On conformal anomalies and invariants in arbitrary dimensions

General solution of the Wess-Zumino consistency condition

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Durban, Third Mandelstam workshop, 15 January 2019

From the works [0706.0340] and [1809.05445], the last one in collaboration with Jordan FRANCOIS and Serge LAZZARINI
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In 1973, Derek Capper and Michael J. Duff discovered that the invariance under Weyl rescaling of the metric tensor

\[ g_{\mu\nu}(x) \rightarrow \Omega^2(x) g_{\mu\nu}(x) \]

displayed by classical massless field systems in interaction with gravity no longer survives in the quantum theory.

\[ \rightarrow \text{Weyl (or conformal) anomaly} \]
Examples of spin-1, spin-1/2 and spin-0 field theories:

- \( S[A_\mu, g_{\mu\nu}] = \frac{1}{4} \int d^4x \sqrt{-g} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \)
  where \( F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \)

- \( S[\Psi, e^a_\mu] = \frac{1}{2} \int d^n x \, e(\bar{\Psi} \gamma^a \nabla_a \Psi - \nabla_a \bar{\Psi} \gamma^a \Psi) \)

- \( S[\phi, g_{\mu\nu}] = \frac{1}{2} \int \sqrt{-g} \left[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \xi(n) \mathcal{R} \Phi^2 \right] d^n x \)
  with \( \xi(n) = \frac{1}{4} \left[ (n - 2)/(n - 1) \right] \).
Spacetime indices → Greek letters, e.g. Riemann tensor
\[ R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} + \ldots, \] Christoffel symbols \( \Gamma^\mu_{\nu\rho} \), Ricci tensor
\[ R_{\alpha\beta} = R^\mu_{\alpha\mu\beta} \] and scalar curvature \( R = g^{\alpha\beta} R_{\alpha\beta} \); Curvature two-form
\[ R^\mu_{\nu} = \frac{1}{2} R^\mu_{\nu\rho\sigma} dx^\rho dx^\sigma. \]

Frame (tangent bundle) indices → Latin letters.

The frame fields are \( e_a = e^\mu_a \partial_\mu \) in coordinates \( x^\mu \). Determinant \( e = \det e^a_\mu \) where \( e^a_\mu e^{\nu}_a = \delta^\nu_\mu \).

For Dirac spinors : Clifford algebra \( \{ \gamma_a, \gamma_b \} = 2\eta_{ab} \) where \( \gamma_a \) denote Dirac’s matrices and \( \eta = \text{diag}(+,-,-,-) \); \( \nabla_a \Psi = e^\mu_a (\partial_\mu - \frac{i}{2} \omega^{bc}_\mu \Sigma_{bc}) \Psi \), where \( \Sigma_{bc} = \frac{i}{4} [\gamma_b, \gamma_c] \) and \( \omega^{bc}_\mu = \omega^{bc}_\mu(e) \) is the Levi-Civita spin-connection.
Trace of stress-tensor

- These matter systems coupled to gravity are invariant under the local Weyl rescalings

\[
\begin{align*}
g_{\mu \nu} & \rightarrow \Omega^2(x) \, g_{\mu \nu} \\
e^a_\mu & \rightarrow \Omega \, e^a_\mu \\
\Psi & \rightarrow \Omega^{(1-n)/2} \, \Psi \\
\phi & \rightarrow \Omega^{(2-n)/2} \, \phi
\end{align*}
\] (1)

- This is reflected in the (on-shell) tracelessness of the corresponding (matter) symmetric stress-tensors: (1) \( \Rightarrow g^{\mu \nu} T_{\mu \nu} = 0 \).
LOCAL SYMMETRIES

Clearly, by construction these actions are also invariant under diffeomorphisms.

To summarize, the local symmetries of these conformally invariant massless systems coupled to gravity are

LOCAL SYMMETRIES:

- Diffeomorphism invariance
- Weyl rescaling invariance
It turns out that, after regularization, both symmetries cannot survive at the same time. One always chooses to maintain diffeomorphism invariance (conservation of energy-momentum). This is done at the price of a Weyl anomaly

\[ A = g^{\mu\nu} \langle T_{\mu\nu} \rangle_{\text{reg}} \neq 0 \]

Note: Weyl anomalies are also called “Trace anomalies” or “Conformal anomalies” for obvious reasons.
Generating functionals

- Generating functional of Green's functions:
  \[ Z[J] = \int D\Phi e^{\frac{i}{\hbar} \int d^n x [\mathcal{L}(\Phi, \partial \Phi) + J(x)\Phi(x)]} \]

- Generating functional of connected Green's functions:
  \[ W[J] = -i \ln Z[J] \]

- The generating functional of 1PI Green's functions
  \[ \Gamma[\Phi_c] = W[J] - \int d^n x \Phi_c(x)J(x), \quad \Phi_c(x) := \frac{\delta W[J]}{\delta J(x)} \]

The functional \( \Gamma \) is also called quantum action or effective action. In following, write \( \Phi \) in place of \( \Phi_c \).

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1. **An anomaly in QFT ...**

Anomalies occur when quantization spoils symmetries of the classical action, i.e. if $\Gamma[\Phi]$ cannot be made invariant under infinitesimal transformations $s$ by a suitable choice of local counterterms.

2. **... is an infinitesimal variation**

To lowest order in $\hbar$ the variation $A = s \Gamma[\Phi]$ is local. It is an anomaly if it cannot be written as $A = s C$ for any local functional $C$. 
Consistency conditions

Because an anomaly is a variation

\[ A = s \Gamma[\Phi] \]

it is not arbitrary but constrained to obey consistency conditions. Similar to integrability conditions \( \nabla \times F = 0 \) which a gradient \( F = \nabla \varphi \) has to satisfy.

\[ \Rightarrow \text{An anomaly must satisfy the} \]

Wess-Zumino consistency conditions [1971]
The analysis of WZ consistency conditions simplifies in the Becchi-Rouet-Stora-Tyutin (BRST) formulation.

\[
\Rightarrow \text{one introduces a ghost for each gauge parameter;}
\]

\[
\Rightarrow \text{one suitably defines the transformations of the ghosts so that}
\]

\[
[s^2] = 0
\]

**Local cohomology of \( s \)**

The WZ consistency conditions take the simple form

\[
sA = 0, \quad A \neq sC
\]

where \( A \) and \( C \) are **local** functionals \( A = \int a^n_1([\Phi], x), \quad C = \int b^n_1([\Phi], x) \) and \( s \) is the BRST differential.
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Central equations for candidate anomalies in QFT: Wess-Zumino (WZ) consistency conditions. By using these conditions, the general structure of all the known anomalies (except the conformal one) had been determined by purely algebraic methods featuring descent equations à la Stora-Zumino.

Determining the general solution of the WZ consistency conditions is tantamount to computing the cohomology of the corresponding Becchi-Rouet-Stora-Tyutin (BRST) differential $s$ in the space of local functionals with ghost number one.
Stora-Zumino descent of equations?

With $A = \int a_1^n$, the WZ conditions get translated to

\[
sa_1^n + d a_2^{n-1} = 0, \quad a_1^n \sim a_1^n + s c_0^n + d c_1^{n-1}
\]  

(2)

with the total exterior derivative $d = dx^\mu \frac{\partial}{\partial x^\mu}$. One has

\[
s^2 = 0, \quad d^2 = 0,
\]

\[
\{s, d\} := sd + ds = 0.
\]

Acting on (2) with $s$ and using the above relations:

\[
d (s a_2^{n-1}) = 0 \quad \text{algebraic Poincaré lemma} \quad \Rightarrow \quad sa_2^{n-1} + d a_3^{n-2} = 0.
\]

Apply $s$ again on this equations, ...
... one obtains the following descent equations

\[ s a_1^n + d a_2^{n-1} = 0 , \]
\[ s a_2^{n-1} + d a_3^{n-2} = 0 , \]
\[ \vdots \]
\[ s a_q^{n-q+1} + d a_{q+1}^{n-q} = 0 , \]
\[ s a_{q+1}^{n-q} = 0 \quad (0 \leq q \leq n) . \]

If \( q = 0 \), the descent is trivial: \( s a_1^n = 0 \).

**Dubois-Violette, Talon, Viallet (1985)**

- In order to find \( a_1^n \in H^{1,n}(s|d) \), find the \( a_{q+1}^{n-q} \in H(s) \) that can be lifted up to a top form.
Cohomological consideration, although without any descent equation analysis → pioneering works by Bonora, Cotta-Ramusino, Reina, Pasti and Bregola [1983–1985]. Results up to even dimension $n = 6$.

They found:

(i) Euler term

$$
\varepsilon_1^n = \sqrt{-g} \omega (R^{\mu_1\nu_1} \ldots R^{\mu_m\nu_m}) \varepsilon_{\mu_1\nu_1 \ldots \mu_m\nu_m},
$$

plus

(ii) strictly Weyl-invariant scalar densities. In $n = 4$ e.g.

$$
a_1^4 = \omega \sqrt{-g} g^{\sigma\tau} g^{\lambda\kappa} W_{\rho\sigma\lambda} W_{\mu\tau\kappa} d^4 x
$$

where $W_{\rho\sigma\lambda} :$ conformally invariant Weyl tensor, traceless part of Riemann curvature tensor $R^{\mu}_{\rho\sigma\lambda}$. 


Using **dimensional regularization**, Deser and Schwimmer confirmed the structure obtained by Bonora et al. The Euler term from class (i) was called **type-A Weyl anomaly**, while the terms of (ii) were called **type-B anomalies**;

From the structure of the poles in the variation of the effective action, they observed that the **type-A anomaly** appears in a similar way to the **non-Abelian chiral anomaly** in Yang-Mills gauge theory. That the type-A anomaly should arise via some **descent equations** was therefore conjectured.
Apart from $g_{\mu\nu}$, the other fields of the problem are the Weyl ghost $\omega$ and the diffeomorphisms ghosts $\xi^\mu$, $gh(\xi^\mu) = gh(\omega) = 1$.

The BRST transformations on the fields $\Phi^A = \{g_{\mu\nu}, \omega, \xi^\mu\}$ read

\[
\begin{align*}
    s_D g_{\mu\nu} &= \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}, \\
    s_W g_{\mu\nu} &= 2\omega g_{\mu\nu}, \\
    s_D \xi^\mu &= \xi^\rho \partial_\rho \xi^\mu, \\
    s_D \omega &= \xi^\rho \partial_\rho \omega, \\
    s_W \xi^\mu &= 0 = s_W \omega.
\end{align*}
\]

where the BRST differentials $s_W$ and $s_D$ implement the Weyl transformations and the diffeomorphisms, respectively.
Upon quantization one always chooses to preserve diffeomorphism invariance. With $s = s_W + s_D$, decomposing $s a + d b = 0$, $a \sim a + s c + d f$ w.r.t. the Weyl ghost degree gives the WZ consistency conditions for the Weyl anomalies in terms of local forms:

\[
\begin{align*}
\begin{cases}
  s_W a_1^n + d b_2^{n-1} = 0, & a_1^n \neq s_W p_0^n + d f_1^{n-1}, \\
  s_D a_1^n + d c_2^{n-1} = 0, & \forall p_0^n \text{ s.t. } s_D p_0^n + d h_1^{n-1} = 0.
\end{cases}
\end{align*}
\]
Denoting \( \tilde{s}_W = s_W + d \) and similarly for \( s_D \), the problem (*) amounts to determining the \( \tilde{s}_D \)-invariant \((n + 1)\)-local total forms \( \alpha(W) \) satisfying

\[
\tilde{s}_W \alpha(W) = 0, \quad \alpha(W) \neq \tilde{s}_W \zeta(W) + \text{constant},
\]

(3)

where \( \zeta(W) \) must be \( \tilde{s}_D \)-invariant.

Using very general results obtained in [Friedemann Brandt, CMP 1996], we know that the solution of (3) will take the form

\[
\alpha(W) = 2\omega \tilde{C}^{N_1} \ldots \tilde{C}^{N_n} a_{N_1 \ldots N_n}(\mathcal{F}).
\]

(4)
The space $\mathcal{T}$ is generated by the (invertible) metric $g_{\mu\nu}$ together with the $W$-tensors $\{W_{\Omega_i}\}$, $i \in \mathbb{N}$ that contain $W^\mu_{\nu\rho\sigma}$, $\nabla_\tau W^\mu_{\nu\rho\sigma}$ and tower $\{\mathcal{D}_{\alpha_1} \ldots \mathcal{D}_{\alpha_n} W^\mu_{\nu\rho\sigma}\}$, $n \in \mathbb{N}$, where $[\mathcal{D}, \mathcal{D}] \sim \text{Weyl + Cotton}$.

One can write the Weyl tensor as

$$W^\mu_{\nu\rho\sigma} = R^\mu_{\nu\rho\sigma} - 2\left(\delta^\mu_{[\rho} P_{\sigma]\nu} - g_{[\rho} P^\mu_{\sigma]\nu}\right),$$

where the Schouten tensor $P_{\mu\nu}$ is

$$P_{\mu\nu} = \frac{1}{n-2} \left(\mathcal{R}_{\mu\nu} - \frac{1}{2(n-1)} g_{\mu\nu} \mathcal{R}\right).$$
Generalized connections

The generalized connections $\tilde{C}^N$ in (4):

$$\{\tilde{C}^i_N\} = \{2\omega, dx^\nu, \tilde{C}^\mu_\nu, \tilde{\omega}_\alpha\},$$

$$\tilde{C}^\mu_\nu = \Gamma^\mu_{\nu\rho} dx^\rho, \quad \tilde{\omega}_\alpha = \omega_\alpha - P_{\alpha\rho} dx^\rho, \quad \omega_\alpha = \partial_\alpha \omega.$$

Note that $\tilde{\omega}_\alpha$ is a local total form of degree 1, the sum of piece $\omega_\alpha$ with ghost number 1 but form degree 0 plus $P_{\alpha\rho} dx^\rho$ of ghost degree zero but form degree 1:

$$TotalDeg = formdeg + gh$$
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**Theorem**: Let $\psi_{\mu_1...\mu_{2p}}$ be the local total form

$$
\psi_{\mu_1...\mu_{2p}} = \frac{\omega}{\sqrt{-g}} \varepsilon^{\alpha_1...\alpha_r}_{\nu_1...\nu_r, \mu_1...\mu_{2p}} \tilde{\omega}_{\alpha_1} ... \tilde{\omega}_{\alpha_r} \; dx^{\nu_1} ... dx^{\nu_r},
$$

$$
p = m - r, \quad m = n/2, \quad 0 \leq r \leq m
$$

and let $W^{\mu\nu}$ denote the tensor-valued two-form $W^{\mu\nu} = W^\rho_{\mu} g^{\rho\nu}$, then the local total forms $\Phi_{r}^{[n-r]} (0 \leq r \leq m)$

$$
\Phi_{r}^{[n-r]} = \frac{(-1)^p}{2p} \frac{m!}{r!p!} \psi_{\mu_1...\mu_{2p}} \; W^{\mu_{1}\mu_{2}} \; ... \; W^{\mu_{2p-1}\mu_{2p}}
$$

obey a descent equations so that the following relations hold:

$$
\tilde{s}_W \alpha = 0 = \tilde{s}_W \beta
$$

with

$$
\alpha := \sum_{r=1}^{m} \Phi_{r}^{[n-r]}, \quad \beta := \Phi_{0}^{[n]}.
$$
**Theorem (A)**

The top form-degree component $a_1^n$ of $\alpha$ (cf. Theorem 1) satisfies the WZ consistency conditions for the Weyl anomalies. The WZ conditions for $a_1^n$ give rise to a non-trivial descent and $a_1^n$ is the unique anomaly with such a property, up to the addition of trivial terms and anomalies satisfying a trivial descent.

**Theorem (B)**

The top form-degree component $e_1^n$ of $(\alpha + \beta)$ is proportional to the Euler density of the manifold $\mathcal{M}_n$:

$$e_1^n = \frac{(-1)^m}{2^m} \sqrt{-g} \omega (R^{\mu_1\nu_1} \ldots R^{\mu_m\nu_m}) \varepsilon_{\mu_1\nu_1 \ldots \mu_m\nu_m}.$$  

The anomaly $\beta = \Phi^{[n]}_0$ — a contraction of a product of Weyl tensors — satisfies a trivial descent. It is a type-B anomaly.
Universal structure of Weyl anomalies established in a purely algebraic manner, independently of any regularization scheme and in arbitrary dimensions $n$. The type-A Weyl anomaly: The unique Weyl anomaly satisfying a non-trivial descent of equations.
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From anomalies to invariants

- Conformal anomalies are related to global conformal invariant. The Deser-Schwimmer paper triggered the interest of mathematicians working in the field of conformal geometry.

- Global conformal invariants are given by the integral over a $n$-dimensional (pseudo) Riemannian manifold $\mathcal{M}_n(g)$ of linear combinations of strictly Weyl-invariant scalar densities with scalar densities that are invariant under Weyl rescalings only up to a total derivative.

- What is the general structure of the latter?
  - relevant for deformations of Weyl-invariant Lagrangians densities.
By the assumption of *locality*, a global invariant is a ghost-zero scalar density whose Hodge dual $a^{0,n}$ obeys the cocycle equation

$$sa^{0,n} + db^{1,n-1} = 0.$$ 

The local conformal invariants are (the integral of) scalar densities that are strictly Weyl invariant. They can be built using various techniques, be them algebraic or geometric [tractor calculus].

The global invariants are scalar densities that are Weyl invariant only up to a total derivative $\Rightarrow$ Produce a non-trivial descent equations.
Non-trivial descent equations:

\[
\begin{align*}
  sa^{0,n} + da^{1,n-1} &= 0 \\
  sa^{1,n-1} + da^{2,n-2} &= 0 \\
  &\quad \vdots \\
  sa^{p-1,n-p+1} + da^{p,n-p} &= 0 \\
  sa^{p,n-p} &= 0 
\end{align*}
\]

It stops either because \( p = n \) or because one encounters an \( s \)-cocycle \( a^{p,n-p} \).

Decomposing the first equation wrt Weyl-ghost degree:

\[
\begin{cases}
  s_D a^{0,n} + df^{1,n-1} = 0, \\
  s_W a^{0,n} + dg^{1,n-1} = 0,
\end{cases}
\]

\( a^{0,n} \neq db^{0,n-1} \).
The classification of global conformal invariants is also given by the cohomology of the associated BRST differential in top form degree $n$, but this time, at ghost number zero, i.e., $H^{0,n}(s|d)$. The two cohomological groups $H^{1,n}(s|d)$ (anomalies) and $H^{0,n}(s|d)$ present some similarities but also important differences. The latter group is the larger!

The conjecture of Deser and Schwimmer on the structure of Weyl anomalies led the mathematician Spyros Alexakis to study the problem of the classification of global conformal invariants.

Pursuing the cohomological analysis

- From
  \[
  \begin{align*}
  s_D a^{0,n} + df^{1,n-1} &= 0, \\
  s_W a^{0,n} + dg^{1,n-1} &= 0,
  \end{align*}
  \]

  Find the cocycles of the differential \( s_W \) modulo \( d \), in the cohomology of the diffeomorphism-invariant local \( n \)-forms.

- The latter cohomology class already been worked out in [Brandt-Dragon-Kreuzer89] and [Barnich-Brandt-Henneaux95].

- Denote by \( f_K := \text{Tr}(R^m(K)) \), \( K \in \{1, \ldots, r = [n/2]\} \), the invariant polynomials of the Lorentz algebra \( so(1,n-1) \) and \( q^0_K \) the corresponding Chern-Simons \( (2m(K) - 1) \)-forms obeying \( dq^0_K = f_K \). The general solution of the first equation above decomposes into two main classes:
• Two main classes:

\[ a^{0,n} = \sqrt{-g} \left( L(\nabla, R, g)\,d^n x + \sum_{m} \sum_{\text{K}:m(\text{K})=m} q^K_0 \frac{\partial}{\partial f_\text{K}} P_m(f_1, \ldots, f_r) \right) \]

• The second class only contributes for spacetimes of dimensions \( n = 4p - 1 \), \( p \in \mathbb{N}^* \). Taking \( n = 7 \) as a definite example, the second class gives two structures

\[
\text{Tr}(\Gamma d\Gamma + \frac{2}{3} \Gamma^3)\text{Tr}(R^2) \equiv L^3_{CS} \text{Tr}(R^2) \text{ and } L^7_{CS} = \text{Tr}(I_7),
\]

\[ I_7 = \Gamma (d\Gamma)^3 + \frac{8}{5} (d\Gamma)^2 \Gamma^3 + \frac{4}{5} \Gamma (d\Gamma)^2 + 2 \Gamma^5 d\Gamma + \frac{4}{7} \Gamma^7, \]

where \( \Gamma \) denotes the matrix-valued 1-form \( dx^\mu \Gamma^\alpha_{\beta\mu} \) whose components \( \Gamma^\alpha_{\beta\mu} \) are the Christoffel symbols and \( \text{Tr}(\cdot) \) denotes the matrix trace.

\( \text{Tr}R^2 \equiv R^\alpha_\beta R^\beta_\alpha \) for \( R^\alpha_\beta = \frac{1}{2} dx^\mu dx^\nu R^\alpha_{\beta\mu\nu} \) the curvature 2-form.
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**Lemma 1:**

Let \( \psi_{\mu_1...\mu_{2p}} \) be the local total form

\[
\psi_{\mu_1...\mu_{2p}} = \frac{1}{\sqrt{-g}} \varepsilon^{\alpha_1...\alpha_r \nu_1...\nu_r \mu_1...\mu_{2p}} \tilde{\omega}_{\alpha_1} \ldots \tilde{\omega}_{\alpha_r} \, dx^{\nu_1} \ldots dx^{\nu_r},
\]

\[
p = m - r, \quad m = n/2, \quad r \in \{0, \ldots, m\}.
\]

Then, the local total forms

\[
\Phi_{[n-r]}^{r} = \frac{(-1)^p}{2^p} \frac{m!}{r! \, p!} \psi_{\mu_1...\mu_{2p}} \, W^{\mu_1 \mu_2} \ldots W^{\mu_{2p-1} \mu_{2p}}
\]

satisfy non-trivial descent equations and give solutions

\[
\tilde{s}_W \alpha = 0 = \tilde{s}_W \beta \quad \text{for}
\]

\[
\alpha = \sum_{r=1}^{m} \Phi_{[n-r]}^{r} \quad \text{and} \quad \beta = \Phi_{0}^{[n]}.
\]
The top form-degree component $a^{0,n}$ of $\alpha$ in Lemma 1 satisfies the cocycle condition for the conformal invariants. It gives rise to a non-trivial descent in $H(s_W|d)$. The invariant $\beta = \Phi_0^n$ satisfies a trivial descent and is obtained by taking contractions of products of Weyl tensors ($m$ of them in dimension $n = 2m$). The top form-degree component $e^{0,n}$ of $\alpha + \beta$ is proportional to the Euler density of the manifold $\mathcal{M}_n$:

$$e^{0,n} = \frac{(-1)^m}{2^m} \sqrt{-g} \varepsilon_{\alpha_1\beta_1...\alpha_m\beta_m} (R^{\alpha_1\beta_1} \wedge \ldots \wedge R^{\alpha_m\beta_m})$$

It is the only conformal invariant of the class I that satisfies a non-trivial descent in $H(s_W|d)$. 
**Lemma 3 [Invariants of class II]**

Let $\alpha_{[2m-1]}^{4p-1}$ be the total $(4p - 1)$-form of degree $2m - 1$ in the connection 1-form $\Gamma$, defined by

$$\alpha_{[2m-1]}^{4p-1} := -\frac{1}{2m-1} \text{Tr} \left( [\omega dx - R]^{2p-m} \Gamma^{2m-1} \right), \quad m = 1, 2, \ldots, 2p,$$

$$\alpha_{[0]}^{4p-1} := 2\omega (d\omega)^{2p-1},$$

where $[\omega dx - R]$ stands for the matrix-valued total 2-form with components $\omega^\alpha dx^\beta - R^\alpha_\beta$ and $\Gamma$ denotes the matrix-valued 1-form with $\Gamma^\alpha_\beta$ for components. Then, the total form

$$\tilde{\alpha}^{4p-1} := \alpha_{[0]}^{4p-1} + \sum_{m=1}^{2p} \alpha_{[2m-1]}^{4p-1}$$

obeys the equation

$$\tilde{s}_W \tilde{\alpha}^{4p-1} = \text{Tr} R^{2p}.$$
By decomposing the equation $\tilde{s}_W \tilde{\alpha}^{4p-1} = \text{Tr} R^{2p}$ with respect to the form degree, we obtain, in dimension $n = 4p - 1$, the descent equations

$$\text{Tr} R^{2p} = dL^n_{CS},$$

$$s_W L^n_{CS} + da^{1,n-1} = 0,$$

$$s_W a^{1,n-1} + da^{2,n-2} = 0,$$

$$\vdots$$

$$s_W a^{2p-1,2p} + da^{2p,2p-1} = 0,$$

$$s_W a^{2p,2p-1} = 0, \quad a^{2p,2p-1} \equiv \alpha^{4p-1}[0]. \quad (5)$$

Equation (5) is the WZ consistency condition for a conformal anomaly in a submanifold of co-dimension 1 wrt $\mathcal{M}_{4p-1}$. The consistent Weyl anomaly is the integral, over this co-dimension one submanifold $\mathcal{M}_{4p-2}$, of

$$a^{1,n-1} = \sum_{m=1}^{2p} \frac{(-1)^m}{2m-1} \ dx^\mu g_{\mu \alpha} [\Gamma^{2m-1} R^{2p-m-1}]^\alpha_\beta \ g^{\beta \sigma} \omega_\sigma.$$

Finally, descent equations associated with a product of the type $L_{CS}^{4p-1} f_{K_1} \cdots f_{K_m}$ will be exactly the same as the descent associated with $L_{CS}^{4p-1}$, where each element $a^{q,n-q}$ is obtained from the corresponding one in the descent for $L_{CS}^{4p-1}$ upon taking the wedge product with $f_{K_1} \cdots f_{K_m}$. In other words, the products of the type $f_{K_1} \cdots f_{K_m}$ are completely spectators in a descent of $s_W$ modulo $d$. That the $f_K$’s are $s_W$-closed is trivial once one realizes the identity $\text{Tr}(R^{m(K)}) \equiv \text{Tr}(W^{m(K)})$ that is obtained from the relation $R^{ab} = W^{ab} + 2e^{[a} P^{b]}$ where $e^a$ are the vielbein 1-forms and $P^a$ is the Schouten 1-form.
**Action and Field Equations**

- Given a pseudo-Riemannian spacetime $\mathcal{M}_{4p-1}$ of dimension $n = 4p - 1$ with an orientation, consider the functional

\[
I[g_{\mu\nu}] = \frac{1}{2p} \int_{\mathcal{M}_{4p-1}} L^{4p-1}_{CS}.
\]

- The Euler-Lagrange derivative (wrt the metric) of the functional is

\[
\mathcal{E}^{\mu\nu} := \frac{\delta I}{\delta g_{\mu\nu}} = \frac{1}{2^{2p-1}} \nabla^\lambda \mathcal{A}^{(\mu|\nu)} \lambda,
\]

where

\[
\mathcal{A}^{\mu|\nu} \lambda := \varepsilon^{\mu\nu_2\nu_3\ldots\nu_{4p-1}} \left[ R_{\nu_2\nu_3} \ldots R_{\nu_{4p-2}\nu_{4p-1}} \right]^\nu \lambda.
\]

and $[R_{\nu_2\nu_3} \ldots R_{\nu_{4p-2}\nu_{4p-1}}]^\nu \lambda$ denotes the $(2p - 1)$-fold product of the 2-form valued matrix $[R_{\nu_2\nu_3}]^\alpha_\beta \equiv R^\alpha_\beta \nu_2\nu_3$. 

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Classification of Conformal Invariant  
15 January 2019
Weyl and diffeomorphism invariances of the action $I[g_{\mu\nu}]$ get translated into the Noether identities

$$g_{\mu\nu} e^{\mu\nu} \equiv 0 \,, \quad \text{and} \quad \nabla_\mu e^{\mu\nu} \equiv 0 \,.$$  

For the second identity, one must use

$$\varepsilon^{\nu_1 \ldots \nu_4 p-1} \text{Tr}[R_{\nu_1 \nu_2} \cdots R_{\nu_4 p-3 \nu_4 p-2} R_{\nu_4 p-1 \nu}] \equiv 0 \,,$$

(Schouten identity and cyclicity of the trace)

Finally, one has the strict invariance under Weyl transformations:

$$s_W e^{\mu\nu} = -2 \omega e^{\mu\nu} \Leftrightarrow s_W e^{\mu}_{\nu} = 0 \,.$$  

that can be seen by expressing

$$\mathcal{A}^{\mu|\nu}_\lambda = \varepsilon^{\mu \nu_2 \nu_3 \ldots \nu_4 p-1} [W_{\nu_2 \nu_3} \cdots W_{\nu_4 p-2 \nu_4 p-1}]^\nu_\lambda \,.$$
Conclusions

- As a consequence of our decomposition, global conformal invariants are not in one-to-one correspondence with the conformal anomalies. Indeed, multiplying the Lorentz Chern-Simons densities by the Weyl parameter $\sigma(x)$ does not produce any consistent conformal anomaly.

- Our work generalises the analyses devoted to the three-dimensional case $p = 1$ [see TMG and PvN] and completes the results obtained in the book by Alexakis, where the global conformal invariants related to the Lorentz Chern-Simons densities were overlooked.

- Prospect : Higher-derivative TMG?