

## QUANTIFIER ELIMINATION IN VALUED ORE MODULES

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**Abstract.** We consider valued fields with a distinguished isometry or contractive derivation as valued modules over the Ore ring of difference operators. Under certain assumptions on the residue field, we prove quantifier elimination first in the pure module language, then in that language augmented with a chain of additive subgroups, and finally in a two-sorted language with a valuation map. We apply quantifier elimination to prove that these structures do not have the independence property.

**§1. Introduction.** The model theory of Witt vectors with the Witt Frobenius as a distinguished automorphism has been investigated in [5], [28], [6]. The results are of Ax-Kochen-Ershov type. For instance:

Let  $K = W(F)$  be the field of Witt vectors with coefficients in  $F$ , where  $F$  is a  $p$ -closed field of characteristic  $p$ , let  $v$  be the  $p$ -adic valuation, and let  $\sigma$  be the Witt Frobenius (see below). Then  $Th(K, v, \sigma)$  is axiomatized by:

1.  $(K, v)$  is a valued field of characteristic 0 and  $\sigma$  is an isometry, i.e.,  $v(\sigma(x)) = v(x)$ , inducing the ordinary Frobenius  $x \mapsto x^p$  on the residue field  $F$ ;
2. a suitable analog of Hensel's lemma holds for polynomials involving  $\sigma$ ;
3. the residue field satisfies  $Th(F)$ ;
4. the value group is a  $\mathbb{Z}$ -group with least positive element  $v(p)$ .

In this paper we will consider the theory of these fields, and other valued fields with an isometry, in weaker formalisms of (valued) modules. This amounts essentially to restrict ourselves to study *linear* equations of the form  $c_n\sigma^n(X) + \dots + c_1\sigma(X) + c_0X = b$ , where  $c_i, b \in K$ . Ore was the first to study systematically these equations in the case of the usual Frobenius map  $x \mapsto x^p$ . There is a well established analogy between differential fields and fields with a distinguished automorphism, also called difference fields in the literature. It turns out that if a valued field has a derivation  $\partial$  which is contractive, i.e., such that  $v(\partial x) \geq v(x)$ , then many of our results hold as well for these structures. We will treat both cases simultaneously in a suitable formalism. In all main results, a linear form of an analog of Hensel's lemma (alluded to above) plays a crucial role and corresponds to divisibility of the corresponding module. The main results, which apply also in positive characteristic, consist of axiomatizations and quantifier elimination in appropriate languages (see Sections 4

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and 5), with an application to the absence of the independence property. They were sketched in [7].

Some of the basics also hold when  $\sigma$  is just an endomorphism (e.g., the Hensel property) but we have not pursued the matter further.

The plan of the paper is as follows. In Section 2 we introduce the module formalism and we axiomatize the theories of valued fields with an isometry or a contractive derivation as modules over a (Ore) skew polynomial ring. In Section 3 we introduce the formalisms of valued modules, first as two-sorted structures with a valuation map between the module sort and the ordered set sort, and then as modules with a chain of distinguished additive subgroups indexed by their value set which yield an abelian structure. In Section 4 we prove quantifier elimination in the abelian structure formalism, and then in Section 5 for the two-sorted one. In Section 6 we apply quantifier elimination to prove that in the two-sorted language, these structures do not have the independence property (and so their definable sets can be endowed with a VC-dimension, see e.g., [21]). Finally, in Section 7, we observe that all this allows to prove a transfer principle between Witt vectors and power series.

**Notation and terminology.** We will use boldface notation for tuples, e.g.,  $\mathbf{x} = (x_1, \dots, x_n)$ . All our modules will be right modules. For an element  $r$  of a ring,  $\text{ann}(r)$  will denote its annihilator in a given module. For a valued field  $(K, v)$ , we denote its value group by  $vK$ , its valuation ring by  $\mathcal{O}_K$ , its residue field by  $\bar{K}$ . The natural residue map from  $\mathcal{O}_K$  to  $\bar{K}$  is denoted by  $\bar{\phantom{x}}$ , and will be used for various reductions. For example if  $q(X) \in \mathcal{O}_K[X]$ , then  $\bar{q}$  is the reduced element in  $\bar{K}[X]$  obtained by reducing the coefficients. An isometry is an automorphism of  $K$  such that  $v(\sigma(x)) = v(x)$ . A contractive map is a map  $\partial : K \rightarrow K$  such that  $v(\partial(x)) \geq v(x)$ . If  $f$  is a function on a set  $X$  we will sometimes use the notation  $x^f$  for  $f(x)$ ,  $x \in X$ .

Let  $F$  be a perfect field of characteristic  $p > 0$ , we denote by  $W[F]$  the ring of Witt vectors over  $F$ . It can be seen as a ring structure on  $F^\omega$  given by universal polynomials  $s_n(X_0, Y_0, \dots, X_n, Y_n)$ ,  $p_n(X_0, Y_0, \dots, X_n, Y_n)$  with coefficients in the integers and without constant terms:  $(x_0, x_1, \dots) + (y_0, y_1, \dots) = (x_0 + y_0, s_1(x_0, y_0, x_1, y_1), s_2(x_0, \dots), \dots)$  and similarly for the product using the  $p_n$ 's. It is a complete discrete valuation ring of characteristic 0 with uniformizing parameter  $p$  and residue field  $F$ . Its field of fractions is denoted by  $W(F)$ . From the point of view above, the corresponding valuation is  $v(x_0, x_1, \dots) = \min\{n : x_n \neq 0\}$ , and it is apparent that the map  $(x_0, x_1, \dots) \mapsto (x_0^p, x_1^p, \dots)$  is an automorphism of  $W[F]$  which induces an isometry on  $W(F)$ . We call this map the Witt Frobenius. If  $F = \mathbb{F}_p$ , then  $W[\mathbb{F}_p]$  is the ring  $\mathbb{Z}_p$  of  $p$ -adic integers,  $W(\mathbb{F}_p)$  is the field  $\mathbb{Q}_p$  of  $p$ -adic numbers, and the Witt Frobenius coincides with the identity. We refer to [19] for basics on Witt vectors.

A field of characteristic  $p$  is  $p$ -closed if it does not have any finite algebraic extension of degree divisible by  $p$ . G. Whaples showed that this is equivalent to the fact that every polynomial of the form  $\sum c_i x^{p^i} = c$  has a root (for a proof “by hand” see *Afterthought: Maximal fields with valuation* in [20]). In particular, a  $p$ -closed field is perfect.

A ring is said to be right Ore, if any two elements have a common right multiple.

We will assume known classical results on the model theory of modules, and use the usual “pp” for “positive primitive”, e.g., pp-formula, etc. For the model theory of modules we refer to [25].

**§2. Modules.** In order to deal simultaneously with the cases of a valued field with an isometry or with a contractive derivation, it will be convenient for us to follow the formalism of Cohn’s book [9]. Other formalisms are possible to cover the two cases, e.g.,  $D$ -valued fields (cf. [28]) or valued fields with a twist (cf. [26]), but Cohn’s seems more fitting to our context.

**DEFINITION 2.1.** Let  $K$  be a field and  $\sigma$  an automorphism of  $K$ . A map  $\partial : K \rightarrow K$  is called a  $\sigma$ -derivation, if  $\partial(x + y) = \partial(x) + \partial(y)$  and  $\partial(xy) = \partial(x)y^\sigma + x\partial(y)$ , for all  $x, y \in K$ .

When  $\sigma$  is the identity, this is just a standard derivation. Note that since for us  $K$  is always commutative, when  $\sigma$  is not the identity the  $\sigma$ -derivations are all of the form  $\partial(x) = (\sigma(x) - x)c$  for some nonzero  $c \in K$  ([9], Theorem 2.1.3).

We now fix  $(K, v, \sigma, \partial)$ , namely a valued field  $(K, v)$  with distinguished isometry  $\sigma$  and contractive  $\sigma$ -derivation  $\partial$ .

If  $\partial$  is the zero derivation, then we just have a valued field with an isometry, and if  $\sigma$  is the identity map, then we just have a valued field with a contractive derivation.

The operators  $\partial$  and  $\sigma$  have an induced action on  $\bar{K}$ , that will be denoted by  $\bar{\sigma}$  and  $\bar{\partial}$  respectively, namely  $\bar{\sigma}(\bar{a}) = \overline{\sigma(a)}$  and  $\bar{\partial}(\bar{a}) = \overline{\partial(a)}$ , which make  $\bar{\partial}$  into a  $\bar{\sigma}$ -derivation. More generally, given any polynomial  $q[X] \in \mathcal{O}_K[X]$ , we can consider the action of  $q(\sigma)$  on  $K$  and the induced action of  $\bar{q}(\bar{\sigma})$  on  $\bar{K}$ .

**DEFINITION 2.2.** [9] We let  $A = K[t; \sigma, \partial]$  be the skew polynomial ring in the variable  $t$  with the commutation rule for  $k \in K$ :

$$kt = tk^\sigma + \partial(k).$$

When  $\partial = 0$ , we will simply write  $A = K[t; \sigma]$ .

We let  $A_0 = \mathcal{O}_K[t; \sigma, \partial]$  be the subring of  $A$  consisting of polynomials with coefficients from  $\mathcal{O}_K$ , and we let  $\mathcal{S}$  be the set of polynomials from  $A_0$  with at least one coefficient of valuation 0.

In  $A$  we will write the polynomials in the form  $\sum t^i k_i$ .

First, let us assume that  $\partial = 0$ . Then  $K$  can be considered as a module over  $A$ , with  $t$  acting as  $\sigma$ , in the following way:

$$k \cdot \left( \sum_{i=0}^n t^i k_i \right) = \sum_{i=0}^n \sigma^i(k) k_i.$$

Let  $\partial \neq 0$  be a non-trivial  $\sigma$ -derivation, then  $K$  can be considered as a module over  $A$ , with  $t$  acting as  $\partial$ , in the following way:

$$k \cdot \left( \sum_{i=0}^n t^i k_i \right) = \sum_{i=0}^n \partial^i(k) k_i.$$

Note that when  $\sigma$  is not the identity, then as noted above  $\partial$  is of the form  $\partial(x) = (\sigma(x) - x)c$ , and we are back to the first case. If we make the following change of variable, setting  $t' := t \cdot c^{-1} + 1$ , we have that  $A$  is isomorphic to  $K[t'; \sigma]$  with the

commutation rule  $kt' = t'k^\sigma$ . So, we basically have two cases: either  $t$  acts as  $\sigma$  (or a linear polynomial in  $\sigma$ ) with  $\partial = 0$ , or  $t$  acts as a classical derivation with  $\sigma$  being the identity.

Let  $(F((x^{-1})), \tau)$  be the field of Laurent series in  $x^{-1}$  over a field  $F$ , let  $\tau$  be an automorphism of  $F((x^{-1}))$ . Let  $K := F((x^{-1}))(T)$  with the  $T$ -adic valuation, let  $\sigma$  be the automorphism of  $K$  defined by  $\sigma(\sum_i c_i T^i) := \sum_i \tau(c_i) T^i$ . Then, with  $\partial = 0$ ,  $t$ -motives in [15] are a special kind of  $A_0$ -module.

Let  $\bar{A} = \bar{K}[t; \bar{\sigma}, \bar{\partial}]$ . The reduction map  $\bar{\cdot}$  makes  $\bar{K}$  into a corresponding  $\bar{A}$ -module via the action of  $\bar{\sigma}$  or  $\bar{\partial}$ .

The skew polynomial ring  $A$  is right Euclidean and left Euclidean (see [9], Chapter 2). In particular it is an integral domain and right Ore. The degree of an element  $q(t) = \sum_{i=0}^d t^i k_i$  with  $k_i \in K$  and  $k_d \neq 0$ , is equal to  $d$ .

The center of  $A$  is contained in  $Fix(\sigma) \cap K_\partial$ , where  $K_\partial = \{x \in K : \partial(x) = 0\}$  and  $Fix(\sigma) = \{x \in K : \sigma(x) = x\}$ . Note that since a  $\sigma$ -derivation is of the form  $c(\sigma - 1)$ , whenever  $\sigma \neq 1$ , the subfield  $Fix(\sigma) \cap K_\partial$  is either equal to  $K_\partial$  or to  $Fix(\sigma)$ .

DEFINITION 2.3. Let  $q(t) \in A$ . We will say that  $q(t)$  is *irreducible* if it cannot be expressed as a product of two elements of  $A$  of degree bigger than or equal to 1. We will say  $q(t)$  is *separable* if  $q(0) \neq 0$ .

Since  $K$  is a valued field, we can extend the map  $v$  on the ring  $A$  as follows:

$$v(\sum_i t^i k_i) := \min_i \{v(k_i)\}$$

where  $k_i \in K \setminus \{0\}$  and  $v(0) = \infty$  (see [9], Chapter 9). In the case where  $\partial = 0$ , since  $v(k^\sigma) = v(k)$  for any  $k \in K$ , this is a valuation (ibid., pp. 425-426), which can be extended to the fraction field of  $A$  (ibid., Proposition 9.1.1). For the convenience of the reader we will show that this is the case in general.

Note that  $A_0 = \{q \in A : v(q) \geq 0\}$  and  $\mathcal{S} = \{q \in A : v(q) = 0\}$ .

LEMMA 2.1. *The map  $v$  as defined above is a valuation on  $A$ .*

PROOF. We need only check the valuation of a product. By induction on  $n$ , let us first show that

$$at^n = \sum_{i=0}^n t^i \left( \sum_{m_i \in C_i^n} a^{m_i(\sigma, \partial)} \right)$$

where  $m_i(\sigma, \partial)$  denotes a monomial of degree  $n$  in  $\sigma, \partial$  with  $i$  occurrences of  $\sigma$  and  $C_i^n$  denotes the set of such monomials, whose cardinal is the corresponding binomial coefficient. We have

$$\begin{aligned} at^{n+1} &= (at^n) t = \left( \sum_{i=0}^n t^i \left( \sum_{m_i \in C_i^n} a^{m_i(\sigma, \partial)} \right) \right) t \\ &= \sum_{i=0}^n t^{i+1} \left( \sum_{m_i \in C_i^n} a^{m_i(\sigma, \partial)\sigma} \right) + \sum_{i=0}^n t^i \left( \sum_{m_i \in C_i^n} a^{m_i(\sigma, \partial)\partial} \right) \\ &= t^{n+1} a^{\sigma^{n+1}} + \sum_{i=0}^{n-1} t^{i+1} \left( \sum_{m_i \in C_i^n} a^{m_i(\sigma, \partial)\sigma} \right) + \sum_{i=0}^{n-1} t^{i+1} \left( \sum_{m_i \in C_{i+1}^n} a^{m_i(\sigma, \partial)\partial} \right) + a^{\partial^n} \end{aligned}$$

$$= \sum_{i=0}^{n+1} t^i \left( \sum_{m_i \in C_i^{n+1}} a^{m_i(\sigma, \partial)} \right).$$

Now consider

$$\begin{aligned} \sum_{i=0}^n t^i a_i \cdot \sum_{j=0}^m t^j b_j &= \sum_{i=0}^n t^i \sum_{j=0}^m a_i t^j b_j \\ &= \sum_{i=0}^n \sum_{j=0}^m \left( \sum_{\ell=0}^j t^{\ell+i} \left( \sum_{m_\ell \in C_\ell^j} a_i^{m_\ell(\sigma, \partial)} \right) b_j \right) \end{aligned}$$

Let  $0 \leq i_1 < \dots < i_\mu \leq n$  be the indices where  $v(a_i)$ ,  $i \in \{0, \dots, n\}$ , is minimum and let  $0 \leq j_1 < \dots < j_\mu \leq m$  be the indices where  $v(b_j)$ ,  $j \in \{0, \dots, m\}$ , is minimum. Let  $k = i_\mu + j_\mu$ . Then the coefficient of  $t^k$  is equal to  $a_{i_\mu}^{\sigma^{i_\mu}} b_{j_\mu}^{\sigma^{j_\mu}} + \sum_{i,j} a_i^{m_i} \cdot b_j^{m_j}$ , where  $\partial$  occurs at least once in  $m_i$  and  $m_j$  and  $i + j > k$ , so either  $i > i_\mu$  and so  $v(a_i^{m_i}) > v(a_{i_{k_1}})$  or  $j > j_\mu$  and so  $v(b_j^{m_j}) > v(b_{j_\mu})$ . Hence the minimum valuation possible  $v(a_{i_\mu}^{\sigma^{i_\mu}}) + v(b_{j_\mu}^{\sigma^{j_\mu}}) = v(a_{i_\mu}) + v(b_{j_\mu})$  is attained, as wanted.  $\dashv$

We will show that  $A_0$  satisfies a generalized right Euclidean algorithm and so is right Ore. We will state that result in a more general setting.

LEMMA 2.2. *Let  $D$  be a right Ore domain,  $\alpha$  a monomorphism of  $D$  and  $\partial$  a  $\alpha$ -derivation. Then the skew polynomial ring  $D[t; \alpha, \partial]$  satisfies a generalized right Euclidean algorithm. Namely, given any  $q_1(t)$ ,  $q_2(t)$  with  $\text{deg}(q_1) \geq \text{deg}(q_2)$ , there exist  $c \in D \setminus \{0\}$  and  $f, r \in D[t; \alpha, \partial]$  with  $\text{deg}(r) < \text{deg}(q_2)$  such that  $q_1 c = q_2 f + r$ .*

PROOF. (See [18], Theorem 2.14, p.128). By induction on  $\text{deg}(q_1) + \text{deg}(q_2)$ . Write  $q_1 = \sum_{i=0}^n t^i a_i$  and  $q_2 = \sum_{j=0}^m t^j b_j$ , with  $n \geq m$ . Then  $q_1 b_m^{\alpha^{n-m}} - q_2 t^{n-m} a_n$  is an element of  $A_0$  of degree strictly less than  $n$ . So we apply the induction hypothesis to this polynomial. The degree 0 case is handled by the fact that  $D$  is right Ore.  $\dashv$

Note that  $\mathcal{O}_K$  is a commutative integral domain and so it is certainly right Ore.

DEFINITION 2.4. Let  $L_A$  be the language of  $A$ -modules and let  $T_A$  be the  $L_A$ -theory of right  $A$ -modules.

1. Let  $T_d$  be the theory  $T_A$  together with the axioms  $\forall m \exists n (n \cdot q(t) = m)$ , where  $q(t)$  varies over the irreducible polynomials of  $A$ .
2. Let  $T_{d,\sigma}$  be the theory  $T_A$  together with:
  - (2.1)  $\forall m \exists n (m = n \cdot t)$ , &  $\forall m (m \cdot t = 0 \rightarrow m = 0)$ ,
  - (2.2)  $\forall m \exists n (n \cdot q(t) = m)$ , where  $q(t)$  varies over the separable irreducible polynomials of  $A$ .
3. Let  $T_{Ore}$  (respectively  $T_{Ore,\sigma}$ ) be the theory  $T_d$  (respectively  $T_{d,\sigma}$ ) with the axioms  $\exists n \neq 0 (n \cdot q(t) = 0)$ , where  $q(t)$  varies over all irreducible (respectively separable irreducible) polynomials of  $A$ .

Let  $M$  be an  $A$ -module. We will denote by  $Tor(M)$  the torsion part of  $M$ , namely  $Tor(M) = \{m \in M : \exists q(t) \in A, m \cdot q(t) = 0\}$ . Note that since  $A$  is right Ore,  $Tor(M)$  is a submodule of  $M$ . For each  $a \in A$ ,  $\text{ann}_M(a)$  is a  $\text{Fix}(\sigma) \cap K_\partial$ -vector space.

Since  $A$  is right and left Euclidean, the pp-formulas have a very simple form (see for instance [18], Theorem 3.8, p.181).

Recall that a theory of modules admits positive quantifier elimination if any positive primitive formula is equivalent to a finite conjunction of atomic formulas.

**PROPOSITION 2.3.** *The theory  $T_d$  of  $A$ -modules admits positive q.e. Each completion of  $T_d$  admits quantifier elimination.*

**PROOF.** This follows from classical results in the model theory of modules over right and left Euclidean rings and in particular from the fact that any existential formula is equivalent to annihilators conditions on the parameters (see [17], Proposition 2.7).  $\dashv$

We will now determine the completions of  $T_d$ . We will say that a right  $A$ -module is divisible if it is a model of  $T_d$ .

**LEMMA 2.4.** *For any pair of elements  $\{q_1(t), q_2(t)\}$  with  $\deg(q_1(t)) > \deg(q_2(t))$  of  $A$ , we have the following equivalences in any divisible  $A$ -module  $M$ :*

$$q_2(t) \text{ divides } q_1(t) \text{ if and only if } \text{ann}_M(q_2) \subseteq \text{ann}_M(q_1).$$

*Moreover, if  $q_1(t) = q_2(t) \cdot r(t)$  and if the cardinality of the quotient  $\text{ann}(q_1(t))/\text{ann}(q_2(t))$  is finite, then  $|\text{ann}(q_1(t))/\text{ann}(q_2(t))| = |\text{ann}(r(t))|$ .*

**PROOF.** See [17], Lemma 2.9 and Proposition 2.10.  $\dashv$

**COROLLARY 2.5.** *If the subfield  $\text{Fix}(\sigma) \cap K_\partial$  of  $K$  is infinite, then the completions of  $T_d$  are obtained by specifying for which irreducible polynomials  $q(t)$   $\text{ann}(q(t)) \neq \{0\}$ . In particular under this assumption  $T_{\text{Ore}}$  (respectively  $T_{\text{Ore},\sigma}$ ) admits quantifier elimination. If  $\text{Fix}(\sigma) \cap K_\partial$  is finite, then the completions of  $T_d$  are obtained by specifying for each polynomial  $q(t)$  if the cardinal  $|\text{ann}(q(t))|$  is finite and giving its value.*

**PROOF.** First, we observe that if an element  $m$  does not belong to a pp-definable subgroup  $S$ , then  $\lambda \cdot m$  does not belong to that subgroup for any  $\lambda \in \text{Fix}(\sigma) \cap K_\partial$ : let  $\lambda \neq \mu \in \text{Fix}(\sigma) \cap K_\partial$  and assume that  $\lambda \cdot m$  and  $\mu \cdot m$  belong to the same coset, then  $(\lambda - \mu) \cdot m \in S$ , so  $m \in S$  a contradiction.

So, if  $\text{Fix}(\sigma) \cap K_\partial$  is infinite, then the index of a pair of pp-definable subgroups is either 1 or infinite.

Second, by the positive quantifier elimination result, the proper definable subgroups are the annihilators of elements of  $A$ .

So in the first case, in order to determine the completions, by the above lemma, it suffices to determine which irreducible elements  $q(t)$  of  $A$  have a non-zero annihilator.

When  $\text{Fix}(\sigma) \cap K_\partial$  is finite, the completions are determined by the cardinalities of non-trivial quotients of the annihilators of elements of  $A$ , which by the above Lemma can be reduced to the cardinalities of the annihilators.  $\dashv$

Let  $N$  be a pure-injective indecomposable model of  $T_d$ . Let  $J_N := \{a \in A : \exists n \in N \setminus \{0\}, n \cdot a = 0\}$  and suppose  $\text{Tor}(N) \neq \{0\}$ . Denote the subset of separable irreducible elements of  $A$  by  $\mathcal{P}$ .

**PROPOSITION 2.6.** *Let  $N_1, N_2$  be two pure-injective indecomposable models of  $T_d$  such that  $\text{Tor}(N_1) \neq \{0\}$ . Then, there exists  $q_0 \in A$  such that  $J_{N_1} \cap \mathcal{P} = \{q \in A :$*

$\deg(q) = \deg(q_0)$  &  $\exists q_1 \exists q_2 q \cdot q_1 = q_2 \cdot q_0$ , with  $\deg(q_1), \deg(q_2) < \deg(q \cdot q_0)$ .  
 Also, for any  $a_1, a_2 \in J_{N_1} \cap \mathcal{P}$ , we have that  $\text{ann}_{N_1}(a_1) \cong \text{ann}_{N_1}(a_2)$ .

Moreover, if  $J_{N_1} \cap J_{N_2} \cap \mathcal{P} \neq \{0\}$ , then  $N_1 \cong N_2$ .

PROOF. See [10], Lemma 3.14. ⊢

LEMMA 2.7. Let  $M$  be an  $A$ -module, where each element is divisible by  $t$ . Then,  $M$  is  $A$ -divisible iff it is  $\mathcal{S}$ -divisible.

PROOF. Let  $r(t) := \sum_i t^i k_i \in A \setminus A_0$ , let  $\lambda$  be such that  $v(\lambda) = \min_i \{v(k_i)\} \neq 0$ .  
 W.l.o.g., we may assume that  $r(t)$  is separable. Set  $r_1(t) := \lambda^{-1}r(t)$ , we have  $v(r_1(t)) = 0$  i.e.,  $r_1(t) \in \mathcal{S}$ . Let  $m \in M$ . Since  $M$  is  $\mathcal{S}$ -divisible, there exists  $n$  such that  $m = n \cdot r_1(t) = n \cdot \lambda^{-1} \cdot r(t)$ . ⊢

We will now determine conditions which ensure that  $(K, v, \sigma, \partial)$  is a model of  $T_d$  or  $T_{d,\sigma}$ . We will state everything directly in terms of modules. The reader is invited to make the direct translation in terms of linear  $\sigma$ -equations and linear  $\partial$ -equations.

DEFINITION 2.5. Let  $M = (K, v, \sigma, \partial)$  be as above and viewed as an  $A$ -module. We will say that  $M$  has the *linear Hensel property*, if for any  $q(t) \in \mathcal{S}$  and  $m \in \mathcal{O}_K$ , if there is  $y \in \mathcal{O}_K$  such that  $\bar{y} \cdot \bar{q}(t) = \bar{m}$ , then there exists  $x \in \mathcal{O}_K$  such that  $x \cdot q(t) = m$  and  $\bar{x} = \bar{y}$ .

In our basic case of the action of the isometry  $\sigma$ , this is the  $\sigma$ -Hensel scheme in [6] but restricted to linear  $\sigma$ -polynomials. In the case of the action of  $\partial$ , this is the  $D$ -henselian property of [28] but restricted to linear  $D$ -polynomials. A henselian property for polynomial equations involving contractive endomorphisms (even many at a time) has also been considered in [26].

PROPOSITION 2.8. Any  $M = (K, v, \sigma, \partial)$  such that  $(K, v)$  is a complete discrete valued field and  $\bar{M} = (\bar{K}, \bar{\sigma}, \bar{\partial})$  is a divisible  $\bar{A}$ -module, has the linear Hensel property.

PROOF. Let  $q(t) \in \mathcal{O}_K[t; \sigma, \partial]$  such that  $\bar{q}(t) \neq 0$ . Let  $m \in \mathcal{O}_K$ , and assume that there exists  $u_0 \in \mathcal{O}_K$  such that  $\bar{u}_0 \cdot \bar{q}(t) = \bar{m}$ . We want to find  $u \in \mathcal{O}_K$  such that  $u \cdot q(t) = m$  and  $v(u - u_0) > 0$ .

Set  $Q_0 = u_0 \cdot q(t) - m$ , we have that  $v(Q_0) > 0$ .

We look for an element  $x \in \mathcal{O}_K$  such that

$$v((u_0 + xQ_0) \cdot q(t) - m) > v(Q_0).$$

We have  $u_0 \cdot q(t) + xQ_0 \cdot q(t) - m = Q_0 + x \cdot Q_0 q(t) Q_0^{-1} Q_0 = (1 + x \cdot Q_0 q(t) Q_0^{-1}) Q_0$ .  
 Note that  $v(Q_0 q(t) Q_0^{-1}) = v(q(t)) = 0$ .

Since  $\bar{K}$  is  $\bar{A}$ -divisible, there exists  $x_1 \in \mathcal{O}_K$  such that

$$v(1 + x_1 \cdot Q_0 q(t) Q_0^{-1}) > 0$$

So, by letting  $h_1 = x_1 Q_0$  and  $u_1 = u_0 + h_1$ , we get

$$u_1 \cdot q(t) - m = (1 + x_1 \cdot Q_0 q(t) Q_0^{-1}) Q_0$$

and the desired inequality  $v(u_1 \cdot q(t) - m) > v(Q_0) = v(u_0 \cdot q(t) - m)$ .

We now generate, by induction, sequences  $u_n, h_n, Q_n, x_n \in \mathcal{O}_K$  such that  $u_{n+1} = u_n + h_{n+1}, h_{n+1} = x_{n+1} Q_n, Q_n = u_n \cdot q(t) - m$  and that the sequence  $v(Q_n)$  is strictly increasing. Indeed, as above, there exists  $x_{n+1} \in \mathcal{O}_K$  such that  $v(1 + x_{n+1} \cdot Q_n q(t) Q_n^{-1}) > 0$ , and letting  $h_{n+1} = x_{n+1} Q_n, u_{n+1} = u_n + h_{n+1}, Q_{n+1} = u_{n+1} \cdot q(t) - m$ , we get the desired inequality  $v(Q_{n+1}) > v(Q_n)$ . Now, the sequence

$(u_n)_{n \in \omega}$  converges since  $u_{n+1} - u_n = x_{n+1}Q_n$  with  $x_{n+1} \in \mathcal{O}_K$  and  $v(Q_n) \rightarrow +\infty$ . Let  $u = \lim_n u_n$ . Then,  $u \cdot q(t) - m = u \cdot q(t) - u_n \cdot q(t) + (u_n \cdot q(t) - m) = (u - u_n) \cdot q(t) + Q_n$ . Since  $v((u - u_n) \cdot q(t)) \geq v(u - u_n)$ , we get that  $u \cdot q(t) - m = 0$ . Moreover,  $v(u_0 - u) > 0$ . ⊢

**COROLLARY 2.9.** *Assume that  $M = (K, \sigma, \partial, v)$  has the linear Hensel property, each element is  $t$ -divisible and that  $\bar{M}$  is  $\bar{A}$ -divisible. Then  $M$  is a model of  $T_d$ .*

**PROOF.** By the linear Hensel property and  $\bar{A}$ -divisibility,  $M$  is  $\mathcal{F}$ -divisible. Then apply Lemma 2.7. ⊢

**COROLLARY 2.10.** *Let  $F$  be a field of characteristic  $p$  which is  $p$ -closed.*

1. *Let  $K = W(F)$  with  $\sigma$  the Witt Frobenius. Then  $W(F)$  as an  $A$ -module with  $t$  acting as the Witt Frobenius is a model of  $T_{d,\sigma}$ .*
2. *Let  $K = F((x))$ , the field of Laurent series, with  $\sigma$  defined as  $\sigma(\sum c_i x^i) = \sum c_i^p x^i$ . Then  $F((x))$  as an  $A$ -module with  $t$  acting as  $\sigma$  is a model of  $T_{d,\sigma}$ .*

**PROOF.** Axiom (2.1) is direct since we have an automorphism. Note that  $\bar{K} = F$  and  $\bar{\sigma}(x) = x^p$ . We then use the fact that  $p$ -closed implies that every equation  $\sum c_i x^{p^i} = c$ , with  $c_i, c \in F$  has a solution in  $F$ . But this amounts to  $\bar{K}$  being  $\bar{A}$ -divisible, and Axiom (2.2) follows from the linear Hensel property and Lemma 2.7. ⊢

We will return at the end of this paper to the relation between these two models. It is interesting to note that the general setting in e.g., [6] does not cover the case of equal characteristic  $p$ , like the field of Laurent series  $F((x))$  above (but see [3] in that direction.)

We may apply these results to  $F = \tilde{\mathbb{F}}_p$ , the algebraic closure of the prime field  $\mathbb{F}_p$ , and to  $F = k_p = \bigcup_{n \in \omega} \mathbb{F}_{p^{p^n}}$  which is the minimal  $p$ -closed field (N.B. Its theory is decidable ([2])). So, both  $W(\tilde{\mathbb{F}}_p)$ ,  $W(k_p)$  are models of the theory  $T_{d,\sigma}$  with the corresponding skew polynomial ring. Since  $\text{ann}(q(t)) \neq \{0\}$  with  $q(t)$  a separable irreducible polynomial, their theories are axiomatized by  $T_{\text{Ore},\sigma}$  and they admit quantifier elimination.

Note that there are non algebraically closed fields of characteristic  $p$  where each separable additive polynomial has a non trivial zero (see [6]). We don't know whether if we require in addition the field to be  $p$ -closed that it implies it is algebraically closed. We also don't know if it is possible to characterize the elements of  $W(\tilde{\mathbb{F}}_p)$  which belong to the torsion submodule and to  $\text{ann}(q(t))$  for a specific  $q(t)$ .

The question whether the theory  $T_{\text{Ore},\sigma}$  is decidable (at least in the classical sense) only makes sense if the ring  $A$  is countable and such that some part of its existential theory is decidable. Another way to proceed would be to consider decidability questions working with BSS-machines instead of Turing machines (see [8]). In particular, this implies that the elements of the field are given.

**COROLLARY 2.11.** *Let  $K$  be an elementary substructure of  $W(F)$  where  $F$  is a field of characteristic  $p$  which is  $p$ -closed. Assume that  $K$  is recursively presented with decidable word problem. Let  $A := K[t; \sigma]$ . Then, the corresponding theory  $T_{\text{Ore},\sigma}$  is consistent and decidable.*

**PROOF.** One checks that the ring  $A$  satisfies the required condition  $D$  in [25], Chapter 17, p. 334. ⊢



Here are assumptions on  $K$  which ensure that  $\text{Fix}(\sigma)$  is infinite. In case  $K = F((x^{-1}))$ , this is a special case of a theorem of Hellegouarch (see [14], Theorem 1).

**PROPOSITION 2.12.** *Consider  $(K, v, \sigma)$ . Assume that  $K$  has the linear Hensel property with  $t$  acting as  $\sigma$ , and that  $\text{ann}(t - \mu) \neq 0$  in  $\bar{K}$  for all  $\mu \in \bar{K}$ . Then,  $\text{Fix}(\sigma)$  is infinite whenever the valuation is non trivial.*

**PROOF.** We can assume  $K$  has finite characteristic. Let  $x \in K$  with  $v(x) > 0$ . This implies in particular that  $x$  is transcendental over the prime subfield of  $K$ . Since  $\sigma$  is an isometry, we have that  $v(\sigma(x)) = v(x)$ . In other words there exists  $u$  with  $v(u) = 0$  such that  $\sigma(x) = xu$ . Note that  $\bar{u} \neq 0$ . We look for an element  $a \in K$  with  $v(a) = 0$  such that  $\sigma(xa) = xa$ . So,  $xua^\sigma = xa$ , i.e.,  $a^\sigma u = a$  or equivalently  $a^\sigma = au^{-1}$ . By assumption, we can solve non trivially the equation  $a \cdot (t - u^{-1}) = 0$  residually. So by the linear Hensel property, we get that  $K$  has a solution of that equation with  $v(a) = 0$ . Since  $xa$  is again necessarily transcendental over the prime field, this ensures that  $\text{Fix}(\sigma)$  is infinite. -†

Now we give some examples involving a derivation  $\partial$ .

Let  $(F, \partial)$  be a differential field of characteristic 0. Assume its field of constants is algebraically closed. Then, it is known that there exists an extension  $\tilde{F}$  of  $F$  with no new constants such that any finite subset of  $\tilde{F}$  is included in a finite sequence of successive Picard-Vessiot extensions and such that  $\tilde{F}$  has no proper Picard-Vessiot extensions (see [22], Theorem 3.34).

First consider  $(\tilde{F}((x)), v)$  with the  $x$ -adic valuation and  $\partial$  defined by  $\partial(\sum_{i \geq m} x^i \cdot a_i) = \sum_{i \geq m} x^i \cdot \partial(a_i)$ . Then,  $\partial$  is contractive. Second, consider  $(\tilde{F}((x^{-1})), v)$  with the  $x^{-1}$ -adic valuation and  $\partial$  defined by  $\partial(\sum_{i \geq m} a_i \cdot x^{-i}) = \partial(a_m) \cdot x^{-m} + (\sum_{i \geq m} (-i \cdot a_i + \partial(a_{i+1})) \cdot x^{-(i+1)})$  (namely we take the derivative with respect to  $x$ ). Then, again  $\partial$  is contractive.

We obtain the following.

**PROPOSITION 2.13.** *Let  $K$  be either the field of Laurent series  $(\tilde{F}((x)), v, \partial)$  or  $(\tilde{F}((x^{-1})), v, \partial)$ . Then  $K$ , viewed as an  $A$ -module with  $t$  acting as  $\partial$ , is a model of  $T_{Ore}$ .*

**PROOF.** We will prove at the same time that they are models of  $T_d$  and  $T_{Ore}$ . By Proposition 2.8 and Corollary 2.9, to prove that they are models of  $T_d$  it suffices to check that they are  $t$ -divisible and that  $\tilde{F}$  is  $\bar{A}$ -divisible.

First note that any inhomogeneous linear differential equation of order  $n \geq 1$ , of the form  $L(y) = b$ ,  $b \in \tilde{F}$ , can be reduced to the homogeneous equation  $b \cdot \partial(1/b \cdot L(y)) = 0$ . Now  $\tilde{F}$  has no proper Picard-Vessiot extension. So, we get all the solutions in  $\tilde{F}$ , so it is  $\bar{A}$ -divisible and moreover we get non-trivial solutions to any homogeneous linear differential equation.

Then, it remains to check that for any element  $c \in \tilde{F}((x))$  there is a  $a$  such that  $\partial(a) = c$ . In the first case, this follows directly from the fact that  $\tilde{F}$  has no proper Picard-Vessiot extension. In the second case, let  $c = \sum_{i \geq m} c_i \cdot x^{-i}$ , then we look for elements  $a_i \in \tilde{F}$  such that  $\sum_{i \geq m} c_i \cdot x^{-i} = a_m \cdot x^{-m} + \sum_{i \geq m} (-i \cdot a_i + \partial(a_{i+1})) \cdot x^{-(i+1)}$ . Equivalently,  $c_m = \partial(a_m)$  and for  $i > m$ ,  $c_i + (i - 1) \cdot a_{i-1} = \partial(a_i)$ . Again, we use the fact that  $\tilde{F}$  has no proper Picard-Vessiot extension and so we first solve the first equation, finding  $a_m$ , plug it in the second one, find  $a_{m+1}$  etc. -†

A subgroup  $H$  of a group  $G$  is definably connected (in  $G$ ) if it does not contain any proper relatively definable subgroup of finite index (a relatively definable subgroup of  $H$  is the intersection of  $H$  with a definable subgroup of  $G$ ). Recall that in any module, a definable subgroup has a pp-definable subgroup of finite index ([16], p.140) as a consequence of B.H. Neumann’s Lemma. Note that in our modules, a pp-definable subgroup is invariant under multiplication by elements of  $Fix(\sigma) \cap K_\partial$ . Also, in any module, a subgroup, say  $S$ , invariant under multiplication by an infinite set of invertible elements cannot be of finite index in another subgroup. (If  $\mu$  is invertible and if an element  $a$  does not belong to  $S$ , then the elements  $a$  and  $a \cdot (1 - \mu)$  cannot belong to the same coset.) So, if we assume that  $Fix(\sigma) \cap K_\partial$  is infinite (for instance in the case where  $K = W(\mathbb{F}_p)$  with the Witt Frobenius,  $Fix(\sigma) = \mathbb{Q}_p$ ), then no definable subgroup can have a proper definable subgroup of finite index. Since any pp-definable subgroup is  $\emptyset$ -definable, then any definable subgroup is connected and  $\emptyset$ -definable.

PROPOSITION 2.14. Consider  $(K, v, \sigma)$  and assume that  $v$  is non trivial on  $Fix(\sigma) \cap K_\partial$ . Then the valuation ring  $\mathcal{O}_K$  of  $K$  is not definable in the language  $L_A$  of modules.

PROOF. In particular  $Fix(\sigma) \cap K_\partial$  is infinite. By the above,  $\mathcal{O}_K$  would be pp-definable, and so invariant under multiplication by elements of  $Fix(\sigma) \cap K_\partial$ , a contradiction if we multiply by an element of non zero valuation.  $\dashv$

**§3. Valued modules and abelian structures.** We will now fix the structures of valued modules we will be dealing with.

We keep the same notation as in the previous section, with a fixed  $(K, v, \sigma, \partial)$ ,  $A$  the skew polynomial ring  $K[t; \sigma, \partial]$ , etc.

DEFINITION 3.1 (cf. [11]). A valued  $A$ -module is a structure  $(M, \Delta, \leq, +, w, \infty)$ , where  $M$  is an  $A$ -module,  $\infty \in \Delta$ ,  $(\Delta, \leq)$  is a totally ordered set for which  $\infty$  is a maximum,  $+$  is an action of  $vK$  on  $(\Delta, \leq)$  and  $w$  is a surjective map  $w : M \rightarrow \Delta$  such that

1. For all  $\delta_1, \delta_2 \in \Delta, \gamma_1, \gamma_2 \in vK$ , if  $\delta_1 \geq \delta_2$  and  $\gamma_1 \leq \gamma_2$  then  $\delta_1 + \gamma_1 \leq \delta_2 + \gamma_2$ .
2. For all  $m_1, m_2 \in M, w(m_1 + m_2) \geq \min\{w(m_1), w(m_2)\}$ , and  $w(m_1) = \infty$  iff  $m_1 = 0$ .
3. For all  $m \in M, w(m \cdot t) \geq w(m)$ .
4. For all  $m \in M, \forall \lambda \in K, \lambda \neq 0, w(m \cdot \lambda) = w(m) + v(\lambda)$ .

Taking  $M = (K, v, \sigma, \partial)$  and  $w = v$ , it is a valued  $A$ -module in either case of  $t$  acting as  $\sigma$  or  $\partial$ .

From the axioms above, we deduce as usual the following properties:  $w(m) = w(-m)$ , and if  $w(m_1) < w(m_2)$ , then  $w(m_1 + m_2) = w(m_1)$ .

DEFINITION 3.2. We let  $L_w$  be the two-sorted language of valued  $A$ -modules obtained from  $L_A$ , with a sort  $M$  for the underlying module, and a sort  $\Delta$  for the ordered set of valuations, a constant symbol  $\infty$  of sort  $\Delta$ , and unary function symbols  $+ \gamma$  for each  $\gamma \in vK$ . We define the following  $L_w$ -theories:

- let  $T_w$  be the theory of valued  $A$ -modules obtained by translating the required axioms in  $L_w$ .
- let  $T_w^* = T_w \cup T_d$ .

Let  $M \models T_w$ , then we define

$$M_\delta := \{m \in M : w(m) \geq \delta\}$$

This is not only a subgroup of  $M$ , but an  $A_0$ -submodule.

In order to go into the setting of abelian structures, we will now introduce another (less expressive) language. This is the language of [27], where Rohwer was considering the field of Laurent series over the prime field  $\mathbb{F}_p$  with the usual Frobenius map  $y \mapsto y^p$  (which is not an isometry). If  $(M, \Delta, w)$  is a valued  $A$ -module, we have in mind the structure  $(M, (M_\delta)_{\delta \in \Delta})$ .

DEFINITION 3.3. Let  $(\Delta, \leq)$  be a fixed totally ordered set with an action  $+$  of  $vK$  on  $\Delta$  such that for all  $\delta_1, \delta_2 \in \Delta, \gamma_1, \gamma_2 \in vK$ , if  $\delta_1 \leq \delta_2$  and  $\gamma_1 \leq \gamma_2$  then  $\delta_1 + \gamma_1 \leq \delta_2 + \gamma_2$ . We let  $L_V$  be the language consisting of the language  $L_A$  of  $A$ -modules together with a set of unary predicates  $V_\delta$ , indexed by the elements of  $\Delta$ .

DEFINITION 3.4. Let  $T_V$  be the  $L_V$ -theory obtained from  $T_A$  together with the following axioms (1)–(7). Let  $T_V^*$  be the theory  $T_V$  together with axiom scheme (8).

1.  $\forall m \exists n (m = n \cdot t)$ .
2.  $\forall m \exists n (n \cdot q(t) = m)$ , where  $q(t) \in \mathcal{P}$ .
3.  $\forall m (V_{\delta_1}(m) \rightarrow V_{\delta_2}(m))$ , whenever  $\delta_1 \geq \delta_2$ .
4.  $\forall m (V_\delta(m) \rightarrow V_\delta(m \cdot t))$ .
5.  $\forall m_1 \forall m_2 (V_\delta(m_1) \ \& \ V_\delta(m_2) \rightarrow V_\delta(m_1 + m_2))$ .
6.  $\forall m (V_\delta(m) \rightarrow V_{\delta+v(\lambda)}(m \cdot \lambda))$ , where  $\lambda \in K$ .
7.  $\forall m (V_\delta(m) \rightarrow V_{\delta+v(q(t))}(m \cdot q(t)))$ , where  $q(t) \in A$ .
8.  $\forall m \in V_\delta \exists n \in V_\delta n \cdot q(t) = m$ , where  $q(t) \in \mathcal{S}$ .

If  $(M, \Delta, w)$  is a valued  $A$ -module, let  $\mathcal{M}$  be the  $L_V$ -structure

$$(M, +, 0, (\cdot r)_{r \in A}, (M_\delta)_{\delta \in \Delta}),$$

where  $V_\delta$  is interpreted as  $M_\delta$ .

EXAMPLE 1. Let  $F$  be a  $p$ -closed field of characteristic  $p$ . If  $M = K = W(F)$  with  $t$  acting as the Witt Frobenius, or if  $M = K = F((x))$  with  $t$  acting as  $\sum c_i x^i \mapsto \sum c_i^p x^i$ , then  $\mathcal{M} \models T_V^*$ , as in Corollary 2.10.

EXAMPLE 2. Let  $(F, \partial)$  be a differential field of characteristic 0 whose field of constants is algebraically closed. Let  $K$  be either the field of Laurent series  $(\tilde{F}((x)), v, \partial)$  or  $(\tilde{F}((x^{-1})), v, \partial)$  as in Corollary 2.13. Then  $K$ , viewed as an  $A$ -module with  $t$  acting as  $\partial$ , is a model of  $T_V^*$ .

The structure  $\mathcal{M}$  is an abelian structure and one gets as in the classical case of (pure) modules that any formula is equivalent to a boolean combination of pp-formulas and index sentences (namely, sentences telling the index of two pp-definable subgroups in one another; for all this see [25]). Moreover, this elimination is uniform in the class of such structures.

Note that a positive primitive formula  $\phi(\mathbf{x})$  is now of the form:

$$\exists y_1 \exists y_2 \cdots \exists y_n \bigwedge_i (\mathbf{x}, \mathbf{y}) \cdot B = 0 \ \& \ V_{\gamma_i}(t_i(\mathbf{x}, \mathbf{y})),$$

where  $B$  is a matrix with coefficients in  $A$ ,  $\gamma_i \in \Gamma$  and  $t_i(\mathbf{x}, \mathbf{y})$  is a term in the language  $L_A$ .

In the next section, we will prove a positive quantifier elimination result for these abelian structures.

**§4. Quantifier elimination for valued modules considered as abelian structures.**

This section will be devoted to the proof that  $T_V^*$  admits positive quantifier elimination. This means that for any formula  $\phi(x, \mathbf{y})$  which is a conjunction of atomic formulas, the existential formula  $\exists x \phi(x, \mathbf{y})$  is equivalent to a conjunction of atomic formulas. By our previous remark about general pp-elimination in abelian structures, it will imply that any formula is equivalent to a quantifier free formula and index sentences.

The subgroups defined by the unary predicates  $V_\delta$  are  $A_0$ -modules. We will use the fact that  $A_0$  satisfies a generalized right (henceforth g.r.) Euclidean algorithm (cf. Lemma 2.2).

PROPOSITION 4.1.  $T_V^*$  admits positive quantifier elimination.

PROOF. We can proceed by induction on the number of existential quantifiers, so it suffices to consider a formula existential in just one variable  $\exists x \phi(x, \mathbf{y})$ , where  $\phi(x, \mathbf{y})$  is a conjunction of atomic formulas. Then, since  $A$  is right Euclidean, we can always assume that we have at most one equation involving  $x$ .

Let  $\phi(x, \mathbf{y})$  be a positive quantifier-free formula of the form

$$x \cdot r_0 = t_0(\mathbf{y}) \ \& \ \bigwedge_{i=1}^n V_{\delta_i}(x \cdot r_i - t_i(\mathbf{y})) \ \& \ \theta(\mathbf{y})$$

where  $r_i \in A$ ,  $\theta(\mathbf{y})$  is a quantifier-free pp-formula, the  $t_i(\mathbf{y})$  are  $L_A$ -terms, and  $\delta_i \in \Delta$  with  $\delta_1 \geq \delta_2 \geq \dots \geq \delta_n$ . Consider  $\exists x \phi(x, \mathbf{y})$ . It suffices to show that any such formula is equivalent to a positive quantifier-free formula.

First, we note that for each  $i$  there exists a non-zero  $\lambda_i \in K$  such that  $r_i \lambda_i \in \mathcal{S}$ . Then, in  $T_V^*$ , we can replace  $V_{\delta_i}(x \cdot r_i - t_i(\mathbf{y}))$  by  $V_{\delta_i + v(\lambda_i)}(x \cdot r_i \cdot \lambda_i - t_i(\mathbf{y}) \cdot \lambda_i)$  and  $x \cdot r_0 = t_0(\mathbf{y})$  by  $x \cdot r_0 \cdot \lambda_0 = t_0(\mathbf{y}) \cdot \lambda_0$ , so we can always assume that the  $r_i \in \mathcal{S}$ , for all  $i$ .

Note also that we can always assume that  $\text{deg}(r_0) > \text{deg}(r_i)$ , for all  $i$ . Indeed, suppose that  $\text{deg}(r_0) \leq \text{deg}(r_i)$ , for some  $i$ , say  $i = 1$ . By the g.r. Euclidean algorithm in  $A_0$ , there exists  $\lambda \in \mathcal{O}_K$  such that  $r_1 \cdot \lambda = r_0 \cdot r + r'_1$  with  $\text{deg}(r'_1) < \text{deg}(r_0)$  and  $r, r'_1 \in A_0$ . So, we have that  $x \cdot r_1 \cdot \lambda = x \cdot r_0 \cdot r + x \cdot r'_1 = t_0 \cdot r + x \cdot r'_1$ , and we can replace  $V_{\delta_1}(x \cdot r_1 + t_1)$  by  $V_{\delta_1 + v(\lambda)}(x \cdot r'_1 + t_0(\mathbf{y}) \cdot r - t_1(\mathbf{y}))$ .

We will call *normalization* the process of going through the last two reductions and re-indexing if necessary to keep the condition  $\delta_1 \geq \delta_2 \geq \dots \geq \delta_n$ . We will use the notation  $x \cdot r \equiv_\delta u$  to mean that  $V_\delta(x \cdot r - u)$  holds.

First, we will assume that there is one equation present in  $\phi(x, \mathbf{y})$ . We will concentrate on the system formed by this equation and the ‘‘congruences’’.

Consider the system (1):

$$x \cdot r_0 = t_0, \ x \cdot r_1 \equiv_{\delta_1} t_1, \ \dots, \ x \cdot r_n \equiv_{\delta_n} t_n \tag{1}$$

with  $\text{deg}(r_0) > \text{deg}(r_i)$ ,  $r_0, r_i \in \mathcal{S}$ ,  $t_0 = t_0(\mathbf{y})$ ,  $t_i = t_i(\mathbf{y})$ ,  $1 \leq i \leq n$ .

Applying the g.r. Euclidean algorithm, we get some  $\lambda \in \mathcal{O}_K$  and  $s, s_1 \in A_0$  such that  $r_0 \cdot \lambda = r_1 \cdot s + s_1$  with  $\text{deg}(s_1) < \text{deg}(r_1)$ .

We claim that system (1) is equivalent to the following system (2):

$$x \cdot r_1 = t_1, \ x \cdot s_1 \equiv_{\delta_1 + v(\lambda)} t_0 \cdot \lambda - t_1 \cdot s, \ x \cdot r_2 \equiv_{\delta_2} t_2, \ \dots, \ x \cdot r_n \equiv_{\delta_n} t_n \tag{2}$$

(1)  $\rightarrow$  (2). Let  $x$  satisfy (1). Then  $x \cdot r_1 = t_1 + a$  for some  $a \in V_{\delta_1}$ . By the divisibility condition on  $V_{\delta_1}$  there exists  $u \in V_{\delta_1}$  such that  $u \cdot r_1 = a$ . Then, we obtain  $(x - u) \cdot r_1 = t_1$  and  $(x - u) \cdot s_1 = (x - u) \cdot r_0 \cdot \lambda - (x - u) \cdot r_1 \cdot s$ . So,  $(x - u) \cdot s_1 = t_0 \cdot \lambda - t_1 \cdot s + (-u) \cdot r_0 \cdot \lambda$ . Since  $(-u) \cdot r_0 \in V_{\delta_1}$ , we get that  $(-u) \cdot r_0 \cdot \lambda \in V_{\delta_1+v(\lambda)}$ . Since  $\delta_1 \geq \delta_2 \dots$ , we have  $u \in V_{\delta_i}$  and  $u \cdot r_i \in V_{\delta_i}$ , for  $i \geq 2$ , so that  $x - u$  still satisfies the other congruence conditions.

(2)  $\rightarrow$  (1). Let  $x$  satisfy (2). We look for an element  $u \in V_{\delta_1}$  such that  $(x+u) \cdot r_0 = t_0$ , or equivalently  $(x+u) \cdot r_0 \cdot \lambda = t_0 \cdot \lambda$ . Replacing  $r_0 \cdot \lambda$  by  $r_1 \cdot s + s_1$ , we obtain:  $(x+u) \cdot r_1 \cdot s + (x+u) \cdot s_1 = t_0 \cdot \lambda$ . We have:  $t_1 \cdot s + u \cdot r_1 \cdot s + t_0 \cdot \lambda - t_1 \cdot s + a + u \cdot s_1 = t_0 \cdot \lambda$ , for some  $a \in V_{\delta_1+v(\lambda)}$ . So,  $u$  has to satisfy  $u \cdot r_0 \cdot \lambda + a = 0$ , but by the divisibility property of  $V_{\delta_1+v(\lambda)}$ , we can find such an element. It remains to check that  $x + u$  satisfies the other conditions. But,  $x \cdot r_1 = t_1$  and since  $u \in V_{\delta_1}$  we have that  $(x + u) \cdot r_1 \equiv_{\delta_1} t_1$ , and similarly for the other congruence conditions since  $u \in V_{\delta_i}$ .

By this device, we replace a system where the couple  $(r_0, r_1)$  occurred by a system where  $(r_1, s_1)$  occurs with  $deg(r_0) > deg(r_1) > deg(s_1)$ . We might have to normalize the new system.

Second, we will consider the case where there are only congruence relations in the system.

Consider the following system (3):

$$x \cdot r_1 \equiv_{\delta_1} t_1, \dots, x \cdot r_n \equiv_{\delta_n} t_n \tag{3}$$

with  $\delta_1 \geq \delta_2 \geq \dots \geq \delta_n, r_i \in \mathcal{S}, 1 \leq i \leq n$ .

i) We assume that  $deg(r_1) \geq deg(r_2)$ .

Applying the g.r. Euclidean algorithm, we get some  $\lambda \in \mathcal{O}_K$  and  $s, s_2 \in A_0$  such that  $r_1 \lambda = r_2 s + s_2$  with  $deg(s_2) < deg(r_2)$ .

We claim that system (3) is equivalent to the following system (4<sub>1</sub>):

$$x \cdot r_2 = t_2, x \cdot s_2 \equiv_{\delta_2+v(\lambda)} t_1 \cdot \lambda - t_2 \cdot s, x \cdot r_3 \equiv_{\delta_3} t_3, \dots, x \cdot r_n \equiv_{\delta_n} t_n \tag{4_1}$$

(3)  $\rightarrow$  (4<sub>1</sub>). Let  $x$  satisfy (3). Then  $x \cdot r_2 = t_2 + a$  and  $x \cdot r_1 = t_1 + b$  for some  $a \in V_{\delta_2}$  and  $b \in V_{\delta_1} \subseteq V_{\delta_2}$ . By the divisibility condition on  $V_{\delta_2}$  there exists  $u \in V_{\delta_2}$  such that  $u \cdot r_2 = a$ . Then, we obtain  $(x - u) \cdot r_2 = t_2$  and  $(x - u) \cdot s_2 = (x - u) \cdot r_1 \lambda - (x - u) \cdot r_2 s = t_1 \cdot \lambda - t_2 \cdot s + b \cdot \lambda + (-u) \cdot r_1 \cdot \lambda$ . Since  $(-u) \cdot r_1 \in V_{\delta_2}$ , we get that  $(-u) \cdot r_1 \cdot \lambda, b \cdot \lambda \in V_{\delta_2+v(\lambda)}$ . So  $(x - u) \cdot s_2 \equiv_{\delta_2+v(\lambda)} t_1 \cdot \lambda - t_2 \cdot s$ . The other congruence conditions are still satisfied by  $x - u$  since  $u \in V_{\delta_2} \subseteq V_{\delta_i}$ , for  $i \geq 3$ .

(4<sub>1</sub>)  $\rightarrow$  (3). Let  $x$  satisfy (4<sub>1</sub>). We look for an element  $u \in V_{\delta_2}$  such that  $(x + u) \cdot r_1 \equiv_{\delta_1} t_1$  or equivalently  $(x + u) \cdot r_1 \cdot \lambda \equiv_{\delta_1+v(\lambda)} t_1 \cdot \lambda$ . Replacing  $r_1 \cdot \lambda$  by  $r_2 \cdot s + s_2$ , we obtain  $(x + u) \cdot r_2 \cdot s + (x + u) \cdot s_2 \equiv_{\delta_1+v(\lambda)} t_1 \cdot \lambda$ . So we have  $t_2 \cdot s + u \cdot r_2 \cdot s + t_1 \cdot \lambda - t_2 \cdot s + a + u \cdot s_2 \equiv_{\delta_1+v(\lambda)} t_1 \cdot \lambda$ , for some  $a \in V_{\delta_2+v(\lambda)}$ . In fact, we can ask that  $u$  satisfies  $u \cdot r_1 \cdot \lambda + a = 0$ . But by the divisibility property of  $V_{\delta_2+v(\lambda)}$  we can find such an element. It remains to check that  $(x + u) \cdot r_i \equiv_{\delta_i} t_i$ , for  $i \geq 2$ . But this follows from  $x \cdot r_i \equiv_{\delta_i} t_i$  and  $u \in V_{\delta_2} \subseteq V_{\delta_j}, j \geq 3$ .

By this device, we replace a system where the couple  $(r_1, r_2)$  occurred by a system where  $(r_2, s_2)$  occurs with  $deg(r_1) \geq deg(r_2) > deg(s_2)$ . Then, we normalize the system and we note that system (4<sub>1</sub>) is of the same kind as system (1).

ii) Assume that  $\deg(r_1) < \deg(r_2)$ .

Applying the g.r. Euclidean algorithm, we get  $\lambda \in \mathcal{O}_K$ , and  $s, s_2 \in A_0$  such that  $r_2 \cdot \lambda = r_1 \cdot s + s_2$  with  $\deg(s_2) < \deg(r_1)$ .

We claim that (3) is equivalent to the following system (4<sub>2</sub>):

$$x \cdot r_1 = t_1, \quad x \cdot s_2 \equiv_{\delta_2+v(\lambda)} t_2 \cdot \lambda - t_1 \cdot s, \quad x \cdot r_3 \equiv_{\delta_3} t_3, \quad \dots, \quad x \cdot r_n \equiv_{\delta_n} t_n \quad (4_2)$$

(3)  $\rightarrow$  (4<sub>2</sub>). Let  $x$  satisfy (3). Then  $x \cdot r_1 = t_1 + a$  and  $x \cdot r_2 = t_2 + b$ , for some  $a \in V_{\delta_1}$  and  $b \in V_{\delta_2}$ . By the divisibility condition on  $V_{\delta_1}$  there exists  $u \in V_{\delta_1}$  such that  $u \cdot r_1 = a$ . Then, we obtain  $(x - u) \cdot r_1 = t_1$  and  $(x - u) \cdot s_2 = (x - u) \cdot r_2 \cdot \lambda - (x - u) \cdot r_1 \cdot s = t_2 \cdot \lambda + b \cdot \lambda - t_1 \cdot s + (-u) \cdot r_2 \cdot \lambda$ . Since  $(-u) \cdot r_2 \in V_{\delta_1}$ , we get  $(-u) \cdot r_2 \cdot \lambda \in V_{\delta_1+v(\lambda)} \subseteq V_{\delta_2+v(\lambda)}$  and  $b \cdot \lambda \in V_{\delta_2+v(\lambda)}$ . The other congruence conditions are still satisfied by  $x - u$  since  $u \in V_{\delta_1} \subseteq V_{\delta_i}$ , for  $i \geq 2$ .

(4<sub>2</sub>)  $\rightarrow$  (3). Let  $x$  satisfy (4<sub>2</sub>). Replacing  $r_2 \cdot \lambda$  by  $r_1 \cdot s + s_2$ , we obtain  $x \cdot r_1 \cdot s + x \cdot s_2 \equiv_{\delta_2+v(\lambda)} t_2 \cdot \lambda$ . So we have  $t_1 \cdot s + t_2 \cdot \lambda - t_1 \cdot s + a \equiv_{\delta_2+v(\lambda)} t_2 \cdot \lambda$ , for some  $a \in V_{\delta_2+v(\lambda)}$ . So,  $x \cdot r_2 \equiv_{\delta_2} t_2$ .

Note that in each case, we have decreased the degree of each of the coefficients, and we strictly decreased the sum of the degrees of the elements occurring in the congruence conditions. In the end, we will decrease the degree of the element occurring in the congruence relation corresponding to the smallest subgroup  $V_\delta$ .

So, we may assume that we reduce ourselves to a system consisting of possibly a conjunction of congruence conditions on the parameters and:

$$x \cdot r = u(\mathbf{y}), \quad x \cdot \mu \equiv_\delta t'(\mathbf{y}), \quad x \cdot r'_3 \equiv_{\delta'_3} t'_3(\mathbf{y}), \quad \dots, \quad x \cdot r'_\ell \equiv_{\delta'_\ell} t'_\ell(\mathbf{y}) \quad (*)$$

where  $\mu \in \mathcal{O}_K, r, r'_i \in \mathcal{S}, \delta \geq \delta'_3 \geq \dots \geq \delta'_\ell, 3 \leq i \leq \ell \leq n$ . Note that if  $\mu = 0$ , we strictly decreased the number of congruence conditions in our system. If  $\mu \neq 0$ , we replace the first congruence by  $x \equiv_{\delta+v(\mu^{-1})} t'(\mathbf{y}) \cdot \mu^{-1}$ . If  $V_{\delta+v(\mu^{-1})}$  is no longer the smallest subgroup we continue the above procedure with the smallest subgroup. But after a finite number of steps, we will obtain a system of the form (\*) with the second formula of the form  $x \cdot \mu \equiv_\delta t'(\mathbf{y})$ , where we may now assume w.l.o.g. that  $\mu = 1$ .

To finish, we observe that the above system (\*) with  $\mu = 1$  is equivalent to the following congruence conditions on the parameters:

$$t'(\mathbf{y}) \cdot r \equiv_\delta u(\mathbf{y}), \quad t'(\mathbf{y}) \cdot r'_3 \equiv_{\delta'_3} t'_3(\mathbf{y}), \quad \dots, \quad t'(\mathbf{y}) \cdot r'_\ell \equiv_{\delta'_\ell} t'_\ell(\mathbf{y}) \quad (**)$$

Indeed, assume that (\*\*) holds, in particular  $t'(\mathbf{y}) \cdot r \equiv_\delta u(\mathbf{y})$ . So we have that  $t'(\mathbf{y}) \cdot r = u(\mathbf{y}) + b$  for some  $b \in V_\delta$ . By the divisibility property of  $V_\delta$  there exists  $c \in V_\delta$  such that  $c \cdot r = b$ . Then  $(t'(\mathbf{y}) - c) \cdot r = u(\mathbf{y})$ . Since  $V_\delta$  is the smallest subgroup, the element  $t'(\mathbf{y}) - c$  still satisfies the other conditions of (\*) with  $\mu = 1$ .  $\dashv$

**COROLLARY 4.2.** *Let  $F$  be a field of characteristic  $p$  which is  $p$ -closed.*

1. *The  $L_V$ -structure  $(W(F), (W(F)_\delta)_{\delta \in \mathbb{Z}})$  admits quantifier elimination, for the Witt Frobenius action.*
2. *The  $L_V$ -structure  $(F((x)), (F((x))_\delta)_{\delta \in \mathbb{Z}})$  admits quantifier elimination, for the action by  $\sum c_i x^i \mapsto \sum c_i^p x^i$ .*

**COROLLARY 4.3.** *Let  $(F, \partial)$  be a differential field of characteristic 0 whose field of constants is algebraically closed. Let  $K$  be either the field of Laurent series  $(\tilde{F}((x)), v, \partial)$  or  $(\tilde{F}((x^{-1})), v, \partial)$  as in Corollary 2.13. Then the  $L_V$ -structure  $(K, (K_\delta)_{\delta \in \mathbb{Z}})$  admits quantifier elimination, with  $t$  acting as  $\partial$ .*

**§5. Model-completion.** Let  $M$  be a  $A$ -module. Let  $X := \{t^n : n \in \mathbb{N}\}$ , this is a right denominator set (see [12], Lemma 9.1). So, there exists a right ring of fractions  $AX^{-1}$  (ibid., Theorem 9.7). It is isomorphic to the set of equivalence classes in  $A \times X$  of the following equivalence relation:  $(a, t^{n_1}) \sim (a', t^{n_2})$  iff there exist  $s \in A$  and  $n \in \mathbb{N}$  such that  $at^n = a's$  and  $t^{n_1}t^n = t^{n_2}s$ . We may extend the valuation on  $A$  to  $AX^{-1}$ .

The module  $M$  has a right module of fractions  $M_X$  with respect to  $X$  and it is isomorphic to  $M \otimes_A AX^{-1}$ , into which it embeds if  $M$  is  $X$ -torsion-free. Moreover, any element of  $M_X$  has the form  $m \otimes t^{-n}$ , for some  $m \in M$ ,  $n \in \mathbb{N} \setminus \{0\}$ . (ibid., Theorem 9.13, Proposition 9.14).

**PROPOSITION 5.1.**  $T_V^*$  is the model-completion of the theory  $T_{V,X}$  consisting of  $T_V$  together with the axioms  $\forall m (m \cdot t = 0 \rightarrow m = 0)$  and  $\forall m V_\delta(m \cdot t) \leftrightarrow V_\delta(m)$ , for each  $\delta \in \Delta$ .

**PROOF.** Let now  $M$  be a model of  $T_{V,X}$ . We will embed  $M$  in a model of  $T_V^*$ .

First, we extend the predicates  $V_\delta$ ,  $\delta \in \Delta$  on  $M_X$  by  $V_\delta(m \otimes t^{-n})$ ,  $m \in M$ ,  $n \in \mathbb{N} \setminus \{0\}$ , whenever  $V_\delta(m)$ . This is well-defined since  $V_\delta(m \cdot t)$  iff  $V_\delta(m)$ . It is easy to see that  $M_X$  is still a model of  $T_V$ .

Now, consider  $M_X^\omega$  the direct product of  $\omega$  copies of  $M_X$ . Let  $\mathcal{F}$  be the Fréchet filter on  $\omega$ . We endow  $M_X^\omega$  with a structure of an  $A$ -module as follows. Let  $(m_i)_{i \in \omega} \in M_X^\omega$ , define  $(m_i)_{i \in \omega} \cdot t := (m_{i+1} \cdot t)_{i \in \omega}$  and extend this by linearity on  $A$ . Then define  $V_\delta((m_i)_{i \in \omega})$  iff  $V_\delta(m_i)$ , for every  $i \in \omega$ .

Finally, consider the quotient of  $M_X^\omega$  by  $\mathcal{F}$  and the diagonal embedding of  $\Delta$  into  $\Delta^\omega/\mathcal{F}$ . Define  $((m_i)_{i \in \omega})_{\mathcal{F}} \cdot t := ((m_{i+1} \cdot t)_{i \in \omega})_{\mathcal{F}}$  and  $V_\delta((m_i)_{i \in \omega})_{\mathcal{F}}$  iff  $\{i \in \omega : V_\delta(m_i)\} \in \mathcal{F}$ . Let us show that this is a model of  $T_V^*$ . It is a model of  $T_V$  since axiom schemes (3) up to (7) are Horn sentences,  $M_X$  is a model of  $T_V$  and Horn sentences are preserved by reduced product. It remains to check axiom scheme (8). Let  $q(t) = \sum_{j=0}^d t^j \cdot a_j$  with  $a_j \in K$ . Given  $m \in M_X^\omega/\mathcal{F}$ , we wish to find  $n \in M_X^\omega/\mathcal{F}$  such that  $m = n \cdot q(t)$ . Choose a representative of  $m$  in the direct product, say  $m' = (m_i)_{i \in \omega}$  and  $m = m'_{\mathcal{F}}$ . Let  $n' = (n'_i)_{i \in \omega}$  and  $n = n'_{\mathcal{F}}$ . By induction on  $i$ , we show how to choose  $n'_i$ . We will just consider the special case of  $n'_d$ . Suppose we have chosen  $n'_0, \dots, n'_{d-1}$ . Consider  $m'_0 = n'_0 \cdot q(t) = n'_d \cdot t^d \cdot a_d + \sum_{i=0}^{d-1} n'_i \cdot t^i \cdot a_i$ , which is equivalent to  $n'_d \cdot t^d = (m'_0 \cdot a_d^{-1} - \sum_{i=0}^{d-1} n'_i \cdot t^i \cdot a_i \cdot a_d^{-1})$ . Since  $M_X$  is  $X$ -divisible we can choose such an element  $n'_d$ .  $\dashv$

**§6. Two-sorted valued structures.** In this section we revert to the two-sorted language  $L_w$  of valued  $A$ -modules. In  $L_w$ ,  $x, y, z$  will denote variables of sort  $M$  and  $\delta$  will denote variables of sort  $\Delta$ . We keep the same notation as before, with a fixed  $(K, v, \sigma, \partial)$ ,  $A = K[t; \sigma, \partial]$ , etc.

In order to get more precise quantifier elimination results for valued  $A$ -modules  $(M, \Delta, \leq, +, w, \infty)$ , it is useful to look more closely at the structure  $((\Delta, \leq), (+\gamma)_{\gamma \in vK})$ .

**DEFINITION 6.1.** Let  $(\Gamma, +, 0, \leq)$  be a fixed totally ordered abelian group. Let  $L_\Delta$  be the language of sort  $\Delta$  with a binary predicate  $\leq$  and unary function symbols  $+\gamma$  for each  $\gamma \in \Gamma$ . We will write the action of  $+\gamma$  from the right. Let  $T_\Delta$  be the following  $L_\Delta$ -theory.

1.  $\leq$  is a total order.
2.  $\forall \delta ((\delta + \gamma_1) + \gamma_2 = \delta + (\gamma_1 + \gamma_2))$ , where  $\gamma_1, \gamma_2 \in \Gamma$ .
3.  $\forall \delta_1 \forall \delta_2 (\delta_1 \leq \delta_2 \rightarrow \delta_1 + \gamma_1 \leq \delta_2 + \gamma_2)$ , where  $\gamma_1, \gamma_2 \in \Gamma$  and  $\gamma_1 \leq \gamma_2$ .

For a valued  $A$ -module  $(M, \Delta, +, w)$  and  $\Gamma = vK$ , we see that  $((\Delta, \leq), (+\gamma)_{\gamma \in vK}) \models T_\Delta$ .

We will consider two conditions under which the corresponding extensions of  $T_\Delta$  to valued  $A$ -modules admit quantifier elimination.

DEFINITION 6.2. We define the following  $L_\Delta$ -theories.

1. Let  $T_{\Delta, \text{dense}}$  be the theory  $T_\Delta$  together with the axiom that the total order  $\leq$  is dense.
2. Assume that  $\Gamma$  has a smallest strictly positive element 1. Let  $T_{\Delta, \text{discrete}}$  be the theory  $T_\Delta$  together with the axiom

$$\forall \delta_1 \exists \delta_2 \forall \delta_3 (\delta_2 > \delta_1 \ \& \ (\delta_3 > \delta_1 \rightarrow (\delta_2 \leq \delta_3 \ \& \ \delta_2 = \delta_1 + 1)))$$

In the case of  $T_{\Delta, \text{discrete}}$ , we will always make the assumption that we have a constant 1 in the language of the group  $\Gamma$ .

PROPOSITION 6.1 (cf. [11]). *Both theories  $T_{\Delta, \text{dense}}$  and  $T_{\Delta, \text{discrete}}$  admit quantifier elimination.*

PROOF. It suffices that any formula with free variables  $\delta = (\delta_1, \dots, \delta_n)$  of the form

$$\exists \alpha \left( \bigwedge_{\ell} \delta_{i_\ell} + \gamma_{i_\ell} \square \alpha \square \delta_{j_\ell} + \gamma_{j_\ell} \right)$$

where  $\square \in \{\leq, <\}$ ,  $1 \leq i_\ell, j_\ell \leq n$ , and  $\gamma_{k\ell} \in \Gamma$ , be equivalent to a quantifier-free formula in  $\delta$ . It is equivalent to

$$\exists \alpha (\max\{\delta_{i_\ell} + \gamma_{k_\ell}\} \square \alpha \square \min\{\delta_{j_\ell} + \gamma_{t_\ell}\} \ \& \ \theta(\delta))$$

for some quantifier-free formula  $\theta(\delta)$ . In  $T_{\Delta, \text{dense}}$ , it is equivalent to

$$\max\{\delta_{i_\ell} + \gamma_{k_\ell}\} \square \min\{\delta_{j_\ell} + \gamma_{t_\ell}\} \ \& \ \theta(\delta).$$

Note that we have used, as it is often the case, the abbreviations *max* and *min* in place of taking the (finite) disjunctions over all possible configurations of the elements in the ordering.

In  $T_{\Delta, \text{discrete}}$ , if one of the  $\square$  is  $\leq$ , then it is treated similarly as above replacing the  $\square$  in the resulting formula by a strict inequality if there is one. If both  $\square$  stand for  $<$ , then we use the special axiom of  $T_{\Delta, \text{discrete}}$  and the resulting formula is:

$$\max\{\delta_{i_\ell} + \gamma_{k_\ell}\} + 1 < \min\{\delta_{j_\ell} + \gamma_{t_\ell}\} \ \& \ \theta(\delta). \quad \dashv$$

DEFINITION 6.3. We define the following  $L_w$ -theories of valued  $A$ -modules.

1. Let  $T_{w, d}$  consists of the following:

(1.1)  $T_w$ .

(1.2) Divisibility axioms (DG). For each  $q$  in  $\mathcal{S} \cup \{t\}$ :

$$\forall x (x \neq 0 \rightarrow \exists y (x = y \cdot q \ \& \ w(x) = w(y))).$$



(1.3) No residual identities axioms (IR). For each  $p_1, \dots, p_n$  in  $\mathcal{F}$ :

$$\forall \delta \forall x_1 \cdots \forall x_n \exists x \left( \left( \bigwedge_{i=1}^n w(x_i) \geq \delta \right) \rightarrow \bigwedge_{i=1}^n w(x \cdot p_i + x_i) = \delta \right).$$

2. Let  $T_{w,d,dense} = T_{w,d} \cup T_{\Delta,dense}$ .
3. Let  $T_{w,d,discrete} = T_{w,d} \cup T_{\Delta,discrete}$ .

EXAMPLE 3. Let  $F$  be a  $p$ -closed field of characteristic  $p$ . Then, as in Corollary 2.10,  $M = K = W(F)$  with the action of the Witt Frobenius, and  $M = K = F((x))$  with the action of  $\sum c_i x^i \mapsto \sum c_i^p x^i$ , yield models of  $T_{w,d,discrete}$ .

EXAMPLE 4. Let  $(F, \partial)$  be a differential field of characteristic 0. Assume its field of constants is algebraically closed. Let  $K$  be either the field of Laurent series  $(\tilde{F}((x)), v, \partial)$  or  $(\tilde{F}((x^{-1})), v, \partial)$  as in Corollary 2.13. Then  $K$ , with  $t$  acting as  $\partial$ , yields models of  $T_{w,d,discrete}$ .

We will abuse notation and continue to use the  $V_\delta$ , and introduce the new  $V_{\delta+}$ , with  $V_{\delta+}(x) \leftrightarrow w(x) > \delta$ . We will also identify all these with the sets they define in any model, and we recall they are always  $A_0$ -modules.

Note that (IR) implies that  $V_\delta/V_{\delta+} \neq \bigcup_{i=1}^n \{x + V_{\delta+} : x \in V_\delta \ \& \ p_i(x) \equiv 0 \pmod{V_{\delta+}}\}$  and that  $|V_\delta/V_{\delta+}|$  is infinite. When the model of  $T_{w,d}$  is  $K$  itself, it implies that  $\tilde{K}$  does not satisfy any linear  $\bar{\sigma}$ -identities or  $\bar{\delta}$ -identities, according to the action of  $t$ .

DEFINITION 6.4. Let  $r \in A_0$  and  $n \in \omega$ .

1. A residual index formula  $\text{Indr}_{n,r}(\delta)$ , is a  $L_w$ -formula which is existential in the module sort with a free variable in the ordered set sort, of the form

$$\exists x_1 \cdots \exists x_n \left( \bigwedge_{1 \leq i < j \leq n} w(x_i - x_j) = \delta \ \& \ \bigwedge_{i=1}^n (w(x_i \cdot r) > \delta \ \& \ w(x_i) = \delta) \right).$$

2. A residual index sentence is an existential sentence of the form  $\exists \delta \text{Indr}_{n,r}(\delta)$ .

We will often use the following fact. In models of  $T_w \cup \{(DG)\}$ , if  $r \in \mathcal{F}$  and  $\text{Indr}_{n,r}(\delta)$  holds, then there exist  $n$  elements in  $\text{ann}(r) \cap V_\delta$  which belong to different cosets of  $V_{\delta+}$  in  $V_\delta$ . In particular, index sentences (in the theory of modules) are special instances of residual index sentences.

PROPOSITION 6.2. In the theory  $T_{w,d}$ , any existential  $L_w$ -formula in the module sort is equivalent to a formula which is quantifier free in the module sort, existential in the totally ordered set sort, plus some residual index formulas and sentences.

COROLLARY 6.3. Let  $(M, w, \Delta_M) \subset (N, w, \Delta_N)$  be valued  $A$ -modules satisfying  $T_{w,d}$ . Assume that both satisfy the same residual index formulas with parameters in  $\Delta_M$  and that  $(\Delta_M, \Gamma, +)$  is existentially closed in  $(\Delta_N, \Gamma, +)$ . Then  $(M, w, \Delta_M)$  is existentially closed in  $(N, w, \Delta_N)$ . +

COROLLARY 6.4. In the theory  $T_{w,d,dense}$  (respectively  $T_{w,d,discrete}$ ), any  $L_w$ -formula  $\phi(x, \delta)$  is equivalent to the conjunction of a quantifier-free  $L_w$ -formula  $\theta(x, \delta)$  together with residual index formulas and residual index sentences. Therefore the completions are given by residual index sentences and quantifier-free  $L_\Delta$ -sentences.

PROOF. Apply Propositions 6.1 and 6.2. +

Note that if the action of  $v(K)$  is transitive on  $\Delta$ , then the residual index formulas can be translated into statements in a fixed quotient.

**PROOF OF PROPOSITION 6.2.** In the language  $L_w$ , the only interaction between the two sorts  $M$  and  $\Delta$  occurs in valuation equalities and inequalities. Let  $r \in A$  and  $u$  a  $M$ -term where  $x$  does not occur, we will replace each valuation inequality where a term of the form  $w(x \cdot r + u)$  occurs, by a formula  $\exists \delta (w(x \cdot r + u) = \delta \ \& \ \psi)$ , where  $\delta$  is a new variable and  $\psi$  is obtained from the inequality by putting in  $\delta$  for  $w(x \cdot r + u)$ . In this way, we can always arrange that terms of the form  $w(x \cdot r + u)$  only occur in equations of the form  $w(x \cdot r + u) = \delta$  (cf. [29]).

Also, by replacing each inequality  $x \cdot r \neq u$  by  $w(x \cdot r - u) \neq \infty$ , we can always assume there are no such  $x \cdot r \neq u$ . (N.B. We could handle these  $x \cdot r \neq u$  in the elimination process, but it would generate new index formulas of the form  $|ann(r) \cap V_\delta| \geq n$ .)

As in Proposition 4.1 it suffices to consider a formula existential in just one variable  $\exists x \phi(x, y, \delta)$ , where  $\phi(x, y, \delta)$  is quantifier-free in  $x, y$  and existential in  $\delta$ , and since  $A$  is right Euclidean, we can always assume that we have at most one equation of the form  $x \cdot r = t$  involving  $x$ .

We will need some basic cases and further reductions which are treated in the following lemmas.

The next lemma will ensure that we can always assume we have at least one equation  $x \cdot r = t$ .

**LEMMA 6.5.** *Consider a system of the following form:*

$$\exists x \begin{cases} \bigwedge_{i=1}^m w(x \cdot r_i + t_i) = \delta_1, \\ \bigwedge_{i=m+1}^n w(x \cdot r_i + t_i) = \delta_i, \end{cases} \tag{I}$$

where  $r_i \in \mathcal{S}$ ,  $t_i$  are  $L_A$ -terms,  $1 \leq i \leq n$ , and  $\delta_1 > \delta_{m+1} \geq \dots \geq \delta_n$ .

Then (I) is equivalent to the following system (I').

$$\exists x \begin{cases} x \cdot r_1 + t_1 = 0, \\ \bigwedge_{i=2}^m w(x \cdot r_i + t_i) \geq \delta_1, \\ \bigwedge_{i=m+1}^n w(x \cdot r_i + t_i) = \delta_i. \end{cases} \tag{I'}$$

**PROOF.** Suppose  $x$  is a solution of (I). By axiom (DG), there exists  $y$  such that  $y \cdot r_1 = x \cdot r_1 + t_1$ ,  $w(y) = \delta_1$ , and  $w(y \cdot r_1) = \delta_1$ . We get that  $\bigwedge_{i=2}^m w((x - y) \cdot r_i + t_i) \geq \delta_1$  and  $\bigwedge_{i=m+1}^n w((x - y) \cdot r_i + t_i) = \delta_i$ . So  $x - y$  is a solution of (I'). Conversely, suppose that  $x$  is a solution of (I'). Then, thanks to axiom scheme (IR), we may add to  $x$  an element  $u$  with  $w(u) = \delta_1$ ,  $w(u \cdot r_1) = \delta_1$  and  $\bigwedge_{i=2}^m w(u \cdot r_i + (x \cdot r_i + t_i)) = \delta_1$ . □

**LEMMA 6.6.** *Consider a system of the following form:*

$$\exists x \begin{cases} x \cdot r_0 = t_0, \\ \bigwedge_{i=1}^m w(x + u'_i) = \delta_1, \\ \bigwedge_{i=1}^m w(x \cdot r_i + t_i) \geq \delta_1, \\ \bigwedge_{i=m+1}^n w(x \cdot r_i + t_i) \square_i \delta_i, \\ \bigwedge_{i=1}^m w(t_0 + u'_i \cdot r_0) = \delta_1. \end{cases} \tag{II}$$

where  $r_i \in \mathcal{S}$ ,  $t_i$  are  $L_A$ -terms,  $1 \leq i \leq n$ ,  $\square_i$  is one of  $=, >, \geq$ , and  $\delta_1 > \delta_{m+1} \geq \dots \geq \delta_n$ . We claim that (II) is equivalent to

$$\left\{ \begin{array}{l} \bigwedge_{i \neq j \leq m} w(u'_i - u'_j) \geq \delta_1, \\ \bigwedge_{i=1}^m w(u'_1 \cdot r_i - t_i) \geq \delta_1, \\ \bigwedge_{i=m+1}^n w(u'_1 \cdot r_i - t_i) \square_i \delta_i, \\ \bigwedge_{i=1}^m w(t_0 + u'_i \cdot r_0) = \delta_1. \end{array} \right. \quad (\text{II}')$$

PROOF. Suppose  $x$  is a solution of (II). Then  $w(u'_i - u'_j) = w(x + u'_i - x - u'_j) \geq \delta_1$ ,  $w(u'_1 \cdot r_i - t_i) = w(x \cdot r_i + u'_1 \cdot r_i - x \cdot r_i - t_i) \geq \delta_1$  if  $i \leq m$ , and  $w(u'_1 \cdot r_i - t_i) = w(x \cdot r_i + u'_1 \cdot r_i - x \cdot r_i - t_i) = \delta_i$  when necessary if  $i > m$ . Conversely, suppose (II') is satisfied. By Axiom (DG) let  $u''$  such that  $u'' \cdot r_0 = t_0 + u'_1 \cdot r_0$  and  $w(u'') = w(t_0 + u'_1 \cdot r_0) = \delta_1$ . Then  $u'' - u'_1$  is a solution of (II): we have  $(u'' - u'_1) \cdot r_0 = t_0$ , and  $w((u'' - u'_1) + u'_i) = w(u'' - (u'_1 - u'_i)) \geq \delta_1$ , but the strict inequality would imply  $w((u'' - u'_1) \cdot r_0 + u'_i \cdot r_0) = w(t_0 + u'_i \cdot r_0) > \delta_1$ , which is not the case, and so  $w((u'' - u'_1) + u'_i) = \delta_1$ ; on the other hand  $w((u'' - u'_1) \cdot r_i + t_i) = w(u'' \cdot r_i - u'_1 \cdot r_i + t_i) \geq \delta_1$  if  $i \leq m$ , and  $w((u'' - u'_1) \cdot r_i + t_i) = w(u'' \cdot r_i - u'_1 \cdot r_i + t_i) = \delta_i$  when necessary if  $i > m$ .  $\dashv$

LEMMA 6.7. Consider a system of the following form:

$$\exists x \left\{ \begin{array}{l} x \cdot r_0 = t_0, \\ \bigwedge_{i=1}^m w(x + u'_i) = \delta_1, \\ \bigwedge_{i=1}^m w(x \cdot r_i + t_i) \geq \delta_1, \\ \bigwedge_{i=m+1}^n w(x \cdot r_i + t_i) \square_i \delta_i, \\ \bigwedge_{i \in B} w(t_0 + u'_i \cdot r_0) > \delta_1, \\ \bigwedge_{i \notin B} w(t_0 + u'_i \cdot r_0) = \delta_1. \end{array} \right. \quad (\text{III})_B$$

where  $r_i \in \mathcal{S}$ ,  $t_i$  are  $L_A$ -terms,  $1 \leq i \leq n$ ,  $\square_i$  is one of  $=, >, \geq$ , and  $\delta_1 > \delta_{m+1} \geq \dots \geq \delta_n$ , and  $B$  is a nonempty subset of  $\{1, \dots, m\}$ . Then this system is equivalent to a formula which is quantifier free in the module sort, existential in the totally ordered set sort, plus some residual index formulas and sentences.

PROOF. Note that (III)<sub>B</sub> has a solution only if  $\text{ann}(r_0) \neq \{0\}$ . Indeed, let  $x$  be a solution and  $i \in B$ . By Axiom (DG) let  $u$  be such that  $u \cdot r_0 = t_0 + u'_i \cdot r_0$  and  $w(u) = w(t_0 + u'_i \cdot r_0) > \delta_1$ . Then  $x - (u - u'_i) \in \text{ann}(r_0)$  and  $w(x - (u - u'_i)) = w(x + u'_i - u) = \delta_1 \neq \infty$ , so  $x - (u - u'_i) \neq 0$ .

We claim that (III)<sub>B</sub> is equivalent to a disjunction over the possible nonempty subsets  $J$  of  $\{1, \dots, m\}$  of the following systems

$$\exists x \left\{ \begin{array}{l} x \cdot r_0 = t_0, \\ w(x + u'_1) = \delta_1, \\ \bigwedge_{i \in J} w(x + u'_i) = \delta_1, \\ \bigwedge_{i \notin J} w(u'_i - u'_j) = \delta_1, \\ \bigwedge_{i \notin J} \bigvee_{j \in J} w(u'_i - u'_j) > \delta_1, \\ \bigwedge_{i=1}^m w(u'_1 \cdot r_i - t_i) \geq \delta_1, \\ \bigwedge_{i=m+1}^n w(u'_1 \cdot r_i - t_i) \square_i \delta_i, \\ \bigwedge_{i \in B} w(t_0 + u'_i \cdot r_0) > \delta_1, \\ \bigwedge_{i \notin B} w(t_0 + u'_i \cdot r_0) = \delta_1 \end{array} \right. \quad (\text{IV})_{B,J}$$

Indeed, let  $x$  be a solution of (III)<sub>B</sub>. We get a corresponding subset  $J$ ,  $w(u'_1 \cdot r_i - t_i) = w(x \cdot r_i + u'_1 \cdot r_i - x \cdot r_i - t_i) \geq \delta_1$  if  $i \leq m$ , and  $w(u'_1 \cdot r_i - t_i) = w(x \cdot r_i + u'_1 \cdot r_i - x \cdot$

$r_i - t_i) = \delta_i$  when necessary if  $i > m$ . Thus  $x$  is a solution of  $(IV)_B$ . Conversely, let  $x$  be a solution of  $(IV)_{B,J}$ . Let  $i \leq m$ , we have  $w(x + u'_i) = \delta_1$  if  $i \in J$ , and if  $i \notin J$  we have  $j \in J$  such that  $w(u'_i - u'_j) > \delta_1$  and then  $w(x + u'_i) = w(x + u'_j + u'_i - u'_j) = \delta_1$ . On the other hand,  $w(x \cdot r_i + t_i) = w(x \cdot r_i + u'_1 \cdot r_i - u'_1 \cdot r_i + t_i) \geq \delta_1$  if  $i \leq m$ , and  $w(x \cdot r_i + t_i) = w(x \cdot r_i + u'_1 \cdot r_i - u'_1 \cdot r_i + t_i) = \delta_i$  when necessary if  $i > m$ . Whence  $x$  is a solution of  $(IV)_{B,J}$ .

We now claim that  $(IV)_{B,J}$  is equivalent to a disjunction over the nonempty subsets  $J'$  of  $J$  of the following systems

$$\left\{ \begin{array}{l} \text{Indr}_{|J'|,r_0}(\delta_1), \\ \bigwedge_{i \in B} \bigvee_{j \in J'} w(u'_i - u'_j) > \delta_1, \\ \bigwedge_{i \neq j \in J} w(u'_i - u'_j) = \delta_1, \\ \bigwedge_{i \notin J} \bigvee_{j \in J} w(u'_i - u'_j) > \delta_1, \\ \bigwedge_{i=1}^m w(u'_1 \cdot r_i - t_i) \geq \delta_1, \\ \bigwedge_{i=m+1}^n w(u'_1 \cdot r_i - t_i) = \delta_i, \\ \bigwedge_{i \in J'} w(t_0 + u'_i \cdot r_0) > \delta_1, \\ \bigwedge_{i \notin B} w(t_0 + u'_i \cdot r_0) = \delta_1 \end{array} \right. \quad (V)_{B,J,J'}$$

Indeed, suppose  $x$  is a solution of  $(IV)_{B,J}$ . Let  $J' \subseteq J$  be such that  $\bigwedge_{i \in B} \bigvee_{j \in J'} w(u'_i - u'_j) > \delta_1$  and  $\bigwedge_{i \in J'} \bigvee_{j \in B} w(u'_i - u'_j) > \delta_1$ , namely  $J'$  yields the indices of coset representatives of the  $u'_i, i \in B$ , modulo  $V_{\delta_1+}$ , and  $B$  is nonempty at this stage. Then certainly  $\bigwedge_{i \in J'} w(t_0 + u'_i \cdot r_0) > \delta_1$ . Fix  $j \in J'$ . Then  $w((u'_j + x) \cdot r_0) > \delta_1, w((u'_j - u'_i) \cdot r_0) > \delta_1, j \neq i \in J'$ , so that  $\text{Indr}_{|J'|,r_0}(\delta_1)$  holds, and hence  $(V)_{B,J,J'}$  is satisfied. Conversely, suppose  $(V)_{B,J,J'}$  is satisfied. Fix  $j \in J'$ . Then again  $w((u'_j - u'_i) \cdot r_0) > \delta_1, j \neq i \in J'$ . The condition  $\text{Indr}_{|J'|,r_0}(\delta_1)$  ensures that there is  $y$  such that  $w(y) = \delta_1$  and  $w(y \cdot r_0) > \delta_1$  and  $w(y - (u'_j - u'_i)) = \delta_1, j \neq i \in J'$ . By Axiom (DG), let  $z$  be such that  $z \cdot r_0 = t_0 + (u'_j - y) \cdot r_0$  and  $w(z) = w(t_0 + (u'_j - y) \cdot r_0) > \delta_1$ . Then  $(z - u'_j + y) \cdot r_0 = t_0, w((z - u'_j + y) + u'_j) = \delta_1, w((z - u'_j + y) + u'_i) = \delta_1$  if  $j \neq i \in J'$ . It follows that  $w((z - u'_j + y) + u'_i) = \delta_1$  if  $i \in B$ . On the other hand, if  $i \notin B$ , we already have  $w((z - u'_j + y) + u'_i) = w(z + y + (u'_i - u'_j)) \geq \delta_1$ , but we cannot have  $w((z - u'_j + y) + u'_i) > \delta_1$  because it would lead to  $w((z - u'_j + y) \cdot r_0 + u'_i \cdot r_0) = w(t_0 + u'_i \cdot r_0) > \delta_1$  which is not the case. Thus  $z - u'_j + y$  is a solution of  $(IV)_{B,J}$ . This proves the lemma.  $\dashv$

LEMMA 6.8. Consider the system

$$\exists x \left\{ \begin{array}{l} x \cdot r_0 = t_0, \\ w(x \cdot r + t) > \delta_1, \\ \bigwedge_{i=1}^m w(x \cdot r_i + t_i) \square_i \delta_1, \\ \bigwedge_{i=m+1}^n w(x \cdot r_i + t_i) \square_i \delta_i \end{array} \right.$$

where  $r \in A_0, v(r) = v(r_0) = 0, \text{deg}(r) < \text{deg}(r_0)$ , and where  $r_i \in \mathcal{S}, t_i$  are  $L_A$ -terms,  $1 \leq i \leq n, \square_i$  is one of  $=, >, \geq$ , and  $\delta_1 > \delta_{m+1} \geq \dots \geq \delta_n$ . Let  $\mu \in \mathcal{O}_K, s, r' \in A_0$  such that  $r_0 \mu = rs + r'$ , and  $r' = 0$  or  $\text{deg}(r') < \text{deg}(r)$ . Then the

above system is equivalent to the following system

$$\exists x \begin{cases} x \cdot r = -t, \\ w(x \cdot r' - t \cdot s - t_0\mu) > \delta_1 + v(\mu), \\ \bigwedge_{i=1}^m w(x \cdot r_i + t_i) \square_i \delta_1, \\ \bigwedge_{i=m+1}^n w(x \cdot r_i + t_i) \square_i \delta_i. \end{cases}$$

PROOF. (→) Let  $x$  be a solution of the given system. By Axiom (DG) there exists  $u$  such that  $w(u) > \delta_1$  and  $u \cdot r = -x \cdot r - t$ . We get  $w((x + u) \cdot r' - t \cdot s - t_0\mu) = w(u \cdot r_0\mu) > \delta_1 + v(\mu)$ , and  $x + u$  is a solution of the new system.

(←) Let  $y$  be a solution of the new system. We look for an element  $u$  such that  $(y + u) \cdot r_0 \cdot \mu = (y + u) \cdot r \cdot s + (y + u) \cdot r' = t_0 \cdot \mu$ . Equivalently,  $u \cdot r_0 \mu = t_0 \cdot \mu + t \cdot s - y \cdot r'$ . By Axiom (DG), there exists such an element  $u$  with valuation strictly bigger than  $\delta_1$ . Then  $y + u$  is a solution to the given system. ⊖

Recall that it suffices to consider a formula existential in just one variable  $\exists x \phi(x, y, \delta)$ , where  $\phi(x, y, \delta)$  is quantifier-free in  $x, y$  and existential in  $\delta$ , and we can always assume that we have at most one equation of the form  $x \cdot r = t$  involving  $x$ .

As in the previous elimination theorem, we will concentrate on the system of conditions where  $x$  appears. Using the normalization process of Proposition 4.1 if necessary, we could reduce ourselves to consider systems of the following form:

$$\exists x (x \cdot r_0 = t_0 \ \& \ w(x \cdot r_1 + t_1) = \delta_1 \ \& \ \dots \ \& \ w(x \cdot r_n + t_n) = \delta_n)$$

with  $\text{deg}(r_0) > \text{deg}(r_i)$ ,  $r_0, r_i \in A_0, \bar{r}_0 \neq 0, \bar{r}_i \neq 0 \in \bar{K}[t], 1 \leq i \leq n, \infty \neq \delta_1 = \dots = \delta_m > \delta_{m+1} \geq \dots \geq \delta_n$  and the  $t_\ell$  are  $L_A$ -terms. But in order to proceed inductively, we will, by a similar process, consider the slightly more general systems of the form:

$$\exists x \left( x \cdot r_0 = t_0 \ \& \ \bigwedge_{i=1}^m w(x \cdot r_i + t_i) \square_i \delta_1 \ \& \ \bigwedge_{i=m+1}^n w(x \cdot r_i + t_i) \square_i \delta_i \right) \quad (1)$$

where  $\square_i$  is either one of  $=, >, \geq$ , with  $\text{deg}(r_0) > \text{deg}(r_i)$ ,  $r_0, r_i \in A_0, \bar{r}_0 \neq 0, \bar{r}_i \neq 0 \in \bar{K}[t], 1 \leq i \leq n, \infty \neq \delta_1 > \delta_{m+1} \geq \dots \geq \delta_n$  and the  $t_i$  are  $L_A$ -terms.

We define the complexity of such a system as the triple

$$\left( \text{deg}(r_0), \sum_{i \in I_=} \text{deg}(r_i), |I_{>}| \right)$$

in the lexicographic product  $(\mathbb{N}^3, <)$ , where  $I_=$  is the set of those  $i$  such that  $w(x \cdot r_i + t_i) = \delta_1$  occurs in the system, and  $I_{>}$  is the set of those  $i$  such that  $w(x \cdot r_i + t_i) > \delta_1$  occurs in the system. If  $I_= = \emptyset$ , the complexity is set to  $(\text{deg}(r_0), -\infty, |I_{>}|)$ , and if  $r_0 = 0, I_= = \emptyset$  to  $(-\infty, -\infty, |I_{>}|)$ , and we extend the lexicographical order in the natural way.

To keep track of this complexity measure, it is worth recording the main effect of the normalization process of Proposition 4.1 which we will use again. Namely, it does not alter the degree of  $r_0$ , so that the first component of complexity is left unchanged.

We now proceed by induction on complexity of systems of the above form (1). By Lemma 6.5 and the very first remark made at the beginning of the proof, we can assume we have one equation with  $r_0 \neq 0$ .

Basis of induction.

1. No valuation conditions: we are reduced to the classical case of divisible modules and proceed accordingly.

2.  $\deg(r_0) = 0$  : i.e.,  $r_0 \in K$ , then  $x = t_0 \cdot r_0^{-1}$  and we can eliminate  $x$  by substitution.

3.  $I_- \neq \emptyset, I_> = \emptyset$  and for all  $i \in I_-$ ,  $\deg(r_i) = 0$ : we can assume  $r_i = 1, i \in I_-$ , and we then have a system of the form

$$\exists x \begin{cases} x \cdot r_0 = t_0, \\ \bigwedge_{i=1}^m w(x + u'_i) = \delta_1, \\ \bigwedge_{i=1}^m w(x \cdot r_i + t_i) \geq \delta_1, \\ \bigwedge_{i=m+1}^n w(x \cdot r_i + t_i) \square_i \delta_i. \end{cases} \quad (3)$$

where  $u'_i = u'_i(\mathbf{y})$  is a  $M$ -term, and  $\square_i$  is one of  $=, >, \geq$ . This system is equivalent to a disjunction of the following systems over the subsets  $B$  of  $\{1, \dots, m\}$  as follows:

$$\exists x \begin{cases} x \cdot r_0 = t_0, \\ \bigwedge_{i=1}^m w(x + u'_i) = \delta_1, \\ \bigwedge_{i=1}^m w(x \cdot r_i + t_i) \geq \delta_1, \\ \bigwedge_{i=m+1}^n w(x \cdot r_i + t_i) \square_i \delta_i, \\ \bigwedge_{i \in B} w(t_0 + u'_i \cdot r_0) > \delta_1, \\ \bigwedge_{i \notin B} w(t_0 + u'_i \cdot r_0) = \delta_1. \end{cases} \quad (3)_B$$

If  $B = \emptyset$ , then we are done by Lemma 6.6. If  $B \neq \emptyset$ , then we are done by Lemma 6.7.

Induction step(s). We apply the normalization process when necessary. So consider again system (1). By making a disjunction of cases, we can assume that  $I_- \neq \emptyset$  or  $I_> \neq \emptyset$ .

1.  $I_> \neq \emptyset$ : by Lemma 6.8 we get an equivalent system of strictly lower complexity and we are done by induction.

2.  $I_> = \emptyset, I_- \neq \emptyset$ : by the basis of induction we can assume  $\deg(r_0) \geq 1$  and  $\deg(r_i) \geq 1$  for some  $i \in I_-$ .

Suppose first that some  $r_i, i \in I_-$ , with  $\deg(r_i) \geq 1$  divides  $r_0$ , say  $i = \ell$ . Let  $r_0 \lambda = r_\ell s_\ell$ , where  $\lambda \in \mathcal{O}_K, s_\ell \in A_0$ . Since  $v(r_0) = v(r_\ell) = 0$ , we get  $v(\lambda) = v(s_\ell)$  and we can assume that  $\lambda = 1$ , i.e.,  $r_0 = r_\ell s_\ell$ . We introduce temporarily a new module variable  $z$  which will enable us to do induction. Then system (1) is clearly equivalent to the following system:

$$\exists z \exists x \begin{cases} x \cdot r_\ell = z, \\ \bigwedge_{1 \leq i \neq \ell \leq m} w(x \cdot r_i + t_i) \square_i \delta_i, \\ \bigwedge_{i=m+1}^n w(x \cdot r_i + t_i) \square_i \delta_i, \\ z \cdot s_\ell = t_0, \\ w(z + t_\ell) = \delta_1. \end{cases} \quad (2)_z$$

Now, by induction, the formula

$$\exists x \left( x \cdot r_\ell = z \ \& \ \bigwedge_{1 \leq i \neq \ell \leq m} w(x \cdot r_i + t_i) \square_i \delta_i \ \& \ \bigwedge_{i=m+1}^n w(x \cdot r_i + t_i) \square_i \delta_i \right)$$

is equivalent to a disjunction of formulas of the form

$$\bigwedge_j w(z \cdot r'_j + t'_j) \square_j \delta_j \ \& \ \Theta(\mathbf{y}, \boldsymbol{\delta})$$

where  $r'_j \in A_0$ ,  $t'_j(\mathbf{y})$  are  $L_A$ -terms, the  $\delta_j$  are among  $\delta_1, \dots, \delta_n$ ,  $\square_j$  is either one of  $=, >, \geq$ , and  $\Theta(\mathbf{y}, \boldsymbol{\delta})$  is quantifier free in the  $M$ -sort, existential in the  $\Delta$ -sort, plus some residual index formulas and sentences. We can now apply induction to

$$\exists z \left( z \cdot s_\ell = t_0 \ \& \ w(z + t_\ell) = \delta_1 \ \& \ \bigwedge_j w(z \cdot r'_j + t'_j) \square_j \delta_j \right)$$

and we have completed that induction step.

We are left with the case where no nonconstant  $r_i, i \in I_=$ , divides  $r_0$ . Pick any such, say  $i = \ell$ , and do Euclidean division. Let  $r_0\lambda = r_\ell s_\ell + r$ , where  $\lambda \in \mathcal{O}_K, s_\ell, r \in A_0, r \neq 0, \text{deg}(r) < \text{deg}(r_\ell) < \text{deg}(r_0)$ . We claim that system (1) is then equivalent to the disjunction of the following two systems (2a) or (2b):

$$\exists x \begin{cases} x \cdot r_0 = t_0, \\ w(x \cdot r - t_\ell \cdot s_\ell - t_0\lambda) = \delta_1, \\ w(x \cdot r_\ell + t_\ell) \geq \delta_1, \\ \bigwedge_{1 \leq i \neq \ell \leq m} w(x \cdot r_i + t_i) \square_i \delta_i, \\ \bigwedge_{i=m+1}^n w(x \cdot r_i + t_i) \square_i \delta_i \end{cases} \tag{2a}$$

$$\exists x \begin{cases} x \cdot r_0 = t_0, \\ w(x \cdot r - t_\ell \cdot s_\ell - t_0\lambda) > \delta_1, \\ \bigwedge_{i=1}^m w(x \cdot r_i + t_i) \square_i \delta_i, \\ \bigwedge_{i=m+1}^n w(x \cdot r_i + t_i) \square_i \delta_i. \end{cases} \tag{2b}$$

Indeed, suppose  $x$  a solution of (1). We have  $w(x \cdot r_\ell + t_\ell) = \delta_1$ , so that  $w(x \cdot r_\ell \cdot s_\ell + t_\ell \cdot s_\ell) \geq \delta_1$ . If  $w(x \cdot r_\ell \cdot s_\ell + t_\ell \cdot s_\ell) = \delta_1$ , we get  $x \cdot r_\ell \cdot s_\ell = t_0\lambda - x \cdot r$  and  $w(x \cdot r - t_\ell \cdot s_\ell - t_0\lambda) = \delta_1$ , and  $x$  is a solution of (2a). Similarly, if  $w(x \cdot r_\ell \cdot s_\ell + t_\ell \cdot s_\ell) > \delta_1$ , we get  $w(x \cdot r_\ell - t_\ell \cdot s_\ell - t_0\lambda) > \delta_1$ , and  $x$  is a solution of (2b). Conversely, suppose  $x$  is a solution of (2a). We have  $w(x \cdot r_\ell \cdot s_\ell + t_\ell \cdot s_\ell) \geq w(x \cdot r_\ell + t_\ell)$  and  $w(x \cdot r_\ell \cdot s_\ell + t_\ell \cdot s_\ell) = w(x \cdot r - t_\ell \cdot s_\ell - t_0\lambda) = \delta_1$ , so that we cannot have  $w(x \cdot r_\ell + t_\ell) > \delta_1$ . Hence  $w(x \cdot r_\ell + t_\ell) = \delta_1$  and  $x$  is a solution of (1). Note that a solution of (2b) is immediately a solution of (1).

Consider system (2a) and let  $\mu \in \mathcal{O}_K$  such that  $v(\mu) = v(r_\ell)$ . Now if in system (2a) we replace  $w(x \cdot r - t_\ell \cdot s_\ell - t_0\lambda) = \delta_1$  by  $w(x \cdot r\mu^{-1} - t_\ell \cdot s_\ell\mu^{-1} - t_0\lambda\mu^{-1}) = \delta_1 - v(\mu)$ , we obtain an equivalent system of strictly lower complexity than system (1). Indeed, if  $v(r) = 0$  we have replaced a valuation equation involving  $\delta_1$  with another one where the coefficient of  $x$  has strictly smaller degree and the first component of complexity has strictly decreased, and if  $v(r) > 0$  we have replaced it by a valuation equation with a value strictly less than  $\delta_1$  and the second component of complexity has strictly decreased. So we have completed the induction step in case of system (2a).

Consider system (2b). Recall that  $v(r_0) = 0$  and  $r_0\lambda = r_\ell s_\ell + r$ . So by scaling with a suitable element of  $K$  we can assume that either  $v(r_\ell s_\ell) = 0$  or  $v(r) = 0$ . If  $v(r) = 0$ , then by Lemma 6.8 we can obtain an equivalent system of strictly

lower complexity. If  $v(r_\ell s_\ell) = 0$ , then system (2b) is equivalent to the following system (2c):

$$\exists x \begin{cases} x \cdot r_\ell s_\ell = -t_\ell \cdot s_\ell, \\ w(x \cdot r - t_\ell \cdot s_\ell - t_0 \lambda) > \delta_1, \\ \bigwedge_{i=1}^m w(x \cdot r_i + t_i) \square_i \delta_1, \\ \bigwedge_{i=m+1}^h w(x \cdot r_i + t_i) \square_i \delta_i \end{cases} \quad (2c)$$

Indeed, if  $x$  is a solution of (2b), then by Axiom (DG) there exists  $u$  such that  $u \cdot r_\ell s_\ell = -x \cdot r_\ell s_\ell - t_\ell \cdot s_\ell$  and  $w(u) > \delta_1$ , and then  $x + u$  is a solution of (2c). A similar computation works in the other direction. But now in system (2c),  $r_\ell$  divides the coefficient of  $x$  in the first equation and we are back to a case already treated at the beginning, the extra strict valuation inequality being harmless in that process: we have increased the third component of complexity, but divisibility allows to decrease the dominant first component. The induction is now completed. −

**§7. NIP.** As a reference about the independence property see [24], Chapter 12, Section 12.d. We will abbreviate the property of “not having the independence property” by NIP. Recall that the following theories do have NIP: any stable theory, the theory of a chain (ibid., Section 12.f), the theory of abelian totally ordered groups ([13]). It is known that for most valued fields for which the Ax-Kochen-Ershov principle holds, its theory has NIP iff the theory of its residue field does (see [4]). This applies to  $W(F)$ , but leaves  $F((x))$  open if  $F$  is of characteristic  $p > 0$ . In the case of the theories of valued difference fields treated in [6], T. Scanlon observed that quantifier elimination and a *forgetful functor* trick<sup>1</sup> allow one to transfer the NIP from the pure valued fields theories and get that these difference fields do have NIP. So in particular our module theories  $T_{w,d,dense}$  and  $T_{w,d,discrete}$  do have NIP when  $K$  has characteristic 0. Here, we will give a direct proof that these theories have NIP, covering also the case when  $K$  has nonzero characteristic.

Note that by a similar proof the theory of a chain has NIP, we can deduce that the theories  $T_{\Delta,dense}$  and  $T_{\Delta,discrete}$  have NIP too. Indeed, the quantifier elimination result for these theories implies that the types are determined by the quantifier-free parts and so the type of an element over a model is determined by its cut in that model. Therefore a type has at most two co-heirs and so we get the NIP property by Theorem 12.28 of [24].

**PROPOSITION 7.1.** *Let  $(M, w, \Delta)$  be either a model of  $T_{w,d,dense}$ , or  $T_{w,d,discrete}$ . Then the theory of  $M$  has NIP.*

**PROOF.** We will proceed by contradiction. Assume that we have a formula  $\phi(\alpha, \mathbf{y}, \boldsymbol{\delta})$  witnessing the independence property in the single variable  $\alpha$  (e.g., see [24], Theorem 12.18), where  $\alpha$  is of either sort  $M$  or  $\Delta$  and the variables  $\mathbf{y}$  are of sort  $M$  and the  $\boldsymbol{\delta}$  of sort  $\Delta$ . Denote by  $T$  either  $T_{w,d,dense}$  or  $T_{w,d,discrete}$ .

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<sup>1</sup>One transforms a quantifier free formula from the valued difference field language to a formula from the valued field language by replacing a term  $\sigma^i(x)$  by a new variable. Michaux and Rivière observed in [23] that this kind of trick works in ordered differential fields.



By compactness, we have tuples  $(a_i, \mathbf{b}_m, \mathbf{c}_m)_{i \in \omega, m \in 2^\omega}$  in some model of  $T$  such that the following holds:

$$\bigwedge_{i \in m} \phi(a_i, \mathbf{b}_m, \mathbf{c}_m), \bigwedge_{i \notin m} \neg \phi(a_i, \mathbf{b}_m, \mathbf{c}_m), \quad i \in \omega, m \in 2^\omega.$$

First, suppose that  $\alpha$  is of sort  $\Delta$ .

Since  $T$  admits quantifier elimination (Corollary 6.4), we may assume that  $\phi(\alpha, \mathbf{y}, \boldsymbol{\delta})$  is a finite disjunction of quantifier free formulas  $\phi_\ell$ , of the form:

$$\phi_\ell(\alpha, \mathbf{y}, \boldsymbol{\delta}) := \bigwedge_{i=1}^{n(\ell)} w(t_i(\mathbf{y})) = \delta_i \ \& \ \theta_\ell(\mathbf{y}) \ \& \ \delta_{\ell 1} + \gamma_{\ell 1} \ \square \ \alpha \ \square \ \delta_{\ell 2} + \gamma_{\ell 2} \ \& \ \psi_\ell(\boldsymbol{\delta})$$

where  $\square \in \{<, \leq\}$ ,  $\theta_\ell(\mathbf{y})$  is a quantifier-free pp-formula,  $t_i(\mathbf{y})$  are  $L_A$ -terms of sort  $M$ , and  $\psi_\ell(\boldsymbol{\delta})$  is a quantifier free formula (recall that  $T_{\Delta, \text{dense}}$  and  $T_{\Delta, \text{discrete}}$  both admit quantifier elimination).

We introduce new variables  $\beta_h$  of sort  $\Delta$  that we substitute for each  $w(t_i(\mathbf{y}))$  and  $\delta_{\ell j} + \gamma_{\ell j}$ .

Let  $\phi'_\ell(\alpha, \boldsymbol{\beta}, \boldsymbol{\delta})$  be the formula we obtain from  $\phi_\ell(x, \mathbf{y}, \boldsymbol{\delta})$  by making the above substitution and leaving out all the subformulas not involving either  $\alpha$ ,  $\boldsymbol{\delta}$  or  $\boldsymbol{\beta}$ . Set  $\phi'(\alpha, \boldsymbol{\beta}, \boldsymbol{\delta}) := \bigvee_\ell \phi'_\ell(\alpha, \boldsymbol{\beta}, \boldsymbol{\delta})$ .

Let  $\mathbf{d}_m$  be the value of  $\boldsymbol{\beta}$  obtained by making the above substitutions with the values  $\mathbf{b}_m$  of  $\mathbf{y}$  and  $\mathbf{c}_m$  of  $\boldsymbol{\delta}$ .

Therefore the tuples  $(a_i, \mathbf{d}_m, \mathbf{c}_m)_{i \in \omega, m \in 2^\omega}$  satisfy the following

$$\bigwedge_{i \in m} \phi'(a_i, \mathbf{d}_m, \mathbf{c}_m), \bigwedge_{i \notin m} \neg \phi'(a_i, \mathbf{d}_m, \mathbf{c}_m), \quad i \in \omega, m \in 2^\omega$$

contradicting the property that no chain has the independence property.

Now we suppose that  $\alpha$  is of sort  $M$ . Then we may assume that  $\phi(\alpha, \mathbf{y}, \boldsymbol{\delta})$  is a disjunction  $\bigvee_{\ell=1}^k \phi_\ell(\alpha, \mathbf{y}, \boldsymbol{\delta})$  of quantifier-free formulas of the form:

$$\phi_\ell(\alpha, \mathbf{y}, \boldsymbol{\delta}) :=$$

$$\bigwedge_{i=1}^{n(\ell)} w(\alpha \cdot r_i + t_i(\mathbf{y})) = \delta_i \ \& \ \alpha \cdot r_0 = t_0(\mathbf{y}) \ \& \ \theta_\ell(\mathbf{y}) \ \& \ \psi_\ell(\boldsymbol{\delta})$$

with the same notation as before and  $r_0, r_i \in A$ .

We vary  $j \in n$  and by the pigeonhole principle, there exist pairwise distinct elements  $j_4, j_3, j_2 \in n$  and some atomic formula  $\chi$  occurring in  $\phi$  such that, setting  $S_1 = n - \{j_3, j_4\}$ ,  $S_3 = n - \{j_2, j_4\}$ ,  $S_2 = n - \{j_2, j_3\}$ , we have that  $\bigwedge_{i=2,3} \neg \chi(a_{j_2}, \mathbf{b}_{S_i}, \mathbf{c}_{S_i})$ ,  $\bigwedge_{i=1,2} \neg \chi(a_{j_3}, \mathbf{b}_{S_i}, \mathbf{c}_{S_i})$  and  $\neg \chi(a_{j_4}, \mathbf{b}_{S_1}, \mathbf{c}_{S_1})$  are satisfied. Indeed, suppose that we have at most  $d$  atomic formulas occurring in  $\phi$  and to each sequence of the form  $(a_{j_i}, \mathbf{b}_{S_i}, \mathbf{c}_{S_i})_i$  we associate a sequence of  $\pm 1$  according to the fact that  $(a_{j_i}, \mathbf{b}_{S_i}, \mathbf{c}_{S_i})$  satisfies the atomic formulas in that list or not. Since  $d$  is fixed, and since we can choose  $n$  big enough in order to have sufficiently many choices for  $j_4, j_3, j_2 \in n$ , we have a configuration as above.

Let  $j_1 \in S_1 \cap S_2 \cap S_3$ .

Either  $\chi$  is of the form  $\alpha \cdot r_0 = t_0(\mathbf{y})$  and then we get  $(a_{j_1} - a_{j_2}) \cdot r_0 = 0$  and  $a_{j_1} \cdot r_0 + t_0(\mathbf{b}_{S_2}) = 0$  but  $a_{j_2} \cdot r_0 + t_0(\mathbf{b}_{S_2}) \neq 0$ , which is a contradiction.

Or,  $\chi$  is of the form  $w(\alpha \cdot r_i + t_i(\mathbf{y})) = \delta_i$ . We will denote by  $c_{S_j}$  the  $i^{\text{th}}$  component of  $c_{S_j}$ . Then we have  $w(a_{j_1} \cdot r_i + t_i(\mathbf{b}_{S_1})) = c_{S_1}$  and  $w(a_{j_2} \cdot r_i + t_i(\mathbf{b}_{S_1})) = c_{S_1}$ . So,  $w((a_{j_1} - a_{j_2}) \cdot r_i) \geq c_{S_1}$ .

Similarly, we get  $w(a_{j_1} \cdot r_i + t_i(\mathbf{b}_{S_3})) = c_{S_3}$  and  $w(a_{j_3} \cdot r_i + t_i(\mathbf{b}_{S_3})) = c_{S_3}$ . So,  $w((a_{j_1} - a_{j_3}) \cdot r_i) \geq c_{S_3}$ .

On the other hand since  $j_3 \notin S_1$ ,  $w(a_{j_3} \cdot r_i + t_i(\mathbf{b}_{S_1})) \neq c_{S_1}$  and  $w(a_{j_3} \cdot r_i + t_i(\mathbf{b}_{S_1})) \geq \min\{w((a_{j_1} - a_{j_3}) \cdot r_i), w(a_{j_1} \cdot r_i + t_i(\mathbf{b}_{S_1}))\}$ , and since  $j_2 \notin S_3$ ,  $w(a_{j_2} \cdot r_i + t_i(\mathbf{b}_{S_3})) \neq c_{S_3}$  and  $w(a_{j_2} \cdot r_i + t_i(\mathbf{b}_{S_3})) \geq \min\{w((a_{j_2} - a_{j_1}) \cdot r_i), w(a_{j_1} \cdot r_i + t_i(\mathbf{b}_{S_3}))\}$ .

So we get  $c_{S_1} \geq c_{S_3}$ . Suppose now that  $c_{S_1} > c_{S_3}$ , then  $w(a_{j_2} \cdot r_i + t_i(\mathbf{b}_{S_3})) = w((a_{j_2} - a_{j_1}) \cdot r_i + a_{j_1} \cdot r_i + t_i(\mathbf{b}_{S_3})) = \min\{w((a_{j_2} - a_{j_1}) \cdot r_i), w(a_{j_1} \cdot r_i + t_i(\mathbf{b}_{S_3}))\} = c_{S_3}$ , a contradiction.

So we have  $c_{S_1} = c_{S_3}$  which denote henceforth  $c$ , and by the above, we necessarily get that  $w(a_{j_3} \cdot r_i + t_i(\mathbf{b}_{S_1})) > c$  and  $w(a_{j_2} \cdot r_i + t_i(\mathbf{b}_{S_3})) > c$ .

Using these two strict inequalities, we get that  $w((a_{j_1} - a_{j_3}) \cdot r_i) = c$  and  $w((a_{j_2} - a_{j_3}) \cdot r_i) = c$ .

Since  $j_1 \in S_2$ ,  $w(a_{j_1} \cdot r_i + t_i(\mathbf{b}_{S_2})) = c_{S_2}$ .

Suppose  $c_{S_2} < c$ , then  $w(a_{j_3} \cdot r_i + t_i(\mathbf{b}_{S_2})) = \min\{w((-a_{j_1} + a_{j_3}) \cdot r_i), w(a_{j_1} \cdot r_i + t_i(\mathbf{b}_{S_2}))\} = c_{S_2}$ , which contradicts the fact that  $j_3 \notin S_2$ . Therefore,  $c_{S_2} \geq c$ . Suppose  $c_{S_2} > c$ . First note that  $w((-a_{j_1} + a_{j_4}) \cdot r_i) \geq c_{S_2}$ , since  $w((-a_{j_1} + a_{j_4}) \cdot r_i) \geq \min\{w(a_{j_1} \cdot r_i + t_i(\mathbf{b}_{S_2})), w(a_{j_4} \cdot r_i + t_i(\mathbf{b}_{S_2}))\}$ . Then,  $w(a_{j_4} \cdot r_i + t_i(\mathbf{b}_{S_1})) = \min\{w((-a_{j_1} + a_{j_4}) \cdot r_i), w(a_{j_1} \cdot r_i + t_i(\mathbf{b}_{S_1}))\} = c$ , a contradiction since  $j_4 \notin S_1$ .

So, we get that  $c_{S_2} = c$ .

Now, we have  $w(a_{j_3} \cdot r_i + t_i(\mathbf{b}_{S_2})) \geq \min\{w((a_{j_3} - a_{j_1}) \cdot r_i), w(a_{j_1} \cdot r_i + t_i(\mathbf{b}_{S_2}))\} = c$  and  $w(a_{j_2} \cdot r_i + t_i(\mathbf{b}_{S_2})) \geq \min\{w((a_{j_2} - a_{j_1}) \cdot r_i), w(a_{j_1} \cdot r_i + t_i(\mathbf{b}_{S_2}))\} = c$ . Hence  $w(a_{j_3} \cdot r_i + t_i(\mathbf{b}_{S_2})) > c$  and  $w(a_{j_2} \cdot r_i + t_i(\mathbf{b}_{S_2})) > c$ . But therefore,  $w((a_{j_2} - a_{j_3}) \cdot r_i) > c$ , a contradiction.  $\dashv$

It has been observed that formulas without the independence property are closed under boolean combinations (see e.g., [1]). So we could have directly considered only atomic formulas.

**§8. Ultraproducts.** Let  $\mathcal{U}$  be a non-principal ultrafilter over the set of prime numbers  $p$ . Let  $F_p$  be a  $p$ -closed field of characteristic  $p$  of cardinality at most  $\aleph_1$ . By the Ax-Kochen-Ershov theorem the ultraproducts  $\prod_{\mathcal{U}} W(F_p)$  and  $\prod_{\mathcal{U}} F_p((x))$  are elementarily equivalent as valued fields, and  $\prod_{\mathcal{U}} W(\mathbb{F}_p)$  and  $\prod_{\mathcal{U}} \mathbb{F}_p((x))$  just as well. They are also  $\aleph_1$ -saturated. If we assume the continuum hypothesis<sup>2</sup>, they are of cardinality  $\aleph_1$  and so isomorphic. Similarly  $\prod_{\mathcal{U}} W(\mathbb{F}_p)$  and  $\prod_{\mathcal{U}} \mathbb{F}_p((x))$  are isomorphic as valued fields, say via  $\varphi$ . Let  $\sigma_{pwf}$  be the Witt Frobenius on  $W(F_p)$  and let  $\sigma_{pc}$  be the automorphism sending  $\sum a_i x^i$  to  $\sum a_i^p x^i$  in  $F_p((x))$ . Consider the valued fields with isometry  $(W(F_p), \sigma_{pwf})$ , and  $(F_p((x)), \sigma_{pc})$ . Denote respectively by  $\sigma_{wf}$  and  $\sigma_c$  the induced automorphisms on the ultraproducts of these fields.

Let  $A = \prod_{\mathcal{U}} W(\mathbb{F}_p)[t]$ , the (ordinary) polynomial ring over  $\prod_{\mathcal{U}} \mathbb{Q}_p$ . Consider  $\prod_{\mathcal{U}} W(F_p)$  as an  $A$ -module with  $t$  acting as  $\sigma_{wf}$  on  $\prod_{\mathcal{U}} W(F_p)$ . Consider also  $\prod_{\mathcal{U}} F_p((x))$  as an  $A$ -module via  $\varphi$  and  $t$  acting as  $\sigma_c$ , namely  $m \cdot \sum t^i c_i = \sum \sigma_c^i(m) \varphi(c_i)$ .

<sup>2</sup>S. Shelah has constructed a model of ZFC where these ultraproducts are not isomorphic.

We will consider their theories first as modules, then as valued modules with the usual valuation map. As modules they are elementarily equivalent if for all  $q(t) \in A$ ,  $\text{ann}(q)$  is non-zero in  $\prod_{\mathbb{Z}} W(F_p)$  iff it is non-zero in  $\prod_{\mathbb{Z}} F_p((x))$  (see Corollary 2.5). Since both structures satisfy the linear Hensel property and since the residue fields are isomorphic to  $\prod_{\mathbb{Z}} F_p$  and since both  $\bar{\sigma}_{pwf}$  and  $\bar{\sigma}_{pc}$  act as the standard Frobenius on the residue fields, we get the result.

Concerning their theories as valued modules, first we have to check that they are models of the schemes (DG) and (IR), and by remarks on completions in Corollary 6.4, since in this case the image by the map  $w$  of the module is equal to the value group of the ring, it suffices to examine the cardinalities of the annihilators in the quotient  $V_0/V_0^+$  or in the subgroup  $V_0$ ; namely either in  $\prod_{\mathbb{Z}} F_p$  or in the subgroup  $\prod_{\mathbb{Z}} W[F_p]$  (respectively  $\prod_{\mathbb{Z}} F_p[[x]]$ ). Since  $\prod_{\mathbb{Z}} F_p$  is infinite, it does not satisfy any identities, so (IR) holds, and the axiom (DG) still holds using the linear Hensel property and the fact that  $F_p$  is  $p$ -closed. So we have elementary equivalence as valued modules as well. This follows also from [6], but it might be appropriate to notice that it already follows from the *linear theory*.

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