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Theory and Methodology

Leximin optimality and fuzzy set-theoretic operations

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Abstract

The leximin ranking of vectors of values taken from a totally ordered set is sometimes encountered in fields like operational research, social choice or numerical analysis, but has seldom been studied in connexion with fuzzy optimization. In this paper we prove that a leximin-optimal solution to a vector ranking problem on the unit hypercube can be obtained as the limit of optimal solutions to a problem of fuzzy multiple criteria optimization where fuzzy sets are aggregated either using a triangular norm or a generalized mean or an ordered weighted average (OWA) operation.

Keywords: Fuzzy sets; Triangular norm; OWA operator; Leximin ranking; Optimization

1. Introduction

Fuzzy optimization after Bellman and Zadeh (1970) is based on the maximin strategy. Namely, given n criteria or flexible constraints described by fuzzy sets $F_1, \ldots, F_n$ defined on a solution space Ω, an optimal solution in fuzzy optimization is defined as any $\omega^* \in \Omega$ which maximizes $\min_{i=1,\ldots,n} \mu_{F_i}(\omega)$, where μ_{F_i} is the membership function of F_i, usually ranging on the unit interval.

This strategy has been criticized because the minimum operator does not discriminate among solutions ω, ω' as soon as they equally satisfy the least satisfied constraints or criteria. In order to improve this situation, there have been proposals to turn the minimum operation into a triangular norm (such as the product) or an ordered weighted average (OWA) (Yager, 1988). However doing so, the ordering between solutions noticeably differs from the one obtained in the maximin approach.

In contrast several refinements of the maximin ordering have been pointed out (Fargier et al., 1993; Dubois et al., 1996) that improve the discrimination power without questioning the maximin ordering. Among these refinements, the leximin ordering is the most well known. To our knowledge it was first exhibited in the numerical

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analysis literature by Rice (1962) and by Descloux (1963). More recently, it has been proposed in the operations research literature by Behringer (1981) for solving bottleneck optimization problems, and in the social choice literature (Sen, 1986; Moulin, 1988) for representing egalitarist social welfare functions. An authoritative survey on lexicographic orders and decision rules is due to Fishburn (1974). However, he does not mention the leximin ordering nor does he refer to the numerical analysis literature.

The *leximin* ordering is defined as follows: let $\overline{x} = (x_1, \ldots, x_n) \in [0,1]^n$ and $\overline{y} = (y_1, \ldots, y_n) \in [0,1]^n$. Then rearrange $\overline{x}$ and $\overline{y}$ in increasing order ($x_{i_1} \leqslant \cdots \leqslant x_{i_n}$ and $y_{j_1} \leqslant \cdots \leqslant y_{j_n}$)

$$\overline{x} >^{\text{lex}} \overline{y} \quad \text{iff } \{\exists k \in \{1,\ldots,n\} : \forall l < k, \; x_{i_l} = y_{j_l} \text{ and } x_{i_k} > y_{j_k}\}. \tag{1}$$

This ordering is such that its opposite $\overline{x} \geqslant^{\text{lex}} \overline{y}$ iff $\neg(\overline{y} >^{\text{lex}} \overline{x})$ is a complete pre-ordering.

In this paper we show that leximin-optimal solutions to fuzzy optimization problems can be approached by means of optimal solutions to fuzzy optimization problems based on triangular norms, generalized means, as well as OWAs.

2. Leximin as a strict Chebyshev norm

The oldest approach to the leximin notion is known in Numerical Analysis as the strict Chebyshev norm. Let us recall some of their definitions.

The approximation of a function $f \in \mathscr{H}$ of a real variable by means of elements of a subspace $\mathscr{L} \subseteq \mathscr{H}$ consists in finding the function $\hat{f} \in \mathscr{L}$ minimizing the distance to f:

$$\|f - \hat{f}\| = \min_{h \in \mathscr{L}} \|f - h\|,$$

where $\|\cdot\|$ is a functional norm in $\mathscr{H}$, a function space.

In the sequel of this section, we only consider functions known in some discrete points $\mathscr{B} = \{a_i\}_{i=1,n}$. For each function $h \in \mathscr{H}$, one can obtain a vector $\overline{x}(h) \in \mathbb{R}^n$ whose element x_i equals $h(a_i)$. Let $\mathscr{L}^n = \{\overline{x}(h) : h \in \mathscr{L}\}$. The functional approximation problem can therefore be solved by finding a vector $\overline{x}^* \in \mathscr{L}^n$ as close as possible to $\overline{y}(f) \in \mathbb{R}^n$, where $y_i = f(a_i)$. And the distance is now computed according to a vector norm.

The most well-known norms are the Hölder ones (L_p-norms):

$$\|\overline{y}(f) - \overline{x}(h)\|_p = \left(\sum_{i=1}^{n} |y_i - x_i|^p\right)^{1/p}, \quad 1 \leqslant p \leqslant +\infty.$$

The Euclidian distance corresponds to the L_2-norm, while the L_∞-norm, also called the Chebyshev norm, is very particular

$$\|\overline{y}(f) - \overline{x}(h)\|_\infty = \max_{i=1,\ldots,n} |y_i - x_i|.$$

Rice (1962) had to cope with the non-uniqueness of the best Chebyshev approximation to a given function $f(x)$ from a subspace $\mathscr{L}$ of the continuous functions on a set of points $\mathscr{B}$. He introduced the em strict Chebyshev approximation. Very interestingly, the definition he gave is constructive (see also Duris and Temple, 1973).

> Briefly, once $\mathscr{L}_1$, the set of the Chebyshev approximations to f from $\mathscr{L}$ on $\mathscr{B}$, has been determined, one defines the set of critical points $\mathscr{B}_1$. The latter is the minimal subset of $\mathscr{B}$ containing all the points where the approximation error has to be exactly equal to the Chebyshev norm of the error. In other words, for any best Chebyshev approximation of f, we have that
>
> $$i \in \mathscr{B}_1 \;\Rightarrow\; |y_i - x_i| = \|\overline{y}(f) - \overline{x}(h)\|.$$
>
> Then, one computes $\mathscr{L}_2$ the set of Chebyshev approximations to f from $\mathscr{L}_1$ on $\mathscr{B} - \mathscr{B}_1$. And so on. Finally, the procedure gives the strict Chebyshev approximation, which has been proven to be unique in this case.

The technique for computing leximin-optimal solutions to fuzzy constraint satisfaction problems described in Dubois and Fortemps (1999) is directly related to the above algorithmic definition.

Descloux (1963) showed that the limit of the best L_p-approximations converges to the strict Chebyshev approximation. This result started some works on Polya algorithm that use the following convergence schema. Theoretically, the strict L_∞ solution can be viewed as the limit of the L_p solutions, as p tends to infinity, but the rate of convergence is very slow, namely $1/p$ (Egger and Huotari, 1990). Even if it has not turned out to be tractable, because of the very large values of p needed, it makes sense to study the behavior of the convergence when parameterized t-norms are considered.

It is easy to relate such an approximation scheme to the fuzzy constraint satisfaction problem. Each feasible solution to a fuzzy constraint problem is viewed as a vector $\overline{x}$. Maximizing $\min_i x_i$ in the feasibility domain $\mathscr{L}$ comes down to computing the best approximation to the vector $\overline{1} = (1, 1, \ldots, 1)$ in the sense of the Chebyshev norm

$$\|\overline{1} - \overline{x}^*\|_\infty = \min_{\overline{x} \in \mathscr{L}} \|\overline{1} - \overline{x}\|_\infty,$$

where the norm has to be computed in the n-dimensional space corresponding to the constraints $C_i \in \mathscr{C}$, and the strict Chebyshev approximations correspond to leximin-optimal solutions.

Alternatively we can consider the dual problem of minimizing $\max_i \mu_{F_i}(\omega)$. It comes down to computing the Chebyshev approximation $\overline{x}^* \in V \subseteq [0,1]^n$ to the vector $(0, 0, \ldots, 0)$. This time it comes down to the same kind of ordering as leximin but the x_i's are increasingly ordered in this case. The corresponding ordering can be called *leximax*:

$$\overline{x} <^{\text{lexmax}} \overline{y} \iff \begin{cases} \text{if } x_{i_1} \geqslant \cdots \geqslant x_{i_n} \text{ and } y_{j_1} \geqslant \cdots \geqslant y_{j_n} \\ \text{then } \exists k \in 1, \ldots, n : \forall l < k,\ x_{i_l} = y_{j_l} \text{ and } x_{i_k} < y_{j_k}. \end{cases}$$

Clearly a leximin-optimal solution to the problem: maximize $\min_{i=1,\ldots,n} x_i$ for $\overline{x} \in V \subseteq [0,1]^n$ is a leximax optimal solution to the problem: minimize $\max_{i=1,\ldots,n} y_i$ for $\overline{y} \in V' \subseteq [0,1]^n$ where $V' = \{\overline{y} : \forall i,\ y_i = 1 - x_i,\ \overline{x} \in V\}$. Maximax and minimin optimization problems can be lexicographically refined in a similar way.

3. Fuzzy set aggregation operations

Aggregation operators such as t-norms and t-conorms are commonly used in the fuzzy field instead of min and max. A t-norm T is a semigroup of the unit interval (associative, commutative, with identity 1) which is non-decreasing in each place. Continuous Archimedean t-norms are such that $\forall a \in (0,1),\ aTa < a$. Any Archimedean continuous t-norm T can be written as follows (Schweizer and Sklar, 1983):

$$aTb = \phi^{-1}(\min(\phi(0),\ \phi(a) + \phi(b))), \tag{2}$$

where $\phi : [0,1] \to [0, \phi(0)]$ is a continuous and decreasing function, such that $\phi(1) = 0$. Similar to Hölder norms, parameterized families of strict t-norms T_p, where $p \in \mathbb{R}$ is a parameter, can be written as

$$T_p(\overline{x}) = \phi_p^{-1}\left(\sum_{i=1}^{n} \phi_p(x_i)\right) \tag{3}$$

for which $\phi_p(0) = +\infty$. For our purpose, we only consider parameterized families of t-norms such that

$$\lim_{p \to \infty} T_p(\overline{x}) = |\overline{x}|_\infty = \min_i x_i,$$

e.g., the Frank family $(q = 1/p)$ (Frank, 1979):

$$T_p(\overline{x}) = \log_q \left[1 + \frac{\prod_{i=1}^{n}(q^{x_i} - 1)}{(q-1)^{(n-1)}}\right]; \tag{4}$$

Schweizer and Sklar (1961) family:

$$T_p(\overline{x}) = \max\left(0,\ \sqrt[-p]{\sum_i x_i^{-p} - (p+1)}\right); \tag{5}$$

or even some families of nilpotent t-norms such as Yager (1980) family:

$$T_p(\overline{x}) = 1 - \min\left(1, \left[\sum_i (1 - x_i)^p\right]^{1/p}\right) \tag{6}$$

under some condition of non-saturation.

More generally, for any parameter $p \in (0, +\infty)$ and any generator ϕ of a triangular norm, let T_p be

the triangular norm additively generated by ϕ_p using (2). Then $\lim_{p\to\infty} T_p = \min$ (see Klement et al., 1999).

Another family to which the main result of this paper applies is the family of generalized arithmetic means (for instance, Fodor and Roubens, 1994)

$$M_p(\overline{x}) = \Phi_p^{-1}\left(\frac{1}{n}\sum_i \Phi_p(x_i)\right), \tag{7}$$

where Φ_p is any continuous strictly monotonic function on $[0,1]$. For instance, a well-known family, called the "root-power mean" studied in Dujmovic (1974, 1975), is obtained with $\Phi_p(a) = a^p$, which converges to min when $p \to -\infty$.

A leximin-optimal solution to the problem of maximizing $\min_{i=1,\ldots,n} x_i$, on some subset V of $[0,1]^n$, is denoted $\overline{x}^\infty$ and indices are reordered such that $x_1^\infty \leqslant \cdots \leqslant x_n^\infty$. The following result is noticeable.

Lemma 1. *If $\overline{x}^\infty$ is a leximin-optimal solution to the problem of maximizing $\min_{i=1,\ldots,n} x_i$, on a convex set V, if $\overline{x} \in V, \overline{x} \neq \overline{x}^\infty$ and if $k = \inf\{i : x_i \neq x_i^\infty\}$ then $\exists m \geqslant k$ such that $x_m < x_k^\infty$. Moreover $\overline{x}^\infty >^{\text{lex}} \overline{x}$.*

Proof. Assume that $\forall i \geqslant k : x_i \geqslant x_k^\infty$. Build the solution $z = \overline{x} + \overline{x}^\infty/2$. z lies in V since V is convex.

- $\forall i < k$, we have $z_i = x_i^\infty$ since $x_i = x_i^\infty$.
- For $i = k$, $z_k = x_k + x_k^\infty/2 > x_k^\infty$, since $x_k \neq x_k^\infty$ and $x_k \geqslant x_k^\infty$.
- $\forall i > k$ such that $x_i^\infty = x_k^\infty$, $z_i = x_k^\infty$.
- $\forall i > k$ such that $x_i^\infty > x_k^\infty$, $z_i > x_k^\infty$.

Therefore, $\overline{z}$ should be leximin-better than $\overline{x}^\infty$, which is impossible since $\overline{x}^\infty$ is optimal. Thus, $\exists m \geqslant k$ such that $x_m < x_k^\infty$.

When considering the first $(k-1)$ components, let K be the number of x_i^∞ less than or equal to x_m. Since $\overline{x}^\infty$ is increasingly ordered, the overall number of x_i^∞ less than or equal to x_m is K. But in $\overline{x}$, this number is at least equal to $(K+1)$, since the $(k-1)$ first components of $\overline{x}$ are equal to those of $\overline{x}^\infty$. When comparing $\overline{x}$ and $\overline{x}^\infty$ after reordering $\overline{x}$ increasingly, it is obvious that $x_{i_{K+1}}^\infty > x_{j_{K+1}}$. So, $\overline{x}^\infty >^{\text{lex}} \overline{x}$. □

From this lemma, it is clear that the leximin solution to maximizing $\min_{i=1,\ldots,n} x_i$ for $\overline{x} \in V \subseteq [0,1]^n$ is unique, when V is convex.

If $\overline{x} \in [0,1]^n$, let $\Pi_0(\overline{x}) = \overline{0} = (0,\ldots,0)$ and for all $0 < k \leqslant n$, let $\Pi_k(\overline{x}) = (x_1,\ldots,x_k,0\ldots,0)$, the vector whose first k components are equal to those of $\overline{x}$, while the remaining ones are set equal to zero.

Lemma 2. *If $0 \leqslant k \leqslant n$ and $\epsilon > 0$, there is a $\delta > 0$ such that, if $\overline{y} \in [0,1]^n$ and $\|\Pi_k(\overline{y})\|_\infty < \delta$, then there is a vector $\overline{x} \in [0,1]^n$ such that $\Pi_k(\overline{x}) = \Pi_k(\overline{y})$ and $\|\overline{x}\|_1 < \epsilon$.*

See Egger and Huotari (1992) for the proof based on the topological equivalence of norms.

Let $\{f_p\}_{p\in[0,+\infty)}$ be a two-place function family in $[0,1]$. We shall assume that f_p can recursively be extended to a function from $[0,1]^n$ to $[0,1]$ as follows:

> *Constructiveness.* For any $n \geqslant 2$ and $1 \leqslant i \leqslant n$, for any $\overline{x} \in [0,1]^i$ and any $\overline{y} \in [0,1]^{(n-i)}$, denoting $\overline{z} = (\overline{x},\overline{y})$ the vector obtained by the concatenation of $\overline{x}$ and $\overline{y}$, $\exists g_{(i,n)}$ such that
>
> $$f_p(\overline{z}) = g_{(i,n)}\big(f_p(\overline{x}), f_p(\overline{y})\big). \tag{8}$$

We shall assume moreover that $\lim_{p\to+\infty} f_p = \min$, that f_p is commutative and continuous.

Let V be a closed convex subset of $[0,1]^n$ and assume that $\forall \overline{x} \in V$,

$$f_p(\overline{x}) = f_p(x_1,\ldots,x_n) \neq 0 \text{ and } \neq 1. \tag{9}$$

The last property we need is that of *cancellativeness*: and, for any $n \geqslant 2$ and $1 \leqslant i \leqslant n$, for any $\overline{x},\overline{x}' \in [0,1]^i$ and any $\overline{y} \in [0,1]^{(n-i)}$, such that $\overline{z} = (\overline{x},\overline{y})$, $\overline{z}' = (\overline{x}',\overline{y}) \in V$,

$$f_p(\overline{x}) < f_p(\overline{x}') \;\Rightarrow\; f_p(\overline{z}) < f_p(\overline{z}'). \tag{10}$$

Let $\overline{x}^p$ be an optimal solution to the problem of maximizing $f_p(\overline{x})$ for $\overline{x} \in V$.

Lemma 3. *If $0 \leqslant k \leqslant n$, for a converging sequence $\{p_v\}$ such that $\lim_{v\to\infty} p_v = \infty$ and if $\lim_{v\to\infty} \|\overline{x}^v - \overline{x}\|_1 = 0$, then $\lim_{v\to\infty} f_{p_v}(x_{k+1}^v,\ldots,x_n^v) = f_\infty(x_{k+1},\ldots,x_n) = \min(x_{k+1},\ldots,x_n)$.*

This lemma is obvious due to the continuity of f_p.

Theorem 4. *Let $\overline{x}^p$ be an optimal solution to the maximization of $f_p(\overline{x})$ over a convex subset V of $[0,1]^n$, where f_p is a continuous and commutative aggregation operation satisfying* (8)–(10) *such that* $\lim_{p\to+\infty} f_p(\overline{x}) = \min_i x_i$. *Then,* $\lim_{p\to\infty} \overline{x}^p$ *is the leximin solution to the maximization of* $\min_i x_i$.

Proof. Since $0 < f_p(\overline{x}^p) < 1$, by the Bolzano–Weierstrass Theorem, $\{\overline{x}^p, p \in (1,+\infty)\}$ contains a convergent sequence. The rest of the proof is by induction.

Since $\overline{x}^\infty$ is such that $x_1^\infty \leqslant \cdots \leqslant x_n^\infty$, there exist numbers r_i such that $r_1 < r_2 < \cdots < r_l$ and $\overline{x}^\infty = (r_1,\ldots,r_1,\quad r_2,\ldots,r_2,\ldots,\quad r_l,\ldots,r_l)$. Let $C_0 = \emptyset$ and for $1 \leqslant m \leqslant l$, let $C_m = \{i : x_i^\infty \leqslant r_m\}$. Because of the index reordering, C_m has the form $\{1,2,\ldots,i(m)\}$.

Suppose that the limit holds for each i in C_k and that $\{\overline{x}^{p_\nu}\}$ is a convergent subsequence of $\{\overline{x}^p\}$ such that $\lim_{\nu\to\infty} \overline{x}^{p_\nu} = \overline{x}$. What about $i \in C_{k+1}$?

Assume there exists i in $C_{k+1} \setminus C_k$ such that $x_i \neq x_i^\infty$. Because of Lemma 1, there exists $m \geqslant i$ such that

$$x_m < x_i^\infty = r_{k+1}.$$

Let j be the cardinality of $C_k = \{1,2,\ldots,i(k)\}$. Therefore, remembering that $f_\infty = \min$,

$$\begin{aligned} f_\infty(x_{j+1},\ldots,x_n) &\leqslant x_m \\ &< r_{k+1} = x_{j+1}^\infty \\ &\leqslant f_\infty(x_{j+1}^\infty,\ldots,x_n^\infty). \end{aligned} \tag{11}$$

Because, $\overline{x}^{p_\nu}$ converges to $\overline{x}^\infty$, at least for the j first elements, we have

$$\left\| \Pi_j(\overline{x}^{p_\nu} - \overline{x}^\infty) \right\|_\infty \to 0. \tag{12}$$

By Lemma 2, it exists a sequence $\{\overline{u}^\nu\} \subset [0,1]^n$ such that $\Pi_j(\overline{u}^\nu) = \Pi_j(\overline{x}^{p_\nu} - \overline{x}^\infty)$ and $\|\overline{u}^\nu\|_1 \to 0$, as $\nu \to \infty$.

Let $\overline{\omega}^\nu = \overline{u}^\nu + \overline{x}^\infty$. Then, $\Pi_j(\overline{\omega}^\nu) = \Pi_j(\overline{x}^{p_\nu})$, for all ν and $\lim_{\nu\to\infty} \|\overline{\omega}^\nu - \overline{x}^\infty\|_1 = 0$.

Because of Lemma 3, as $\nu \to \infty$,

$$f_{p_\nu}(\omega_{j+1}^\nu,\ldots,\omega_n^\nu) \to f_\infty(x_{j+1}^\infty,\ldots,x_n^\infty),$$
$$f_{p_\nu}(x_{j+1}^{p_\nu},\ldots,x_n^{p_\nu}) \to f_\infty(x_{j+1},\ldots,x_n),$$

since $\lim_{\nu\to\infty} \overline{x}^{p_\nu} = \overline{x}$. Applying these limit results to Eq. 11, there exists an index σ such that when the j first components are ignored

$$f_{p_\sigma}(x_{j+1}^{p_\sigma},\ldots,x_n^{p_\sigma}) < f_{p_\sigma}(\omega_{j+1}^\sigma,\ldots,\omega_n^\sigma).$$

Since, $\Pi_j(\overline{\omega}^\nu) = \Pi_j(\overline{x}^{p_\nu})$, namely $\overline{\omega}^\nu$ and $\overline{x}^{p_\nu}$ have their first j components equal, Eqs. (8) and (10) give

$$f_{p_\sigma}(\overline{x}^{p_\sigma}) < f_{p_\sigma}(\overline{\omega}^\sigma). \tag{13}$$

And this contradicts the definition of $\overline{x}^{p_\sigma}$ which is the solution of the maximization of f_{p_σ}.

Therefore, $\forall i \in C_{k+1}, \lim_{p\to\infty} x_i^p = x_i^\infty$. Note that this part of the proof already applies when $k = 0$, so that the result holds for $i \in C_1$. So the result holds $\forall i \in \{1,2,\ldots,n\}$. □

Theorem 4 is a generalization of the theorem proved in Egger and Huotari (1992) for Hölder norms to a large class of aggregation operators.

Particular cases of functions f_p to which the results apply are the continuous Archimedean t-norms (for which the function $g_{(i,n)}$ is f_p itself) and generalized means described by Eq. (7).

Of course dual results concerning the leximax can be obtained for suitable families of aggregation operators that converge to the maximum operation.

4. Leximin as a limit of OWA operation-based optimizations

An OWA operation (Yager, 1988; Yager and Kacprzyk, 1988) is defined as follows:

Let $w_1,\ldots,w_n$ be a set of weights such that $\sum_i w_i = 1$. Let $\overline{x} \in [0,1]^n$ and denote $x_{j_1},\ldots,x_{j_n}$ reordered in the increasing order. Then

$$\mathrm{OWA}_{\overline{w}}(\overline{x}) = \sum_{i=1}^n w_i x_{j_i}. \tag{14}$$

In other words, it consists of a weighted average of the reordered components of $\overline{x}$. Such a procedure is relevant when the order of the components is more important than their initial labelling. It is a form of quantified aggregation expressing that some possibly ill-defined proportion of criteria must be satisfied.

This class of aggregation operations covers the entire interval between min and max operations, including the arithmetic mean, the median and the order-statistics. Fodor and Roubens (1994) characterize it as the family of operators which are neutral – the components are treated independently of their label – monotonic, idempotent – $F(x,\ldots,x) = x$ – and stable under positive linear transformations with the same unit for ordered values. On the other hand, Grabisch et al. (1995) prove that any OWA aggregator can be expressed in an equivalent way as a Choquet integral

$$\sum_{i=1}^{n} x_{j_i}[\mu(A_i) - \mu(A_{i+1})], \tag{15}$$

where $\mu(A_i) = \sum_{k=1}^{i} w_k$; μ is a monotonic set-function, $A_i = \{i, i+1, \ldots, n\}$ and $\mu(A_i)$ only depends on the cardinality of A_i.

A strong link between OWA operations and the leximin ordering has been recently established in Dubois et al. (1996a). Namely, the following result holds.

Theorem 5. *For any finite subset L of $[0,1]$ (containing 0 and 1), there exists an OWA operation with a weight pattern $\overline{w}$ such that for any $\overline{x}, \overline{y} \in L^n$,*

$$\overline{x} >^{\text{lex}} \overline{y} \Rightarrow \text{OWA}_{\overline{w}}(\overline{x}) > \text{OWA}_{\overline{w}}(\overline{y}), \tag{16}$$

$$\overline{x} \geqslant^{\text{lex}} \overline{y} \Rightarrow \text{OWA}_{\overline{w}}(\overline{x}) \geqslant \text{OWA}_{\overline{w}}(\overline{y}). \tag{17}$$

Proof. First, it is obvious that $\overline{x} =^{\text{lex}} \overline{y}$ implies that $\forall \overline{w}$, $\text{OWA}_{\overline{w}}(\overline{x}) = \text{OWA}_{\overline{w}}(\overline{y})$.

Let $0 < \epsilon < \min_{a,b \in L, a \neq b} |a - b|$. Let us exhibit a suitable weight pattern $\overline{w}$. Let $\overline{x}, \overline{y}$ such that $\overline{x} >^{\text{lex}} \overline{y}$. We start from this assumption of strict inequality and look for conditions on $\overline{w}$ that will make the following inequality satisfied:

$$\sum_{k=1}^{n} w_k x_{i_k} > \sum_{k=1}^{n} w_k y_{j_k}. \tag{18}$$

Let $K = \min\{k : x_{i_k} \neq y_{j_k}\}$. Inequality (18) becomes

$$w_K x_{i_K} + \sum_{k=K+1}^{n} w_k x_{i_k} > w_K y_{j_K} + \sum_{k=K+1}^{n} w_k y_{j_k}. \tag{19}$$

Since the vectors are increasingly ordered, the worst case is encountered when $\forall k > K,\ x_{i_k} = x_{i_K}$ and $\forall k > K,\ y_{j_k} = 1$.

Therefore, the following inequality has to hold:

$$w_K x_{i_K} + x_{i_K} \sum_{k=K+1}^{n} w_k > w_K y_{j_K} + \sum_{k=K+1}^{n} w_k. \tag{20}$$

The greatest value that y_{j_K} can reach is less than $(x_{i_K} - \epsilon)$, since $\overline{x} >^{\text{lex}} \overline{y}$. Therefore, the previous relation holds as soon as we have

$$\epsilon w_K + x_{i_K} \sum_{k=K+1}^{n} w_k > \sum_{k=K+1}^{n} w_k. \tag{21}$$

Again, we are looking for the worst case, which occurs when x_{i_K} is equal to ϵ. Finally, we have that

$$w_K = \frac{1-\epsilon}{\epsilon} \sum_{l=K+1}^{n} w_l. \tag{22}$$

This recurrence relation gives the following weight pattern:

$$\begin{aligned} &w_n = \epsilon^{n-1}, \\ &w_{n-1} = (1-\epsilon)\epsilon^{n-2}, \\ &\quad\vdots \\ &w_2 = (1-\epsilon)\epsilon, \\ &w_1 = (1-\epsilon). \end{aligned} \tag{23}$$

□

Let us consider again a convex subset V of $[0,1]^n$ and $\overline{x}^{\infty}$ a leximin-optimal vector in V. Denote OWA_{ϵ} for $\epsilon \in (0,1)$ the ordered weighted average operation using the weight pattern in Theorem 5. Let $\overline{x}^{\epsilon}$ be a vector which maximizes

$\mathrm{OWA}_\epsilon(\overline{x})$ for $\overline{x} \in V$. From the above theorem, it is clear that if $\mathrm{OWA}_\epsilon(\overline{x}^\infty) < \mathrm{OWA}_\epsilon(\overline{x}^\epsilon)$, there exists $\epsilon' < \epsilon$ such that $\mathrm{OWA}_{\epsilon'}(\overline{x}^\infty) > \mathrm{OWA}_{\epsilon'}(\overline{x}^\epsilon)$, since $\overline{x}^\infty >^{\mathrm{lex}} \overline{x}^\epsilon$. To do so, it is enough to choose $\epsilon' < \min(\epsilon, \min_{i,j:x_i^\infty \neq x_j^\epsilon} |x_i^\infty - x_j^\epsilon|)$. Indeed, it is obvious that if $\overline{x} >^{\mathrm{lex}} \overline{y}$ and $\mathrm{OWA}_\epsilon(\overline{x}) > \mathrm{OWA}_\epsilon(\overline{y})$ then for any $\epsilon' < \epsilon$, $\mathrm{OWA}_{\epsilon'}(\overline{x}) > \mathrm{OWA}_{\epsilon'}(\overline{y})$.

As in the case of aggregation operations dealt with in the previous section, OWA operations enable the leximin-optimal solution to be attained in the limit.

Theorem 6. *Let $\{\overline{x}_l\}_{l=1,2,\ldots}$ be a sequence of OWA_{ϵ_l}-maximal points of a closed subset V of $[0,1]^n$, for $\{\epsilon_l\}_{l=1,2,\ldots}$ a decreasing sequence of positive numbers converging to 0. Any point $\overline{x}$ adherent to the sequence $\{\overline{x}_l\}$ is leximin-optimal in V.*

Proof. Recall that $\overline{x}$ is adherent to the sequence $\{\overline{x}_l\}$, if $\forall \varepsilon > 0$, $\exists k$ such that $\|\overline{x} - \overline{x}_k\| < \varepsilon$.

By characterization of an adherent point, there exists a subsequence $\{\overline{y}_l\}_{l=1,\ldots}$ of $\{\overline{x}_l\}$ converging to $\overline{x}$. Since each $\overline{y}_l$ is OWA-maximal for the appropriate weight pattern based on some ϵ_l, we have

$$\mathrm{OWA}_{\epsilon_l}(\overline{y}_l) \geqslant \mathrm{OWA}_{\epsilon_l}(\overline{v}) \quad \forall \overline{v} \in V.$$

Since the sequence $\{\overline{y}_l\}$ converges to $\overline{x}$, $\|\overline{y}_l - \overline{x}\| \to 0$. And therefore because the OWA operators are continuous, there exists K, such that $\forall k \geqslant K$,

$$\mathrm{OWA}_{\epsilon_k}(\overline{y}_k) \geqslant \mathrm{OWA}_{\epsilon_k}(\overline{v}) \quad \forall \overline{v} \in V$$

and

$$\mathrm{OWA}_{\epsilon_k}(\overline{x}) \geqslant \mathrm{OWA}_{\epsilon_k}(\overline{v}) \quad \forall \overline{v} \in V.$$

This implies that $\overline{x} \geqslant^{\mathrm{lex}} \overline{v}, \forall \overline{v} \in V$. Indeed, assume that there exists $\overline{v} \in V$, such that $\overline{v} >^{\mathrm{lex}} \overline{x}$. Then, because of Theorem 5, for k large enough (i.e. ϵ_k sufficiently small), we have

$$\mathrm{OWA}_{\epsilon_k}(\overline{x}) < \mathrm{OWA}_{\epsilon_k}(\overline{v}).$$

And this is absurd. □

In the previous theorem, we prove that any convergent subsequence of the $\{\overline{x}^\epsilon\}_{\epsilon \to 0}$ in a closed subset V converges to a leximin-optimal solution. It is now clear that if the leximin-optimal solution is unique (in particular when V is convex), the whole sequence $\{\overline{x}^\epsilon\}_{\epsilon \to 0}$ converges to this leximin-optimal solution.

However, one could wonder if, when V is not convex, any leximin-optimal solution can be reached as the limit of a convergent subsequence of $\{\overline{x}^\epsilon\}_{\epsilon \to 0}$. Unfortunately, a small example can show that there may exist solutions which are not reachable by such a convergent scheme. Indeed, consider the subset V represented in Fig. 1 and bounded by the two curves.

In this bicriteria case, the two leximin-optimal solutions are $\overline{x}^\infty$ and $\overline{x}'^\infty$, which are symmetric with respect to the main diagonal. Looking for an OWA-optimal solution leads to a piecewise linear objective optimization, since the objective can be written $(1-\epsilon)\underline{z} + \epsilon\overline{z}$ with $\underline{z} = \min\{x_1, x_2\}$ and $\overline{z} = \max\{x_1, x_2\}$. For instance, if $\epsilon = \frac{2}{7}$, it gives the objective z_1 combining both partial objective function z_{11} on the South-East part of the figure and z_{12} on the North-West part. z_{11} and z_{12} (plotted

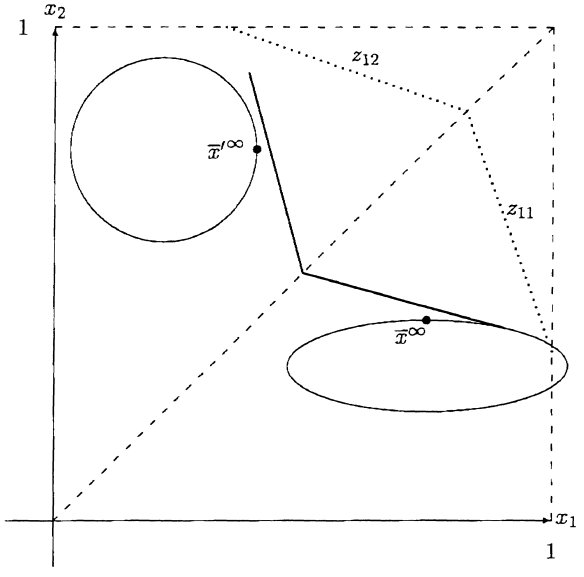

Fig. 1. Example of non-convex subset V with two leximin-optimal solutions.

in dotted lines) have symmetric slopes with respect to the main diagonal of the figure.

Because we are looking for a maximal solution w.r.t. the z_1 criterion, it is obvious that the OWA-optimal solution will be on the lower curve for this value of ϵ, since the function z_{11} will reach first the lower curve. In other words, the values that can be attained on the other curve are smaller than the best value on the South-East curve. It will be the case also for any other value of ϵ (e.g. $\epsilon = \frac{1}{6}$ gives the partial objectives in solid lines). Therefore, all the OWA-optimal solutions of any sequence $\{\overline{x}^{\epsilon}\}$ will be on the lower circle. The limit of such a sequence as $\epsilon \to 0$ will always be $\overline{x}^{\infty}$. Only solution $\overline{x}^{\infty}$ is reachable by the Polya algorithm with OWA operators.

5. Concluding remarks

This paper has shown that the leximin ordering is part of the landscape of fuzzy set theoretical operations and is a natural refinement of the minimum operation in this setting. Note that the result in Section 2 applies to families of continuous t-norms or generalized means that converge to the minimum, regardless if the fact that these families are increasing or decreasing in the parameter. Moreover, the proof of the representability of the leximin ordering by OWA operations in the limit cannot use the result of Section 2, because the property (8) does not apply to OWA operations. Indeed, $\mathrm{OWA}(x_1, \ldots, x_n)$ cannot be computed from $\mathrm{OWA}(x_1, \ldots, x_i)$ and $\mathrm{OWA}(x_{i+1}, \ldots, x_n)$.

The results of this paper enable a lot of variants of Polya algorithm to be imagined for computing leximin-optimal solutions to fuzzy optimization problems.

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