{N}_q^+} \varphi_q(u), \ \inf_{u\in\mathcal{N}_q^-} \varphi_q(u), \ \inf_{u\in\mathcal{M}_q} \varphi_q(u)$$

are attained.

Proof. We give a proof for $\mathcal{M}_q$. The arguments are the same for $\mathcal{N}_q^+$ and $\mathcal{N}_q^-$. Let us define $c := \inf_{\mathcal{M}_q} \varphi_q$ and consider $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}_q$ such that $\varphi_q(u_n) \to c$.

Since $\varphi_q(v) = \left(\frac{1}{p} - \frac{1}{q}\right) \|v\|$ for any $v \in \mathcal{N}_q$, we obtain that $(u_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$. So, up to a subsequence, there exist u, v and w such that $u_n \rightharpoonup u$, $u_n^+ \rightharpoonup v$ and $u_n^- \rightharpoonup w$ in $W_0^{1,p}(\Omega)$. By Sobolev's embedding Theorem and as the function $u \mapsto u^\pm$ is continuous, we obtain that $u^+ = v$ and $u^- = w$.

By Proposition 2.3, the Nehari manifold $\mathcal{N}_q$ is closed in $W_0^{1,p}(\Omega)$. We obtain that

$$\lambda \int_\Omega |u^+|^q = \lambda \lim_{n\to+\infty} \int_\Omega |u_n^+|^q = \lim_{n\to+\infty} \|u_n^+\|^p > 0.$$

So u is a sign-changing function.

It remains to verify that $u \in \mathcal{M}_q$ and $u_n \to u$ in $W_0^{1,p}(\Omega)$. In fact, it suffices to prove that $u_n^+ \to u^+$ and $u_n^- \to u^-$ in $W_0^{1,p}(\Omega)$. Otherwise we can assume w.l.o.g. that u_n^+ does not converge to u^+. Then $\|u^+\|^p < \liminf_{n\to+\infty} \|u_n^+\|^p$, which implies that $\mathrm{d}\varphi_q(u^+)(u^+) < 0$. So u^+ does not belong to the Nehari manifold. By

Proposition 2.1, there exist $0 < \alpha < 1$ and $0 < \beta \leq 1$ such that $\alpha u^+ + \beta u^-$ belongs to $\mathcal{M}_q$. In fact, we have

$$\varphi_q(\alpha u^+ + \beta u^-) < \liminf_{n\to+\infty}(\varphi_q(\alpha u_n^+) + \varphi_q(\beta u_n^-)) \leq \liminf_{n\to+\infty} \varphi_q(u_n) = c,$$

which is a contradiction. So the minimum of the energy on $\mathcal{M}_q$ is attained in u. □

The following results show that the functions found in Proposition 2.4 are solutions of the problem (1). Remark that, as the positive part and the negative part of a solution belong to the Nehari manifold and as the energy of the positive or negative part is strictly less than the energy of the solution, we obtain that the functions which minimize energy on the positive Nehari manifold or negative Nehari manifold are ground state solutions of the problem (1). We will make use of the following lemma, also known as Miranda's theorem.

Lemma 2.5. *Let $B \subseteq \mathbb{R}^n$ be a closed ball, let $f : B \to \mathbb{R}^n$ be a continuous function. If f points inside B on ∂B, then f possesses a zero in B.*

Proof. A proof of this theorem can be found for instance in [1]. □

Proposition 2.6. *If $u_q \in \mathcal{M}_q$ (resp. $\mathcal{N}_q^+$ or $\mathcal{N}_q^-$) is such that $\varphi_q(u_q) = \inf_{u\in\mathcal{M}_q} \varphi_q(u)$ (resp. $\inf_{u\in\mathcal{N}_q^+} \varphi_q(u)$ or $\inf_{u\in\mathcal{N}_q^-} \varphi_q(u)$), then u_q is a critical point for φ_q.*

Proof. We give the proof for $\mathcal{M}_q$. The arguments are essentially the same for the two other cases. We only need to think that a minimum on $\mathcal{N}_q^+$ or $\mathcal{N}_q^-$ is a minimum on $\mathcal{N}_q$. So, for the two other cases, we do not need that the deformation used in the next part of the proof stays in the positive Nehari or negative Nehari manifold.

Let fix $c := \min_{\mathcal{M}_q} \varphi_q$. Let us suppose that u_q is not a critical point for φ_q. Since φ_q is of class $\mathcal{C}^1$, it is possible to find a ball B with $u_q \in B$ and such that, for $\varepsilon > 0$,

$$c - \varepsilon \leq \varphi_q(u) \leq c + \varepsilon \quad , \forall\, u \in B,$$

and

$$\| \mathrm{d}\varphi_q(u)\|_{(W_0^{1,p})'} \geq \frac{\varepsilon}{2}, \quad \forall\, u \in B,$$

Let us consider the quarter of a hyperplane π defined as

$$\pi := \{\alpha u_q^+ + \beta u_q^- \mid \alpha, \beta > 0\}.$$

Notice that, from Proposition 2.1, u_q is the unique global maximum of φ_q on π. By the deformation Lemma (see [10], Proposition 5.1.25), there exists a deformation Γ such that

1. $\varphi_q(\Gamma(t,u)) < c$ for $u \in B \cap \pi$ and $t \in [0,1]$,
2. $\Gamma(t,u) = u$ for $u \in \partial B \cap \pi$ and $t \in [0,1]$, and
3. $||\Gamma(t,u) - u|| \leq 8t$ for $u \in B \cap \pi$ and $t \in [0,1]$.

Because of the compactness of $B \cap \pi$, it is possible to find $t^* > 0$ such that $\Gamma(t^*, u)$ is a sign-changing function for every $u \in B \cap \pi$.

Now, we consider the application

$$\psi : \pi \to \mathbb{R} \times \mathbb{R} : v \mapsto (\,\mathrm{d}\varphi_q(\Gamma(t^*, v)^+)(\Gamma(t^*, v)^+),\ \mathrm{d}\varphi_q(\Gamma(t^*, v)^-)(\Gamma(t^*, v)^-)).$$

Since $\Gamma(t^*, v) = v$ on ∂B, we obtain that the vector field points inwards on ∂B. Using Lemma 2.5 we obtain that there exists $w \in B \cap \pi$ such that $\Gamma(t^*, w) \in \mathcal{M}_q$. This is a contradiction because $\varphi_q(\Gamma(t^*, w)) < c$. □

3. Convergence results

In this Section we study the asymptotic behavior of ground state solutions u_q (resp. least energy nodal solutions) of the Problem (1) when q goes to p. We prove that there exist suitable positive constants C_1 and C_2 such that

$$C_1 \left(\frac{\lambda_1}{\lambda}\right)^{\frac{1}{q-p}} \le \|u_q\| \le C_2 \left(\frac{\lambda_1}{\lambda}\right)^{\frac{1}{q-p}}$$

if u_q is a ground state solution, and

$$C_1 \left(\frac{\lambda_2}{\lambda}\right)^{\frac{1}{q-p}} \le \|u_q\| \le C_2 \left(\frac{\lambda_2}{\lambda}\right)^{\frac{1}{q-p}}$$

if u_q is a least energy nodal solution. We are able to state the following result.

Theorem 3.1. *As $q \to p$, the ground state solutions of Problem (1):*
(i) diverge to infinity, up to a subsequence, if $\lambda < \lambda_1$;
(ii) converge to a first eigenfunction of the p-Laplacian, up to a subsequence, if $\lambda = \lambda_1$;
(iii) converge to zero, up to a subsequence, if $\lambda > \lambda_1$.

Theorem 3.2. *As $q \to p$, the least energy nodal solutions of Problem (1):*
(i) diverge to infinity, up to a subsequence, if $\lambda < \lambda_2$;
(ii) converge to a second eigenfunction of the p-Laplacian, up to a subsequence, if $\lambda = \lambda_2$;
(iii) converge to zero, up to a subsequence, if $\lambda > \lambda_2$.

We mention that the case $\lambda < \lambda_1$ in Theorem 3.1 was already investigated in [11].

Let us first remark that statements (i) and (iii) of Theorems 3.1 and 3.2 can be derived from (ii) as follows. If v_q is a ground state solution of (1) for $\lambda = \lambda_1$, then $u_q := \left(\frac{\lambda_1}{\mu}\right)^{\frac{1}{q-p}} v_q$ will be a ground state solution for $\lambda = \mu$. So for $\lambda < \lambda_1$, as $q \to p$, $u_q := \left(\frac{\lambda_1}{\lambda}\right)^{\frac{1}{q-p}} v_q$ goes to infinity, while for $\lambda > \lambda_1$ it goes to zero. The proof of Theorem 3.2 (i) and (iii) is virtually identical. It remains to consider the case $\lambda = \lambda_1$ for ground state solutions, and $\lambda = \lambda_2$ for least energy nodal solutions. Remark that the energy functional of problem (1) is given by

$$\varphi_q(u) := \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{q}\int_\Omega |u|^q$$

where $\lambda = \lambda_1$ (resp. λ_2). We denote by $\mathcal{N}_{\lambda,q}$ the associated Nehari manifold and $\mathcal{M}_{\lambda,q}$ the associated nodal Nehari set. The family $(u_{q,1})_{q>p}$ will denote a family of ground state solutions for the problem (1) with $\lambda = \lambda_1$, while $(u_{q,2})_{q>p}$ will be a family of least energy nodal solutions for the same problem with $\lambda = \lambda_2$. We prove that, up to a subsequence, $(u_{q,1})_{q>p}$ (resp. $(u_{q,2})_{q>p}$) converge in $L^p(\Omega)$ to a first (resp. second) eigenfunction of $-\Delta_p$.

Let us fix a first eigenfunction e_1 and a second eigenfunction e_2 of $-\Delta_p$.

Lemma 3.3. *For $p < s < p^*$, the quantities $\sup_{p<q<s} t_q^*$, $\sup_{p<q<s} t_q^+$ and $\sup_{p<q<s} t_q^-$ are finite, where t_q^*, t_q^+ and t_q^- are the unique positive real numbers such that $t_q^* e_1 \in \mathcal{N}_{\lambda_1,q}$ and $t_q^+ e_2^+ + t_q^- e_2^- \in \mathcal{M}_{\lambda_2,q}$.*

Proof. We will provide it only for t_q^*. The arguments are the same for the two other cases. Let us fix $s \in (p, p^*)$ and $q \in (q, s)$. Since $(t_q^*)^p ||\nabla e_1||_p^p = (t_q^*)^q \lambda_1 ||e_1||_q^q$, we have

$$t_q^* = \left(\frac{||\nabla e_1||_p^p}{\lambda_1 ||e_1||_q^q}\right)^{\frac{1}{q-p}} > 0.$$

It is enough to show that t_q^* converges, as $q \to p$. We have

$$\lim_{q\to p} \ln\left(\frac{||\nabla e_1||_p^p}{\lambda_1 ||e_1||_q^q}\right)^{\frac{1}{q-p}} = \lim_{q\to p} \frac{1}{q-p}\left(\ln\|\nabla e_1\|_p^p - \ln(\lambda_1 ||e_1||_q^q)\right).$$

Since $\lambda_1 ||e_1||_p^p = \|\nabla e_1\|_p^p$, we apply the Theorem of De L'Hôpital to obtain

$$\lim_{q\to p} \frac{\ln(\|\nabla e_1\|_p^p) - \ln(\lambda_1 ||e_1||_q^q)}{q-p} = \lim_{q\to p} \frac{-\int_\Omega (\ln|e_1|)|e_1|^q}{\int_\Omega |e_1|^q} = \frac{-\int_\Omega (\ln|e_1|)|e_1|^p}{\int_\Omega |e_1|^p}$$

so that

$$\lim_{q\to p} t_q^* = \exp \frac{-\int_\Omega (\ln|e_1|)|e_1|^p}{\int_\Omega |e_1|^p}.$$

□

Proposition 3.4. *The families $(u_{q,1})_{q>p}$ and $(u_{q,2})_{q>p}$ are uniformly bounded in $W_0^{1,p}(\Omega)$.*

Proof. We only give the proof for the family $(u_{q,2})_{q>p}$. The arguments are easier for the other family. As $u_{q,2}$ belongs to the Nehari manifold, $\mathrm{d}\varphi_q(u_{q,2})(u_{q,2}) = 0$, which means $||\nabla u_{q,2}||_p^p = \lambda_2 ||u_{q,2}||_q^q$. On one hand we have

$$\begin{aligned}\left(\frac{1}{p}-\frac{1}{q}\right)||\nabla u_{q,2}||_p^p &= \varphi_q(u_{q,2}) \\ &= \inf_{u\in\mathcal{M}_q} \varphi_q(u) \\ &\leq \varphi_q(t_q^+ e_2^+ + t_q^- e_2^-) \\ &= \varphi_q(t_q^+ e_2^+) + \varphi_q(t_q^- e_2^-).\end{aligned}$$

On the other hand we have

$$\varphi_q(t_q^+ e_2^+) = \frac{1}{p}(t_q^+)^p ||\nabla e_2^+||_p^p - \frac{\lambda_2}{q}(t_q^+)^q ||e_2^+||_q^q = \left(\frac{1}{p}-\frac{1}{q}\right)(t_q^+)^p ||\nabla e_2^+||_p^p$$

and analogously for $\varphi_q(t_q^- e_2^-)$. So we obtain

$$||\nabla u_{q,2}||_p^p \leq (t_q^+)^p ||\nabla e_2^+||_p^p + (t_q^-)^p ||\nabla e_2^-||_p^p.$$

From the uniform boundedness of t_q^+ and t_q^-, for q close to p, the claim follows.

□

The two following results prove that the sequence of ground state solutions (resp. least energy nodal solutions) of problem (1) stays away from the zero function.

Theorem 3.5. *Let $(u_{q,1})_{q>p}$ be a family of ground state solutions of the Lane-Emden problem (1) for $\lambda = \lambda_1$. Then*

$$\liminf_{q\to p} ||\nabla u_{q,1}||_p > 0.$$

Proof. Fix $r > 0$ such that $p < r < p^*$, and set $s := \frac{r(q-p)}{q(r-p)}$. By interpolation of Hölder's inequality we obtain

$$||u_{q,1}||_q^p \leq ||u_{q,1}||_p^{p-ps} ||u_{q,1}||_r^{ps}.$$

By definition of λ_1 we have

$$\lambda_1 ||u_{q,1}||_p^p \leq ||\nabla u_{q,1}||_p^p.$$

On the other hand, since $(u_{q,1})_{q>p}$ belongs to the Nehari manifold $\mathcal{N}_q$, we have

$$||\nabla u_{q,1}||_p^p = \lambda_1 ||u_{q,1}||_q^q$$

and, since $r < p^*$, by Sobolev's embedding Theorem, we know that there exists a constant C such that

$$||u_{q,1}||_r^p \leq C ||\nabla u_{q,1}||_p^p.$$

So it follows that

$$||\nabla u_{q,1}||_p \geq \lambda_1^{\frac{-p+q-sq}{pq-p^2}} C^{-\frac{sq}{pq-p^2}}$$

which means, recalling the definition of s,

$$||\nabla u_{q,1}||_p \geq \lambda_1^{\frac{1}{p-r}} C^{\frac{r}{p(p-r)}}.$$

Since this estimate does not depend on q, we obtain the claim.

□

Theorem 3.6. *Let $(u_{q,2})_{q>p}$ be a family of least energy nodal solutions of the Lane-Emden problem (1) for $\lambda = \lambda_2$. Then*

$$\liminf_{q\to p} ||\nabla u_{q,2}||_p > 0.$$

Proof. Since $u_{q,2}$ is sign-changing we can write $u_{q,2} = u^+_{q,2} + u^-_{q,2}$, with $u^+_{q,2}, u^-_{q,2} \neq 0$. Define

$$A := \left\{ v \in W^{1,p}_0(\Omega) \setminus \{0\} \,|\, v = \alpha u^+_{q,2} + \beta u^-_{q,2},\ (\alpha,\beta) \neq (0,0) \right\}.$$

It can be proved that $A \in \Gamma_2$ as defined in the introduction. Hence by definition of λ_2 we have

$$\lambda_2 \leq \max_{(\alpha,\beta)\neq(0,0)} \frac{|\alpha|^p ||\nabla u^+_{q,2}||^p_p + |\beta|^p ||\nabla u^-_{q,2}||^p_p}{|\alpha|^p ||u^+_{q,2}||^p_p + |\beta|^p ||u^-_{q,2}||^p_p} \leq \max\left\{ \frac{||\nabla u^+_{q,2}||^p_p}{||u^+_{q,2}||^p_p}, \frac{||\nabla u^-_{q,2}||^p_p}{||u^-_{q,2}||^p_p} \right\}.$$

The last inequality follows from the fact that

$$\frac{a+b}{c+d} \leq \frac{a}{c} \;\Leftrightarrow\; \frac{b}{d} \leq \frac{a}{c} \qquad \text{for any } a,\, b,\, c,\, d > 0.$$

Let us assume, without loss of generality, that the maximum is attained for $u^+_{q,2}$. Then we have

$$\lambda_2 ||u^+_{q,2}||^p_p \leq ||\nabla u^+_{q,2}||^p_p.$$

Fix $r > 0$ such that $p < r < p^*$ and set $s := \frac{r(q-p)}{q(r-p)}$. By interpolation of Hölder's inequality we obtain

$$||u^+_{q,2}||^p_q \leq ||u^+_{q,2}||^{p-ps}_p ||u^+_{q,2}||^{ps}_r.$$

On the other hand, since $(u^+_{q,2})_{q>p}$ belongs to the Nehari manifold $\mathcal{N}_q$, we have

$$||\nabla u^+_{q,2}||^p_p = \lambda_2 ||u^+_{q,2}||^q_q$$

and since $r < p^*$ by Sobolev's embedding Theorem we get

$$||u^+_{q,2}||^p_r \leq C ||\nabla u^+_{q,2}||^p_p.$$

So it follows that

$$||\nabla u^+_{q,2}||_p \geq \lambda_2^{\frac{-p+q-sq}{pq-p^2}} C^{-\frac{sq}{pq-p^2}}$$

and if we recall the definition of s

$$||\nabla u^+_{q,2}||_p \geq \lambda_2^{\frac{1}{p-r}} C^{\frac{r}{p(p-r)}}.$$

From the relation

$$||\nabla u_{q,2}||_p \geq ||\nabla u^+_{q,2}||_p$$

and since the estimate does not depend on q we obtain the claim. □

Theorem 3.7. *Let $(u_{q,1})_{q>p}$ be a family of ground state solutions of the Lane-Emden problem (1) for $\lambda = \lambda_1$ (resp. $(u_{q,2})_{q>p}$ be a family of least energy nodal solutions for $\lambda = \lambda_2$). Then, up to a subsequence, $u_{q,1} \to u_*$ (resp. $u_{q,2} \to u_*$) in $L^p(\Omega)$ as $q \to p$, where the function u_* is a first (resp. second) eigenfunction of the p-Laplacian.*

Proof. We give the proof for the family of least energy nodal solutions. The idea is the same for the family of ground state solutions. Let $v \in W_0^{1,p}(\Omega)$. Because of the uniform boundedness of the $(u_{q,2})_{q>p}$ in $W_0^{1,p}(\Omega)$, there exists $u_* \in W_0^{1,p}(\Omega)$ such that $u_{q,2} \rightharpoonup u_*$ in $W_0^{1,p}(\Omega)$ and $u_{q,2} \to u_*$ in $L^p(\Omega)$ for $q \to p$ (up to a subsequence). By Lebesgue's dominated convergence Theorem we also have

$$|u_{q,2}|^{q-2}u_{q,2} \to |u_*|^{p-2}u_* \text{ in } L^p(\Omega).$$

So

$$\int_\Omega |\nabla u_*|^{p-2}\nabla u_* \nabla v = \lim_{q\to p}\int_\Omega |\nabla u_{q,2}|^{p-2}\nabla u_{q,2}\nabla v =$$

$$= \lim_{q\to p}\lambda_2 \int_\Omega |u_{q,2}|^{q-2}u_{q,2}v = \lambda_2\int_\Omega |u_*|^{p-2}u_* v.$$

By Theorem 3.6 $u_* \neq 0$. Hence, u_* is a second eigenfunction of $-\Delta_p$. □

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