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Parity-violating vertices for spin-3 gauge fields

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The problem of constructing consistent parity-violating interactions for spin-3 gauge fields is considered in Minkowski space. Under the assumptions of locality, Poincaré invariance, and parity noninvariance, we classify all the nontrivial perturbative deformations of the Abelian gauge algebra. In space-time dimensions $n = 3$ and $n = 5$, deformations of the free theory are obtained which make the gauge algebra non-Abelian and give rise to nontrivial cubic vertices in the Lagrangian, at first order in the deformation parameter g. At second order in g, consistency conditions are obtained which the five-dimensional vertex obeys, but which rule out the $n = 3$ candidate. Moreover, in the five-dimensional first-order deformation case, the gauge transformations are modified by a new term which involves the second de Wit-Freedman connection in a simple and suggestive way.

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I. INTRODUCTION

One of the most intriguing open questions in classical field theory is the construction of consistent interactions among massless fields of spin (or helicity) s greater than 2. It is generally believed that, when perturbatively deforming Fronsdal's massless spin-s quadratic Lagrangian [1], the only first-order vertex is cubic and contains s derivatives. This is what comes out of manifestly covariant analyzes [2–6] and a light-front approach [7]. Moreover, in the aforementioned works, it is found that the massless spin-3 field (more generally, the odd-spin fields) carries a color, the fields taking value in an internal anticommutative, invariant-normed algebra. This implies that there must be (self-interacting) *multiplets* of spin-3 fields, analogously to what happens in Yang-Mills theories for spin-1 gauge fields.

In a recent paper [8], the problem of introducing consistent interactions among spin-3 gauge fields has been carefully analyzed in Minkowski space-time $\mathbb{R}^{n-1,1}$ ($n > 3$) using Becchi-Rouet-Stora-Tyutin-(BRST)-cohomological methods. Under the assumptions of locality, Poincaré and parity invariance, all the perturbative, consistent deformations of the Abelian gauge algebra have been determined, together with the corresponding deformations of the quadratic action, at first order in the deformation parameter. On top of the covariant cubic vertex of [2], a new cubic vertex is found which also corresponds to a non-Abelian gauge algebra related to an internal, noncommutative, invariant-normed algebra (like in Yang-Mills's theories). This new cubic vertex brings in five derivatives of the field: it is of the form $\mathcal{L}_1 \sim g_{[abc]}(h^a \partial^2 h^b \partial^3 h^c + h^a \partial h^b \partial^4 h^c)$.

In the present paper, we determine the non-Abelian consistent deformations of the free spin-3 gauge theory in Minkowski space-time $\mathbb{R}^{n-1,1}$, relaxing the parity invariance constraint of [8]. In other words, we look for all non-Abelian deformations of the free theory such that the corresponding first-order vertices involve the completely antisymmetric Levi-Civita density $\varepsilon^{\mu_1 \cdots \mu_n}$, like in Chern-Simons theories. As we show explicitly in Sec. IV, such first-order parity-breaking nontrivial deformations exist only in space-time dimensions $n = 3$ and $n = 5$.

Following the cohomological procedure of [8], we first classify all the possible first-order deformations of the spin-3 gauge algebra that contain one Levi-Civita antisymmetric density (these are the "a_2 terms" in the notation of [8]). We find two such deformations that make the algebra non-Abelian, in dimensions $n = 3$ and $n = 5$. Then, we investigate whether these algebra-deforming terms give rise to consistent first-order vertices. Very interestingly, both of the algebra-deforming terms do lead to nontrivial deformations of the quadratic Lagrangian. The first one is defined in dimension $n = 3$, for spin-3 gauge fields that take value in an internal, anticommutative, invariant-normed algebra $\mathcal{A}$, while the second one is defined in a space-time of dimension $n = 5$ for fields that take value in a commutative, invariant-normed internal algebra $\mathcal{B}$. However, as we demonstrate, consistency conditions at second order in $\bar{\mu}$ imply that the algebras $\mathcal{A}$ and $\mathcal{B}$ must also be nilpotent of order three and associative, respectively. In turn, this means that the $n = 3$ parity-breaking deformation is trivial while the algebra $\mathcal{B}$ is a

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direct sum of one-dimensional ideals—provided the metrics which define the norms in $\mathcal{A}$ and $\mathcal{B}$ are positive-definite, which is demanded by the positivity of energy. Essentially, this signifies that we may consider only one *single* self-interacting spin-3 gauge field in the $n = 5$ case, similarly to what happens in Einstein gravity [9].

The paper is organized as follows. In Sec. II, we set up our conventions and summarize the structure of the free spin-3 gauge theory we are going to deform. We comment on the special $n = 3$ case which was not investigated in [8]. The BRST approach [10,11] to the problem of consistent deformation is briefly recalled in Sec. III, where our principal assumptions are spelled out. We then go through the main results of [8] and adapt them to $n = 3$ when needed; this gives us the opportunity to recall some results and notations needed in the following section where the consistent deformation problem is solved. Sec. IV indeed rests upon Sec. III's results. We first determine the parity-breaking algebra-deforming terms. Then we proceed to determine the corresponding first-order deformations of the free Lagrangian and study some constraints that arise at second order. The cases $n = 3$ and $n = 5$ are treated separately. In Sec. V we summarize our results and comment on further researches. Finally, the appendix gathers together several technical computations related to Schouten identities.

II. FREE THEORY

The local action for a collection $\{h^a_{\mu\nu\rho}\}$ of N noninteracting completely symmetric massless spin-3 gauge fields in flat space-time is [1]

$$S_0[h^a_{\mu\nu\rho}] = \sum_{a=1}^{N} \int \Big[-\frac{1}{2}\partial_\sigma h^a_{\mu\nu\rho}\partial^\sigma h^{a\mu\nu\rho} + \frac{3}{2}\partial^\mu h^a_{\mu\rho\sigma}\partial_\nu h^{a\nu\rho\sigma} + \frac{3}{2}\partial_\mu h^a_\nu \partial^\mu h^{a\nu} + \frac{3}{4}\partial_\mu h^{a\mu}\partial_\nu h^{a\nu} - 3\partial_\mu h^a_\nu \partial_\rho h^{a\rho\mu\nu}\Big] d^n x, \tag{1}$$

where $h^a_\mu := \eta^{\nu\rho} h^a_{\mu\nu\rho}$. The Latin indices are internal indices taking N values. They are raised and lowered with the Kronecker delta δ^{ab}. The Greek space-time indices are raised and lowered with the "mostly plus" Minkowskian metric $\eta_{\mu\nu}$. The free action (1) is invariant under the linear gauge transformations $\delta_\lambda h^a_{\mu\nu\rho} = 3\partial_{(\mu}\lambda^a_{\nu\rho)}$, where $\eta^{\mu\nu}\lambda^a_{\mu\nu} \equiv 0$. The gauge transformations are Abelian and irreducible. Curved (respectively square) brackets on indices denote strength-one complete symmetrization (respective antisymmetrization) of the indices.

The field equations read $\frac{\delta S_0}{\delta h^a_{\mu\nu\rho}} \equiv G^{\mu\nu\rho}_a \approx 0$, where $G^a_{\mu\nu\rho} := F^a_{\mu\nu\rho} - \frac{3}{2}\eta_{(\mu\nu}F^a_{\rho)}$ and $F^a_{\mu\nu\rho}$ is the Fronsdal gauge-invariant tensor $F^a_{\mu\nu\rho} := \Box h^a_{\mu\nu\rho} - 3\partial^\sigma\partial_{(\mu}h^a_{\nu\rho)\sigma} + 3\partial_{(\mu}\partial_\nu h^a_{\rho)}$. We denote $F_\mu = \eta^{\nu\rho}F_{\mu\nu\rho}$.

The gauge symmetries enable one to get rid of some components of $h^a_{\mu\nu\rho}$, leaving it on-shell with N^n_3 independent physical components, where [12] $N^n_3 = \frac{n^3-3n^2-4n+12}{6}$. Notice that there is no propagating physical degree of freedom in $n = 3$.

An important object is the curvature gauge-invariant tensor [12–14]

$$K^a_{\alpha\mu|\beta\nu|\gamma\rho} := 8\partial_{[\gamma}\partial_{[\beta}\partial_{[\alpha}h^a_{\mu]\nu]\rho]}, \tag{2}$$

antisymmetric in the three pairs $\alpha\mu$, $\beta\nu$, and $\gamma\rho$. Actually, the spin-3 curvature is invariant under gauge transformations where the parameters $\lambda^a_{\mu\nu}$ are *not* constrained to be traceless.

Its importance, apart from gauge invariance with unconstrained gauge parameters, stems from the fact that the field equations $G^a_{\mu\nu\rho} \approx 0$ in Fronsdal's constrained approach are dynamically equivalent [15,16] to the following field equation in the unconstrained approach:

$$\eta^{\alpha\beta}K^a_{\alpha\mu|\beta\nu|\gamma\rho} \approx 0. \tag{3}$$

There exists another field equation for completely symmetric gauge fields in the unconstrained approach, which also involves the curvature tensor but is nonlocal [17] (see also [18]). The equivalence between both unconstrained field equations was proved in [15]. One of the advantages of the nonlocal field equation of [17] is that it can be derived from an action principle. The Eq. (3) is obtained from the general n-dimensional bosonic mixed symmetry case [15] by specifying to a completely symmetric rank-3 gauge field and is [16] a generalization of Bargmann-Wigner's equations in $n = 4$ [19]. However, it cannot be directly obtained from an action principle. For a recent work in direct relation to [17,18], see [20].

Notice that when $n = 3$, the Eq. (3) implies that the curvature vanishes on-shell, which reflects the "topological" nature of the theory in the corresponding dimension. This is similar to what happens in 3-dimensional Einstein gravity, where the vacuum field equations $R_{\mu\nu} := R^\alpha{}_{\mu\alpha\nu} \approx 0$ imply that the Riemann tensor $R^\alpha{}_{\mu\beta\nu}$ is zero on-shell. The latter property derives from the fact that the conformally-invariant Weyl tensor identically vanishes in dimension 3, allowing the Riemann tensor to be expressed entirely in terms of the Ricci tensor $R_{\mu\nu}$. Those properties are a consequence of a general theorem (see [21], p. 394) which states that a tensor transforming in an irreducible representation of $O(n)$ identically vanishes if the corresponding Young diagram is such that the sum of the lengths of the first two columns exceeds n.

Accordingly, in dimension $n = 3$ the curvature tensor $K^a_{\alpha\mu|\beta\nu|\gamma\rho}$ can be written [14] as

$$K^a_{\alpha\mu|\beta\nu|\gamma\rho} \equiv \tfrac{4}{3}\big(S^a_{\alpha\mu|[\beta[\gamma}\eta_{\rho]\nu]} + S^a_{\beta\nu|[\gamma[\alpha}\eta_{\mu]\rho]} + S^a_{\gamma\rho|[\alpha[\beta}\eta_{\nu]\mu]}\big), \tag{4}$$

where the tensor $S^a_{\alpha\mu|\nu\rho}$ is defined, in dimension $n = 3$, by

$$S^a_{\alpha\mu|\nu\rho} = 2\partial_{[\alpha}F^a_{\mu]\nu\rho} - \tfrac{3}{2}[2\partial_{[\alpha}F^a_{\mu]}\eta_{\nu\rho} - \partial_\rho F^a_{[\alpha}\eta_{\mu]\nu} - \partial_\nu F^a_{[\alpha}\eta_{\mu]\rho} + \partial_\alpha F^a_{(\nu}\eta_{\rho)\mu} - \partial_\mu F^a_{(\nu}\eta_{\rho)\alpha}].$$

It is antisymmetric in its first two indices and symmetric in its last two indices. For the expression of $S^a_{\alpha\mu|\nu\rho}$ in arbitrary dimension $n \geqslant 1$, see [14] where the curvature tensor $K^a_{\alpha\mu|\beta\nu|\gamma\rho}$ is decomposed under the (pseudo-)orthogonal group $O(n-1, 1)$. The latter reference contains a very careful analysis of the structure of Fronsdal's spin-3 gauge theory, as well as an interesting "topologically massive" spin-3 theory in dimension $n = 3$.

III. BRST FORMULATION

In the present section we summarize the BRST-cohomological procedure [10,11] where consistent couplings define deformations of the solution of the master equation. For more details, we refer, in particular, to [8] where the BRST approach is applied to the spin-3 gauge theory at hand.

A. Basic assumptions

We assume, as in the traditional perturbative Noether deformation procedure, that the deformed action can be expressed as a power series in a coupling constant g, the zeroth-order term in the expansion describing the free theory: $S = S_0 + gS_1 + \mathcal{O}(g^2)$.

We require that the deformed Lagrangian be invariant under the Poincaré group, but explicitly breaks parity symmetry by the presence of a completely antisymmetric Levi-Civita density $\varepsilon^{\mu_1\cdots\mu_n}$ in the deformed Lagrangian.

We reject *trivial* deformations arising from field-redefinitions that reduce to the identity at order g^0 and compute only consistent deformations, in the sense that the deformed theory possesses the same number of (possibly deformed) independent gauge symmetries, reducibility identities, etc., as the system we started with. Finally, crucial in the cohomological approach [10,11] is the locality requirement: the deformed action $S[\phi] = S_0[\phi] + gS_1[\phi] + \ldots$ must be a *local* functional. The deformations of the gauge transformations, etc., must be local functions, as well as the allowed field redefinitions.

A local function of some set of fields ϕ^i is a smooth function of the fields ϕ^i and their derivatives $\partial\phi^i, \partial^2\phi^i, \ldots$ up to some *finite* order, say k, in the number of derivatives. Such a set of variables $\phi^i, \partial\phi^i, \ldots, \partial^k\phi^i$ will be collectively denoted by $[\phi^i]$. Therefore, a local function of ϕ^i is denoted by $f([\phi^i])$. A local p-form $(0 \leqslant p \leqslant n)$ is a differential p-form the components of which are local functions. A local functional is the integral of a local n-form.

B. BRST spectrum and differential

According to the general rules of the BRST-antifield formalism, for the spin-3 gauge theory under consideration, the spectrum of fields (including ghosts) and antifields together with their respective ghost and antighost numbers is given by [8]

(a) the fields $h^a_{\mu\nu\rho}$, with ghost number 0 and antighost number 0;
(b) the ghosts $C^a_{\mu\nu}$, with ghost number 1 and antighost number 0;
(c) the antifields $h^{*\mu\nu\rho}_a$, with ghost number -1 and antighost number 1;
(d) the antifields $C^{*\mu\nu}_a$, with ghost number -2 and antighost number 2.

The Grassmannian parity of the (anti)fields is given by their ghost number modulo two.

The BRST differential s of the free theory is generated by the functional

$$W_0 = S_0[h^a] + \int (3h^{*\mu\nu\rho}_a \partial_\mu C^a_{\nu\rho}) d^n x.$$

More precisely, W_0 is the generator of the BRST differential s of the free theory through $sA = (W_0, A)_{a.b.}$, where the antibracket $(\ ,\ )_{a.b.}$ is defined by $(A, B)_{a.b.} = \frac{\delta^R A}{\delta\Phi^I} \times \frac{\delta^L B}{\delta\Phi^*_I} - \frac{\delta^R A}{\delta\Phi^*_I}\frac{\delta^L B}{\delta\Phi^I}$. The functional W_0 is a solution of the *master equation* $(W_0, W_0)_{a.b.} = 0$.

In the context of the free spin-3 gauge theory, the BRST-differential s decomposes into $s = \delta + \gamma$. The Koszul-Tate differential δ decreases the antighost number by one unit, while the first piece γ, the differential along the gauge orbits, leaves it unchanged. Both δ and γ, and consequently the differential s, increase the ghost number by one unit.

The action of the differentials δ and γ gives zero on all the fields of the formalism except in the few following cases:

$$\delta h^{*\mu\nu\rho}_a = G^{\mu\nu\rho}_a,$$
$$\delta C^{*\mu\nu}_a = -3\left(\partial_\rho h^{*\mu\nu\rho}_a - \frac{1}{n}\eta^{\mu\nu}\partial_\rho h^{*\rho}_a\right),$$
$$\gamma h^a_{\mu\nu\rho} = 3\partial_{(\mu}C^a_{\nu\rho)}.$$

C. BRST deformation

As shown in [10], the usual Noether procedure can be reformulated within a BRST-cohomological framework. Any consistent deformation of the gauge theory corresponds to a solution $W = W_0 + gW_1 + g^2W_2 + \mathcal{O}(g^3)$ of the deformed master equation $(W, W)_{a.b.} = 0$. Consequently, the first-order nontrivial consistent local deformations $W_1 = \int a^{n,0}$ are in one-to-one correspondence with elements of the cohomology $H^{n,0}(s|d)$ of the zeroth order BRST differential $s = (W_0, \cdot)$ modulo the total derivative d, in maximum form-degree n and in ghost number

0. That is, one must compute the general solution of the cocycle condition

$$sa^{n,0} + db^{n-1,1} = 0, \tag{5}$$

where $a^{n,0}$ is a topform of ghost number zero and $b^{n-1,1}$ a $(n-1)$-form of ghost number one, with the understanding that two solutions of (5) that differ by a trivial solution should be identified $a^{n,0} \sim a^{n,0} + sp^{n,-1} + dq^{n-1,0}$ as they define the same interactions up to field redefinitions. The cocycles and coboundaries $a, b, p, q, \ldots$ are local forms of the field variables (including ghosts and antifields).

The corresponding second-order interactions W_2 must satisfy the consistency condition $sW_2 = -\frac{1}{2}(W_1, W_1)_{a.b.}$ [10]. This condition is controlled by the local BRST cohomology group $H^{n,1}(s|d)$.

D. Cohomology of γ

The groups $H^*(\gamma)$ have recently been calculated [22] for massless spin-s gauge fields represented by completely symmetric (and double traceless when $s > 3$) rank s tensors. In the special case $s = 3$, the result reads:

Proposition 1—The cohomology of γ is isomorphic to the space of functions depending on

(a) the antifields $h_a^{*\mu\nu\rho}$, $C_a^{*\mu\nu}$ and their derivatives, denoted by $[\Phi^{*i}]$,
(b) the curvature and its derivatives $[K^a_{\alpha\mu|\beta\nu|\gamma\rho}]$,
(c) the symmetrized derivatives $\partial_{(\alpha_1}\ldots\partial_{\alpha_k}F^a_{\mu\nu\rho)}$ of the Fronsdal tensor,
(d) the ghosts $C^a_{\mu\nu}$ and the traceless parts of $\partial_{[\alpha}C^a_{\mu]\nu}$ and $\partial_{[\alpha}C^a_{\mu][\nu,\beta]}$.

Thus, identifying with zero any γ-exact term in $H(\gamma)$, we have $\gamma f = 0$ if and only if $f = f([\Phi^{*i}], [K^a_{\alpha\mu|\beta\nu|\gamma\rho}], \{F^a_{\mu\nu\rho}\}, C^a_{\mu\nu}, \hat{T}^a_{\alpha\mu|\nu}, \hat{U}^a_{\alpha\mu|\beta\nu})$ where $\{F^a_{\mu\nu\rho}\}$ stands for the completely symmetrized derivatives $\partial_{(\alpha_1}\ldots\partial_{\alpha_k}F^a_{\mu\nu\rho)}$ of the Fronsdal tensor, while $\hat{T}^a_{\alpha\mu|\nu}$ denotes the traceless part of $T^a_{\alpha\mu|\nu} := \partial_{[\alpha}C^a_{\mu]\nu}$ and $\hat{U}^a_{\alpha\mu|\beta\nu}$ the traceless part of $U^a_{\alpha\mu|\beta\nu} := \partial_{[\alpha}C^a_{\mu][\nu,\beta]}$.

Let us introduce some useful standard notations and make some remarks.

Let $\{\omega^I\}$ be a basis of the space of polynomials in the $C^a_{\mu\nu}$, $\hat{T}^a_{\alpha\mu|\nu}$ and $\hat{U}^a_{\alpha\mu|\beta\nu}$ (since these variables anticommute, this space is finite-dimensional). If a local form a is γ-closed, we have

$$\gamma a = 0 \quad \Rightarrow$$
$$a = \alpha_J([\Phi^{i*}], [K], \{F\})\omega^J(C^a_{\mu\nu}, \hat{T}^a_{\alpha\mu|\nu}, \hat{U}^a_{\alpha\mu|\beta\nu}) + \gamma b,$$

If a has a fixed, finite ghost number, then a can only contain a finite number of antifields. Moreover, since the *local* form a possesses a finite number of derivatives, we find that the α_J are polynomials. Such a polynomial $\alpha_J([\Phi^{i*}], [K], \{F\})$ will be called an *invariant polynomial*.

Remark 1: The Damour-Deser identity [14] $\eta^{\alpha\beta}K_{\alpha\mu|\beta\nu|\gamma\rho} \equiv 2\partial_{[\gamma}F_{\rho]\mu\nu}$ implies that the derivatives of the Fronsdal tensor are not all independent of the curvature tensor K. Therefore only the completely symmetrized derivatives of F appear in Proposition 1, while the derivatives of the curvature K are not restricted. However, in the sequel we will assume that every time the trace $\eta^{\alpha\beta}K_{\alpha\mu|\beta\nu|\gamma\rho}$ appears, we substitute $2\partial_{[\gamma}F_{\rho]\mu\nu}$ for it. We can then write $\alpha_J([\Phi^{i*}], [K], [F])$ instead of the unconvenient notation $\alpha_J([\Phi^{i*}], [K], \{F\})$.

Remark 2: Proposition 1 must be slightly modified in the special $n = 3$ case. As we said in the introduction, the curvature tensor K can be expressed in terms of the first partial derivatives of the Fronsdal tensor, see (4). Moreover, the ghost variable $\hat{U}^a_{\alpha\mu|\beta\nu}$ identically vanishes because it possesses the symmetry of the Weyl tensor. Thus, in dimension $n = 3$ we have

$$\gamma a = 0 \quad \Rightarrow \quad a = \alpha_J([\Phi^{i*}], [F])\omega^J(C^a_{\mu\nu}, \hat{T}^a_{\alpha\mu|\nu}) + \gamma b. \tag{6}$$

Another simplifying property in $n = 3$ is that the variable $\hat{T}^a_{\alpha\mu|\nu}$ can be replaced by its dual

$$\tilde{T}^{\ a}_{\alpha\beta} := \varepsilon^{\mu\nu}{}_{\alpha}\hat{T}^a_{\mu\nu|\beta} \qquad \left(\hat{T}^a_{\mu\nu|\rho} = -\frac{1}{2}\varepsilon_{\mu\nu}{}^{\alpha}\tilde{T}^a_{\alpha\rho}\right) \tag{7}$$

which is readily seen to be symmetric and traceless, as a consequence of the symmetries of $\hat{T}^a_{\alpha\mu|\nu}$;

$$\tilde{T}^{\ a}_{\alpha\beta} = \tilde{T}^a_{\beta\alpha}, \qquad \eta^{\alpha\beta}\tilde{T}^a_{\alpha\beta} = 0. \tag{8}$$

We now remind [8] the definition of a differential which plays an important role in the classification of the consistent first-order deformations of W_0.

Definition (differential D): The action of the differential D on $h^a_{\mu\nu\rho}$, $h_a^{*\mu\nu\rho}$, $C_a^{*\mu\nu}$, and all their derivatives is the same as the action of the total derivative d, but its action on the ghosts is given by

$$DC^a_{\mu\nu} = \tfrac{4}{3}dx^\alpha\hat{T}^a_{\alpha(\mu|\nu)}, \qquad D\hat{T}^a_{\mu\alpha|\beta} = dx^\rho\hat{U}^a_{\mu\alpha|\rho\beta},$$
$$D(\partial_{(\rho}C^a_{\mu\nu)}) = 0, \qquad D(\partial_{\rho_1\ldots\rho_t}C^a_{\mu\nu}) = 0 \quad \text{if} \quad t \geqslant 2. \tag{9}$$

The operator D coincides with d up to γ-exact terms. It follows from the definitions that $D\omega^J = A^J{}_I\omega^I$ for some constant matrix $A^J{}_I$ that involves dx^μ only.

The differential D is associated with a grading.

Definition (D-degree): The number of $\hat{T}_{\alpha\mu|\nu}$'s plus twice the number of $\hat{U}_{\alpha\mu|\beta\nu}$'s is called the D-degree. It is bounded because there is a finite number of $\hat{T}_{\alpha\mu|\nu}$'s and $\hat{U}_{\alpha\mu|\beta\nu}$'s, which are anticommuting.

The operator D splits as the sum of an operator D_1 that raises the D-degree by one unit, and an operator D_0 that leaves it unchanged. The differential operator D_0 has the same action as d on $h_{\mu\nu\rho}$, $h^{*\mu\nu\rho}$, $C^{*\alpha\beta}$ and all their

derivatives, and gives 0 when acting on the ghosts. The operator D_1 gives 0 when acting on all the variables but the ghosts on which it reproduces the action of D.

E. Cohomological group $H_2^n(\delta|d)$ and $H_k^n(\delta|d, H(\gamma))$, $k \geqslant 2$

We first mention a result proved in [22–24] and needed later.

Proposition 2—A complete set of representatives of $H_2^n(\delta|d)$ is given by the antifields $C_a^{*\mu\nu}$, up to explicitly x-dependent terms. In other terms,

$$\left.\begin{array}{c}\delta a_2^n + db_1^{n-1} = 0,\\ a_2^n \sim a_2^n + \delta c_3^n + dc_2^{n-1}\end{array}\right\} \iff$$

$$\left\{\begin{array}{c} a_2^n = L^a_{\mu\nu}(x) C_a^{*\mu\nu} d^n x + \delta b_3^n + db_2^{n-1},\\ L^a_{\mu\nu}(x) = \lambda^a_{\mu\nu} + P^a_{\mu\nu}(x).\end{array}\right.$$

The tensors $\lambda^a_{\mu\nu}$ and $P^a_{\mu\nu}$ are symmetric and traceless in the indices $(\mu\nu)$. The tensor $\lambda^a_{\mu\nu}$ is constant whereas the tensor $P^a_{\mu\nu}(x)$ depends on the coordinates x^μ explicitly.

Then, we recall that the cohomological groups $H_k^n(\delta|d)$ vanish for $k > 2$, which is a consequence of the fact that the free theory at hand is linear and possesses no reducibility (cf. Theorem 9.1 of [25]).

Finally, the following result is a cornerstone of the cohomological deformation procedure:

Proposition 3—For the free spin-3 gauge theory of Fronsdal, we have $H_k^n(\delta|d, H(\gamma)) \cong 0$, $k > 2$, and the nontrivial elements of $H_2^n(\delta|d, H(\gamma))$ are the same as for $H_2^n(\delta|d)$.

Essentially, using Proposition 2 and $H_k^n(\delta|d) \cong 0$ ($k > 2$), the proof of Proposition 3 goes as follows. It consists in showing that, if the invariant polynomial a_k^n ($k \geqslant 2$) is δ-trivial modulo d, $a_k^n = \delta\mu_{k+1}^n + d\nu_k^{n-1}$, then one can always choose μ_{k+1}^n and ν_k^{n-1} to be invariant.

Proposition 3 is proved in [8] for $n > 3$. We now demonstrate that it also holds for $n = 3$. Without recalling all the details, let us point out the place where, in Sec. 4.6.2 of [8], the proof of Proposition 3 must be adapted to $n = 3$. It is when one makes use of the projector on the symmetries of the Weyl tensor. In [8] we have the equations (4.45) and (4.46)

$$Y_{k+1}^{\mu\nu\rho} = \partial_\alpha\left[T_{k+1}^{\alpha\mu|\nu\rho} + \frac{1}{n-1}\eta^{\nu\rho}T_{k+1}^{\alpha|\mu}\right] + \delta(\ldots), \qquad \Leftrightarrow \partial_\alpha T_{k+1}^{\alpha\mu|\nu\rho} = Y_{k+1}^{\mu\nu\rho} - \frac{1}{n}\eta^{\nu\rho}\eta_{\alpha\beta}Y_{k+1}^{\mu\alpha\beta} + \delta(\ldots), \tag{10a}$$

$$T_{k+1}^{\alpha[\mu|\nu]\rho} + \frac{1}{n-1}T_{k+1}^{\alpha|[\mu}\eta^{\nu]\rho} = \partial_\beta S_{k+1}^{\beta\alpha|\mu\nu|\rho} + \delta(\ldots), \tag{10b}$$

which are instrumental in order to obtain

$$Y_{k+1}^{\mu\nu\rho} = \partial_\alpha\partial_\beta\partial_\gamma\Psi_{k+1}^{\alpha\mu|\beta\nu|\gamma\rho} + \mathcal{G}^{\mu\nu\rho}_{\alpha\beta\gamma}\hat{X}_{k+1}^{\alpha\beta\gamma} + \delta(\ldots), \tag{11}$$

where $Y_{k+1}^{\mu\nu\rho}$ is completely symmetric (the subscript denotes the antighost number), $T_{k+1}^{\alpha\mu|\nu\rho} = -T_{k+1}^{\mu\alpha|\nu\rho} = -T_{k+1}^{\mu\alpha|\rho\nu}$, $\eta_{\nu\rho}T_{k+1}^{\alpha\mu|\nu\rho} \equiv 0$, $S_{k+1}^{\beta\alpha|\mu\nu|\rho}$ is antisymmetric in $(\beta\alpha)$ and $(\mu\nu)$, satisfies $S_{k+1}^{\beta\alpha|[\mu\nu|\rho]} \equiv 0 \equiv \eta_{\nu\rho}S_{k+1}^{\beta\alpha|\mu\nu|\rho}$. The tensor $\Psi_{k+1}^{\alpha\mu|\beta\nu|\gamma\rho}$ possesses the symmetries of the curvature $K^{\alpha\mu|\beta\nu|\gamma\rho}$ and $\mathcal{G}^{\mu\nu\rho}{}_{\alpha\beta\gamma}$ is the second-order differential operator appearing in the equations of motion $0 \approx G^{\mu\nu\rho} = \mathcal{G}^{\mu\nu\rho}{}_{\alpha\beta\gamma}h^{\alpha\beta\gamma}$. Finally, we have

$$\hat{X}_{\alpha\beta\gamma|k+1} := \frac{2}{n-2}\mathcal{Y}^{\sigma\tau\rho}_{\alpha\beta\gamma}\Big(-S^\mu{}_{\sigma|\mu\tau|\rho k+1} + \frac{1}{n}\eta_{\sigma\tau}[S_{\mu\nu|}{}^{\mu\nu}{}_{|\rho k+1} + S_{\mu\nu|}{}^{\mu}{}_{\rho|}{}^{\nu}{}_{k+1}]\Big), \tag{12}$$

where $\mathcal{Y}^{\sigma\tau\rho}_{\alpha\beta\gamma} = \mathcal{Y}^{(\sigma\tau\rho)}_{(\alpha\beta\gamma)}$ projects on completely symmetric rank-3 tensors. The tensors $Y_{k+1}^{\mu\nu\rho}$, $T_{k+1}^{\alpha\mu|\nu\rho}$, $S_{k+1}^{\beta\alpha|\mu\nu|\rho}$, and $\Psi_{k+1}^{\alpha\mu|\beta\nu|\gamma\rho}$ are invariant.

In order to obtain the important Eq. (11), one had to project $\partial_\beta S_{k+1}^{\beta\alpha|\mu\nu|\rho}$ on the symmetries of the Weyl tensor [8]. In dimension 3, this gives zero identically.

If we denote by $W_{k+1}^{\beta|\mu\nu|\alpha\rho}$ the projection $\mathcal{W}^{\mu\nu\alpha\rho}_{\mu'\nu'\alpha'\rho'}S_{k+1}^{\beta\alpha'|\mu'\nu'|\rho'}$ of $S_{k+1}^{\beta\alpha|\mu\nu|\rho}$ on the symmetries of the Weyl tensor, we have of course $W_{k+1}^{\beta|\mu\nu|\alpha\rho} = 0$ because $W_{k+1}^{\beta|\mu\nu|\alpha\rho} \equiv 0$. Then, obviously

$$0 = \tfrac{2}{3}\partial_\alpha\partial_\beta[W_{k+1}^{\mu|\alpha\nu|\beta\rho} + W_{k+1}^{\mu|\alpha\rho|\beta\nu} + W_{k+1}^{\nu|\alpha\mu|\beta\rho} + W_{k+1}^{\nu|\alpha\rho|\beta\mu} + W_{k+1}^{\rho|\alpha\mu|\beta\nu} + W_{k+1}^{\rho|\alpha\nu|\beta\mu}].$$

Substituting for $W_{k+1}^{\mu|\alpha\nu|\beta\rho}$ its expression in terms of $S_{k+1}^{\alpha\beta|\gamma\delta|\rho}$ and using Eqs. (10) we find $0 = Y_{k+1}^{\mu\nu\rho} - \mathcal{G}^{\mu\nu\rho}{}_{\alpha\beta\gamma}\hat{X}_{k+1}^{\alpha\beta\gamma} + \delta(\ldots)$, where $\hat{X}_{k+1}^{\alpha\beta\gamma}$ is still given by (12). The result (11) is thus recovered except for the first Ψ-term. This is linked to the fact that, in $n = 3$, an invariant polynomial depends on the field $h_{\mu\nu\rho}$ only through the Fronsdal tensor $F^{\mu\nu\rho}$, see Eq. (4). In $n = 3$, the Eqs. (4.41) and (4.42) of [8] are changed accordingly. The proof then proceeds as in [8], where one sets Ψ to zero.

IV. PARITY-BREAKING DEFORMATIONS

In this section, we first compute all possible parity-breaking and Poincaré-invariant first-order deformations of the Abelian spin-3 gauge algebra. We find that such deformations exist in three and five dimensions. We then

proceed separately for $n = 3$ and $n = 5$. We analyze the corresponding first-order deformations of the quadratic Lagrangian and find that they both exist. Then, consistency conditions at second order are obtained which make the $n = 3$ deformation trivial and which constrain the $n = 5$ deformation to involve only one *single* gauge field.

A. Deformations of the gauge algebra

The following proposition is proved in [8]:

Proposition 4—Let a be a local topform that is a nontrivial solution of the cocycle equation $sa + db = 0$. Without loss of generality, one can assume that the antighost decomposition of a stops in antighost number two, i.e.

$$a = a_0 + a_1 + a_2.$$

Furthermore, in [8], the most general parity and Poincaré-invariant term a_2 is determined. Here, we allow for parity-breaking terms and prove that new terms a_2 appear when the number of dimensions is 3 and 5. This is summarized in the following theorem.

Theorem 1—Let $a = a_0 + a_1 + a_2$ be a local topform that is a nontrivial solution of the equation $sa + db = 0$. If the last term a_2 is parity-breaking and Poincaré-invariant, then it is trivial except in three and five dimensions. In those cases, modulo trivial terms, it can be written respectively

$$a_2 = f^a{}_{[bc]}\varepsilon^{\mu\nu\rho}C_a^{*\alpha\beta}C^b_{\mu\alpha}\partial_{[\nu}C^c_{\rho]|\beta}d^3x \tag{13}$$

and

$$a_2 = g^a{}_{(bc)}\varepsilon^{\mu\nu\rho\sigma\tau}C_a^{*\alpha}{}_\mu\partial_{[\nu}C^{b\beta}_{\rho]}\partial_{\alpha[\sigma}C^c_{\tau]\beta}d^5x. \tag{14}$$

The structure constants $f^a{}_{[bc]}$ define an internal, anticommutative algebra $\mathcal{A}$ while the structure constants $g^a{}_{(bc)}$ define an internal, commutative algebra $\mathcal{B}$.

Proof: The equation $sa + db = 0$, when $a = a_0 + a_1 + a_2$, is equivalent to the system

$$\gamma a_0 + \delta a_1 + db_0 = 0, \tag{15}$$

$$\gamma a_1 + \delta a_2 + db_1 = 0, \tag{16}$$

$$\gamma a_2 = 0. \tag{17}$$

The last equation implies that, modulo trivial terms, $a_2 = \alpha_I\omega^I$, where α_I is an invariant polynomial and the $\{\omega^I\}$ provide a basis of the polynomials in $C_{\mu\nu}$, $\hat{T}_{\mu\nu|\rho}$, $\hat{U}_{\mu\nu|\rho\sigma}$ (see Sec. III). Let us stress that, as a_2 has ghost number zero and antifield number two, ω^I must have ghost number two. Then, acting with γ on (16) and using the triviality of d, one gets that b_1 should also be an element of $H(\gamma)$, i.e., modulo trivial terms, $b_1 = \beta_I\omega^I$, where the β_I are invariant polynomials.

We further expand a_2 and b_1 according to the D-degree defined in Sec. III D:

$$a_2 = \sum_{i=0}^{M} a_2^i = \sum_{i=0}^{M}\alpha_{I_i}\omega^{I_i}, \qquad b_1 = \sum_{i=0}^{M} b_1^i = \sum_{i=0}^{M}\beta_{I_i}\omega^{I_i},$$

where a_2^i, b_1^i, and ω^{I_i} have D-degree i. The Eq. (16) then reads

$$\sum_i \delta[\alpha_{I_i}\omega^{I_i}] + \sum_i D[\beta_{I_i}\omega^{I_i}] = \gamma(\ldots),$$

or equivalently

$$\sum_i \delta[\alpha_{I_i}]\omega^{I_i} + \sum_i D_0[\beta_{I_i}]\omega^{I_i} \pm \sum_i \beta_{I_i}A^{I_i}_{I_{i+1}}\omega^{I_{i+1}} = \gamma(\ldots),$$

where $A^{I_i}_{I_{i+1}}\omega^{I_{i+1}} = D\omega^{I_i}$, which implies

$$\delta\alpha_{I_i} + d\beta_{I_i} \pm \beta_{I_{i-1}}A^{I_{i-1}}_{I_i} = 0 \tag{18}$$

for each D-degree i, as the elements of the set $\{\omega^I\}$ are linearly independent nontrivial elements of $H(\gamma)$.

The next step is to analyze the Eq. (18) for each D-degree. The results depend on the dimension, so we split the analysis into the cases $n = 3$, $n = 4$, $n = 5$, and $n > 5$.

Dimension 3

(a) *degree zero*: In D-degree 0, the Eq. (18) reads $\delta\alpha_{I_0} + d\beta_{I_0} = 0$, which implies that α_{I_0} belongs to $H_2(\delta|d)$. In antifield number 2, this group has nontrivial elements given by Proposition 2, which are proportional to $C_a^{*\mu\nu}$. The requirement of translation-invar iance restricts the coefficient of $C_a^{*\mu\nu}$ to be constant. On the other hand, in D-degree 0 and ghost number 2, we have $\omega^{I_0} = C^b_{\mu\rho}C^c_{\nu\sigma}$. To get a parity-breaking but Lorentz-invariant a_2^0, a scalar quantity must be build by contracting ω^{I_0}, $C_a^{*\mu\nu}$, the tensor $\varepsilon^{\mu\nu\rho}$ and a product of $\eta_{\mu\nu}$'s. This cannot be done because there is an odd number of indices, so a_2^0 vanishes: $a_2^0 = 0$. One can then also choose $b_1^0 = 0$.

(b) *degree one*: We now analyze Eq. (18) in D-degree 1. It reads $\delta\alpha_{I_1} + d\beta_{I_1} = 0$ and implies that α_{I_1} is an element of $H_2(\delta|d)$. Therefore the only parity-breaking and Poincaré-invariant a_2^1 that can be built is $a_2^1 = f^a{}_{bc}\varepsilon^{\mu\nu\rho}C_a^{*\alpha\beta}C^b_{\alpha\mu}T^c_{\nu\rho|\beta}d^3x$. Indeed, it should have the structure $\varepsilon C^*C\hat{T}$ (or εC^*CT, up to trivial terms), contracted with η's. In an equivalent way, it must have the structure $C^*C\tilde{T}$, contracted with η's, where the variable $\tilde{T}$ was introduced in Eq. (7). Because of the symmetry properties (8) of $\tilde{T}$ which are the same as the symmetries of $C^a_{\mu\nu}$ and $C_a^{*\mu\nu}$, there is only one way of contracting $\tilde{T}$, C, and C^* together: $f^a{}_{bc}C_a^{*\mu\nu}C^b_\mu{}^\rho\tilde{T}^c_{\nu\rho}$. Of course, no Schouten identity (see the appendix) can come into play because of the number and the symmetry of the fields compos-

ing $f^a{}_{bc}C^{*\mu\nu}_a C^b_\mu{}^\rho \tilde{T}^c_{\nu\rho}$. The latter term is proportional to $a^1_2 = f^a{}_{bc}\varepsilon^{\mu\nu\rho}C^{*\alpha\beta}_a C^b_{\alpha\mu}T^c_{\nu\rho|\beta}d^3x$, up to trivial terms. One can now easily compute that $b^1_1 = -3f^a{}_{bc}\varepsilon^{\mu\nu\rho}(h^{*\alpha\beta\lambda}_a - \frac{1}{3}\eta^{\alpha\beta}h^{*\lambda}_a)C^b_{\alpha\mu}\hat{T}^c_{\nu\rho|\beta} \times \frac{1}{2}\varepsilon_{\lambda\sigma\tau}dx^\sigma dx^\tau$.

(c) *degree two*: The Eq. (18) in D-degree 2 is $\delta\alpha_{I_2} + d\beta_{I_2} - \beta_{I_1}A^{I_1}_{I_2} = 0$, with

$$\begin{aligned}-\beta_{I_1}A^{I_1}_{I_2}\omega^{I_2} &= 3f^a{}_{bc}\varepsilon^{\mu\nu\rho}(h^{*\alpha\beta\lambda}_a - \tfrac{1}{3}\eta^{\alpha\beta}h^{*\lambda}_a)\\ &\quad\times(\tfrac{4}{3}\hat{T}^b_{\eta(\alpha|\mu)}\hat{T}^c_{\nu\rho|\beta})\tfrac{1}{2}\varepsilon_{\lambda\sigma\tau}dx^\eta dx^\sigma dx^\tau\\ &= 2f^a{}_{(bc)}\varepsilon^{\mu\nu\rho}(h^{*\alpha\beta\lambda}_a\\ &\quad - \tfrac{2}{3}\eta^{\alpha\beta}h^{*\lambda}_a)\hat{T}^b_{\lambda\mu|\alpha}\hat{T}^c_{\nu\rho|\beta}d^3x.\end{aligned}$$

The latter equality holds up to irrelevant trivial γ-exact terms. It is obtained by using the fact that there are only two linearly independent scalars having the structure $\varepsilon h^*\hat{T}\hat{T}$. They are $\varepsilon^{\mu\nu\rho}h^{*\alpha\beta\gamma}\hat{T}_{\mu\nu|\alpha}\hat{T}_{\rho\beta|\gamma}$ and $\varepsilon^{\mu\nu\rho}h^{*\alpha}\hat{T}_{\mu\nu|}{}^\beta\hat{T}_{\rho\beta|\alpha}$. To prove this, it is again easier to use the dual variable $\tilde{T}$ instead of $\hat{T}$. One finds that the linearly independent terms with the structure $\varepsilon h^*\tilde{T}\tilde{T}$ are $f^a{}_{(bc)}\varepsilon^{\mu\nu\rho}h^{*a}_\mu\tilde{T}^{b\alpha}_\nu\tilde{T}^c_{\rho\alpha}$ and $f^a{}_{(bc)}\varepsilon^{\mu\nu\rho}h^{*a\alpha\beta}_\mu\tilde{T}^b_{\nu\alpha}\tilde{T}^c_{\rho\beta}$; they are proportional to $f^a{}_{(bc)}\varepsilon^{\mu\nu\rho}h^{*\alpha\beta\gamma}_a\hat{T}^b_{\mu\nu|\alpha}\hat{T}^c_{\rho\beta|\gamma}$ and $f^a{}_{(bc)}\varepsilon^{\mu\nu\rho}h^{*\alpha}_a\hat{T}^b_{\mu\nu|}{}^\beta\hat{T}^c_{\rho\beta|\alpha}$.

Since the expression for $\beta_{I_1}A^{I_1}_{I_2}$ is not δ-exact modulo d it must vanish: $f_{abc} = f_{a[bc]}$. One then gets that α_{I_2} belongs to $H_2(\delta|d)$. However, no such parity-breaking and Poincaré-invariant a^2_2 can be formed in D-degree 2, so $a^2_2 = 0 = b^2_1$.

(d) *degree* >2: Finally, there are no a^i_2 for $i > 2$. Indeed, there is no ghost combination ω^{I_i} of ghost number two and D-degree higher than two, because $\hat{U}$ identically vanishes when $n = 3$.

Dimension 4

There is no nontrivial deformation of the gauge algebra in dimension 4.

(a) *degree zero*: Eq. (18) reads $\delta\alpha_{I_0} + d\beta_{I_0} = 0$. It implies that α_{I_0} belongs to $H^4_2(\delta|d)$, which means that α_{I_0} is of the form $k^a{}_{bc}\varepsilon^{\mu\nu\rho\sigma}C^{*\alpha\beta}_a d^4x$ where $k^a{}_{bc}$ are some constants. It is obvious that all contractions of α_{I_0} with two undifferentiated ghosts C in a Lorentz-invariant way identically vanish. One can thus choose $a^0_2 = 0$ and $b^0_1 = 0$.

(b) *degree one*: The equation in D-degree 1 reads $\delta\alpha_{I_1} + d\beta_{I_1} = 0$. The nontrivial part of α_{I_1} has the same form as in D-degree 0. It is however impossible to build a nontrivial Lorentz-invariant a^1_2 because $\omega_{I_1} \sim CT$ has an odd number of indices. So $a^1_2 = 0$ and $b^1_1 = 0$.

(c) *degree two*: In D-degree 2, the equation $\delta\alpha_{I_2} + d\beta_{I_2} = 0$ must be studied. Once again, one has $\alpha_{I_2} = k^a{}_{bc}\varepsilon^{\mu\nu\rho\sigma}C^{*\alpha\beta}_a d^4x$. There are two sets of ω_{I_2}'s : $\hat{T}^b_{\mu\nu|\alpha}\hat{T}^c_{\rho\sigma|\beta}$ and $C^b_{\alpha\beta}\hat{U}^c_{\mu\nu|\rho\sigma}$. *A priori* there are three different ways to contract the indices of terms with the structure $\varepsilon C^*\hat{T}\hat{T}$, but because of Schouten identities (see Appendix A 1) only two of them are independent, with some symmetry constraints for the structure functions. No Schouten identities exist for terms with the structure $\varepsilon C^*C\hat{U}$. The general form of a^2_2 is thus, modulo trivial terms,

$$\begin{aligned}a^2_2 &= \overset{(1)}{k}{}^a{}_{[bc]}\varepsilon^{\mu\nu\rho\sigma}C^{*\alpha\beta}_a\hat{T}^b_{\mu\nu|\alpha}\hat{T}^c_{\rho\sigma|\beta}d^4x\\ &\quad + \overset{(2)}{k}{}^a{}_{(bc)}\varepsilon^{\mu\nu\rho\sigma}C^{*\alpha}_a{}_\mu\hat{T}^b_{\nu\rho|}{}^\beta\hat{T}^c_{\sigma\alpha|\beta}d^4x\\ &\quad + \overset{(3)}{k}{}^a{}_{bc}\varepsilon^{\mu\nu\rho\sigma}C^*_{a\mu\alpha}C^b_{\nu\beta}\hat{U}^c_{\rho\sigma|}{}^{\alpha\beta}d^4x,\end{aligned}$$

and b^2_1 is given by

$$\begin{aligned}b^2_1 = -3\varepsilon^{\mu\nu\rho\sigma}\Big[&\Big(h^{*\lambda\alpha\beta}_a - \frac{1}{4}h^{*\lambda}_a\eta^{\alpha\beta}\Big)\overset{(1)}{k}{}^a{}_{[bc]}\hat{T}^b_{\mu\nu|\alpha}\hat{T}^c_{\rho\sigma|\beta}\\ &+\Big(h^{*\lambda\alpha}_a{}_\mu - \frac{1}{4}h^{*\lambda}_a\delta^\alpha_\mu\Big)\overset{(2)}{k}{}^a{}_{(bc)}\hat{T}^b_{\nu\rho|}{}^\beta\hat{T}^c_{\sigma\alpha|\beta}\\ &+\overset{(3)}{k}{}^a{}_{bc}C^b_{\nu\beta}\hat{U}^c_{\rho\sigma|\alpha}{}^\beta)\Big]\frac{1}{3!}\varepsilon_{\lambda\rho\sigma\tau}dx^\rho dx^\sigma dx^\tau.\end{aligned}$$

(d) *degree three*: Eq. (18) now reads $\delta\alpha_{I_3} + d\beta_{I_3} + \beta_{I_2}A^{I_2}_{I_3} = 0$, with

$$\begin{aligned}\beta_{I_2}A^{I_2}_{I_3}\omega^{I_3} &= -\tfrac{3}{2}\overset{(1)}{k}{}^a{}_{[bc]}\varepsilon^{\mu\nu\rho\sigma}h^{*\lambda}_a\hat{T}^b_{\mu\nu|}{}^\alpha\hat{U}^c_{\lambda\alpha|\rho\sigma}d^4x\\ &\quad - 3\overset{(2)}{k}{}^a{}_{(bc)}\varepsilon^{\mu\nu\rho\sigma}h^{*\alpha\lambda}_a{}_\mu(\hat{T}^b_{\nu\alpha|}{}^\beta\hat{U}^c_{\lambda\beta|\rho\sigma}\\ &\quad - \hat{T}^b_{\nu\rho|}{}^\beta\hat{U}^c_{\lambda\beta|\sigma\alpha})d^4x\\ &\quad + 4\overset{(3)}{k}{}^a{}_{bc}\varepsilon^{\mu\nu\rho\sigma}h^{*\alpha\lambda}_a{}_\mu\hat{T}^b_{\lambda(\beta|\nu)}\hat{U}^c_{\rho\sigma|\alpha}{}^\beta d^4x\\ &= (-\tfrac{3}{2}(\overset{(1)}{k}{}^a{}_{[bc]} + \overset{(2)}{k}{}^a{}_{(bc)})\varepsilon^{\beta\gamma\rho\sigma}h^{*\mu}_a\eta^{\alpha\nu}\\ &\quad - (6\overset{(2)}{k}{}^a{}_{(bc)} + 4\overset{(3)}{k}{}^a{}_{bc})\varepsilon^{\mu\nu\lambda\beta}h^{*\gamma\rho}_a{}_\lambda\eta^{\alpha\sigma})\\ &\quad\times\hat{T}^b_{\beta\gamma|\alpha}\hat{U}^c_{\mu\nu|\rho\sigma}d^4x.\end{aligned}$$

The latter equality is obtained using Schouten identities (see Appendix A 2). It is obvious that the coefficient of $\omega^{I_3} = \hat{T}^b_{\beta\gamma|\alpha}\hat{U}^c_{\mu\nu|\rho\sigma}$ cannot be δ-exact modulo d unless it is zero. This implies that $\overset{(1)}{k}{}^a{}_{[bc]} = \overset{(2)}{k}{}^a{}_{(bc)} = \overset{(3)}{k}{}^a{}_{bc} = 0$. So a^2_2 is trivial and can be set to zero, as well as b^2_1. One now has $\delta\alpha_{I_3} + d\beta_{I_3} = 0$, which has the usual solution for α_{I_3}, but there is no nontrivial Lorentz-invariant a^3_2 because there is an odd number of indices to be contracted.

(e) *degree* ≥ 4: Eq. (18) is $\delta\alpha_{I_4} + d\beta_{I_4} = 0$, thus α_{I_4} is of the form $l^a{}_{bc}\varepsilon^{\mu\nu\rho\sigma}C^{*\alpha\beta}_a d^4x$. There are two different ways to contract the indices: $\varepsilon^{\mu\nu\rho\sigma}C^{*\alpha\beta}_a \times \hat{U}^b_{\mu\nu|\alpha\gamma}\hat{U}^c_{\rho\sigma|\beta}{}^{\gamma}$ and $\varepsilon^{\mu\nu\rho\sigma}C^*_{a\mu\alpha}\hat{U}^b_{\nu\rho|\beta\gamma}\hat{U}^{c\alpha|\beta\gamma}_{\sigma}$, but both functions vanish because of Schouten identities (see Appendix A 3). Thus $a^4_2 = 0$ and $b^4_2 = 0$. No candidates a^i_2 of ghost number two exist in D-degree higher than four because there is no appropriate ω^{I_i}.

Dimension 5

(a) *degree zero*: In D-degree 0, the Eq. (18) reads $\delta\alpha_{I_0} + d\beta_{I_0} = 0$ which means that α_{I_0} belongs to $H^5_2(\delta|d)$. However, a^0_2 cannot be build with such an α_{I_0} because the latter has an odd number of indices while ω^{I_0} has an even one. So, α_{I_0} and β_{I_0} can be chosen to vanish.

(b) *degree one*: In D-degree 1, the equation becomes $\delta\alpha_{I_1} + d\beta_{I_1} = 0$, so α_{I_1} belongs to $H^5_2(\delta|d)$. However, it is impossible to build a nonvanishing Lorentz-invariant a^1_2 because in a product $C^*C\hat{T}$ there are not enough indices that can be antisymmetrized to be contracted with the Levi-Civita density. So α_{I_1} and β_{I_1} can be set to zero.

(c) *degree two*: The Eq. (18) reads $\delta\alpha_{I_2} + d\beta_{I_2} = 0$. Once again, there is no way to build a Lorentz-invariant a^2_2 because of the odd number of indices. So $\alpha_{I_2} = 0$ and $\beta_{I_2} = 0$.

(d) *degree three*: In D-degree 3, the equation is $\delta\alpha_{I_3} + d\beta_{I_3} = 0$, so $\alpha_{I_3} \in H^5_2(\delta|d)$. This gives rise to an a_2 of the form "$g\varepsilon C^*\hat{T}\,\hat{U}\,d^5x$." There is only one nontrivial Lorentz-invariant object of this form: $a_2 = g^a{}_{bc}\varepsilon^{\mu\nu\rho\sigma\tau}C^*_{a\mu\alpha}\hat{T}^b_{\nu\rho|\beta}\hat{U}^{c\alpha\beta|}{}_{\sigma\tau}d^5x$. It is equal to (14) modulo a γ-exact term. One has

$$b^3_1 = \beta_{I_3}\omega^{I_3} = -3g^a{}_{bc}\varepsilon^{\mu\nu\rho\sigma\tau}\left(h^*_{a\mu\alpha}{}^{\lambda} - \frac{1}{5}\eta_{\mu\alpha}h^{*\lambda}_a\right) \times \hat{T}^b_{\nu\rho|\beta}\hat{U}^{c\alpha\beta}{}_{|\sigma\tau}\frac{1}{4!}\varepsilon_{\lambda\gamma\delta\eta\xi}dx^\gamma dx^\delta dx^\eta dx^\xi.$$

(e) *degree four*: The Eq. (18) reads $\delta\alpha_{I_4} + d\beta_{I_4} - \beta_{I_3}A^{I_3}_{I_4} = 0$, with

$$\beta_{I_3}A^{I_3}_{I_4}\omega^{I_4} = -3g^a{}_{[bc]}\varepsilon^{\mu\nu\rho\sigma\tau}h^{*\alpha\lambda}_a{}_{\mu}\hat{U}^b_{\lambda\beta|\nu\rho}\hat{U}^{c\beta}_{\alpha}{}_{|\sigma\tau}d^5x.$$

The coefficient of $\omega^{I_4} \sim \hat{U}\,\hat{U}$ cannot be δ-exact modulo d unless it vanishes, which implies that $g^a{}_{bc} = g^a{}_{(bc)}$. One is left with the equation $\delta\alpha_{I_4} + d\beta_{I_4} = 0$, but once again it has no Lorentz-invariant solution because of the odd number of indices to be contracted. So $\alpha_{I_4} = 0$ and $\beta_{I_4} = 0$.

(f) *degree higher than four*: There is again no a^i_2 for $i > 4$, for the same reasons as in four dimensions.

Dimension n > 5

No new a_2 arises because it is impossible to build a nonvanishing parity-breaking term by contracting an element of $H^5_2(\delta|d)$, i.e. $C^{*\mu\nu}$, two ghosts from the set $\{C^{\mu\nu}, \hat{T}^{\mu\nu|\rho}, \hat{U}^{\mu\nu|\rho\sigma}\}$, an epsilon-tensor $\varepsilon^{\mu_1\cdots\mu_n}$ and metrics $\eta_{\mu\nu}$.

Let us finally notice that throughout this proof we have acted as if α_I's trivial in $H^n_2(\delta|d)$ lead to trivial a_2's. The correct statement is that trivial a_2's correspond to α_I's trivial in $H^n_2(\delta|d, H(\gamma))$ (see e.g. [9] for more details). However, both statements are equivalent in this case, since both groups are isomorphic (Proposition 3).

This ends the proof of Theorem 1.

B. Deformation in 3 dimensions

In the previous section, we determined that the only nontrivial first-order deformation of the free theory in three dimensions deforms the gauge algebra by the term (13). We now check that this deformation can be consistently lifted and leads to a consistent first-order deformation of the Lagrangian. However, we then show that obstructions arise at second order, i.e. that one cannot construct a corresponding consistent second-order deformation unless the whole deformation vanishes.

1. First-order deformation

A consistent first-order deformation exists if one can solve Eq. (15) for a_0, where a_1 is obtained from Eq. (16). The existence of a solution a_1 to Eq. (16) with $a_2 = a^1_2$ is a consequence of the analysis of the previous section. Indeed, the a_2's of Theorem 1 are those that admit an a_1 in Eq. (16). Explicitely, a_1 reads, modulo trivial terms,

$$a_1 = f^a{}_{[bc]}\varepsilon^{\mu\nu\rho}[3(h^{*\alpha\beta\lambda}_a - \tfrac{1}{3}\eta^{\alpha\beta}h^{*\lambda}_a)(\tfrac{1}{3}h^b_{\alpha\mu\lambda}T^c_{\nu\rho|\beta} + \tfrac{1}{2}C^b_{\alpha\mu}\partial_{[\rho}h^c_{\nu]\beta\lambda}) + \tfrac{1}{3}h^{*\lambda}_a T^b_{\lambda\nu|\mu}h^c_\rho + h^*_{a\mu}C^{b\alpha}_{\nu}(-\tfrac{1}{2}\partial^\lambda h^c_{\lambda\alpha\rho} + \partial_{(\alpha}h^c_{\rho)})]d^3x.$$

On the contrary, a new condition has to be imposed on the structure function for the existence of an a_0 satisfying Eq. (15). Indeed a necessary condition for a_0 to exist is that $\delta_{ad}f^d{}_{[bc]} = f_{[abc]}$, which means that the corresponding internal anticommutative algebra $\mathcal{A}$ is endowed with an invariant norm. The internal metric we use is δ_{ab}, which is positive-definite.

The condition is also sufficient and a_0 reads, modulo trivial terms,

$$
\begin{aligned}
a_0 = f_{[abc]}\varepsilon^{\mu\nu\rho}[&\tfrac{1}{4}\partial_\mu h^a_{\nu\alpha\beta}\partial^\alpha h^{b\beta}h^c_\rho + \tfrac{1}{4}\partial_\mu h^a_{\nu\alpha\beta}\partial^\alpha h^{b\beta\gamma\delta}h^c_{\rho\gamma\delta} - \tfrac{5}{4}\partial_\mu h^a_{\nu\alpha\beta}\partial^\alpha h^{b\gamma}h^c_{\rho\gamma}{}^\beta - \tfrac{3}{8}\partial_\mu h^a_\nu\partial^\alpha h^b_\alpha h^c_\rho \\
&+ \tfrac{1}{4}\partial_\mu h^{a\alpha\beta}_\nu\partial^\gamma h^b_\gamma h^c_{\rho\alpha\beta} - \partial_\mu h^a_\nu\partial^\gamma h^b_{\alpha\beta\gamma}h^{c\alpha\beta}_\rho + \tfrac{1}{2}\partial_\mu h^a_{\nu\alpha\beta}\partial^\gamma h^b_{\alpha\gamma\delta}h^{c\beta\delta}_\rho + 2\partial_\mu h^a_\nu\partial^\beta h^{b\gamma}h^{c\beta\gamma}_\rho - \tfrac{1}{4}\partial_\mu h^a_{\nu\alpha\beta}\partial^\gamma h^{b\alpha\beta\delta}h^c_{\rho\gamma\delta} \\
&- \tfrac{1}{4}\partial_\mu h^a_{\nu\alpha\beta}\partial^\gamma h^{b\beta}h^c_{\rho\gamma}{}^\alpha - \tfrac{5}{8}\partial_\mu h^a_\nu\partial_\rho h^{b\beta}h^c_\beta + \tfrac{7}{8}\partial_\mu h^a_{\nu\alpha\beta}\partial_\rho h^{b\alpha\beta\gamma}h^c_\gamma + \tfrac{1}{4}\partial_\mu h^a_{\nu\alpha\beta}\partial_\gamma h^{b\alpha\gamma}_\rho h^{c\beta} + \tfrac{1}{4}\partial_\mu h^a_\nu\partial^\alpha h^b_{\rho\alpha\beta}h^{c\beta} \\
&- \tfrac{1}{4}\partial_\mu h^a_{\nu\alpha\beta}\partial^\gamma h^b_{\rho\gamma\delta}h^{c\alpha\beta\delta} - \tfrac{1}{8}\partial_\mu h^a_\nu\partial^\alpha h^b_\rho h^c_\alpha - \tfrac{1}{8}\partial_\mu h^a_{\nu\alpha\beta}\partial^\gamma h^{b\alpha\beta}_\rho h^c_\gamma]d^3x.
\end{aligned}
$$

To prove these statements about a_0, one writes the most general a_0 with two derivatives, that is Poincaré-invariant but breaks the parity symmetry. One inserts this a_0 into the equation to solve, i.e. $\delta a_1 + \gamma a_0 = db_0$, and computes the δ and γ operations. One takes an Euler-Lagrange derivative of the equation with respect to the ghost, which removes the total derivative db_0. The equation becomes $\frac{\delta}{\delta C_{\alpha\beta}}(\delta a_1 + \gamma a_0) = 0$, which we multiply by $C_{\alpha\beta}$. The terms of the equation have the structure $\varepsilon C\partial^3 hh$ or $\varepsilon C\partial^2 h\partial h$. One expresses them as linear combinations of a set of linearly independent quantities, which is not obvious as there are Schouten identities relating them (see Appendix A 4). One can finally solve the equation for the arbitrary coefficients in a_0, yielding the above results.

2. Second-order deformation

Once the first-order deformation $W_1 = \int(a_0 + a_1 + a_2)$ of the free theory is determined, the next step is to investigate whether a corresponding second-order deformation W_2 exists. This second-order deformation of the master equation is constrained to obey $sW_2 = -\frac{1}{2}(W_1, W_1)_{a.b.}$ (see Sec. III C). Expanding both sides according to the antighost number yields several conditions. The maximal antighost number condition reads

$$
-\tfrac{1}{2}(a_2, a_2)_{a.b.} = \gamma c_2 + \delta c_3 + df_2,
$$

where $W_2 = \int d^3x(c_0 + c_1 + c_2 + c_3)$ and $antigh(c_i) = i$. It is easy to see that the expansion of W_2 can be assumed to stop at antighost number 3 and that c_3 may be assumed to be invariant. The calculation of $(a_2, a_2)_{a.b.}$, where $a_2 = f^a{}_{[bc]}\varepsilon^{\mu\nu\rho}C^{*\alpha\beta}_a C^b_{\mu\alpha}\partial_\nu C^c_{\rho\beta}$, gives

$$
\begin{aligned}
(a_2, a_2)_{a.b.} &= 2\frac{\delta^R a_2}{\delta C^{*\alpha\beta}_a}\frac{\delta^L a_2}{\delta C^a_{\alpha\beta}} \\
&= \gamma\mu + d\nu + 2f^a{}_{bc}f_{ead}\varepsilon^{\mu\nu\rho}\varepsilon_{\alpha\lambda\tau}\Big[\frac{1}{2}C^{*e\sigma\xi}C^{b\alpha}_\mu\hat{T}^c_{\nu\rho|\sigma}\hat{T}^{d\lambda\tau|}{}_\xi + \frac{1}{2}C^{*e\sigma\xi}C^b_{\mu\sigma}\hat{T}^c_{\nu\rho|}{}^\alpha\hat{T}^{d\lambda\tau|}{}_\xi \\
&\quad - \frac{1}{3}C^{*e\alpha\xi}C^{b\sigma}_\mu\hat{T}^c_{\nu\rho|\sigma}\hat{T}^{d\lambda\tau|}{}_\xi - \frac{2}{3}C^{*e\sigma\xi}\hat{T}^{b\lambda}{}_{(\mu|\alpha)}\hat{T}^c_{\nu\rho|\sigma}C^{d\tau}_\xi - \frac{2}{3}C^{*e\sigma\xi}\hat{T}^{b\lambda}{}_{(\mu|\sigma)}\hat{T}^c_{\nu\rho|}{}^\alpha C^{d\tau}_\xi \\
&\quad + \frac{4}{9}C^{*e\alpha\xi}\hat{T}^{b\lambda}{}_{(\mu|\sigma)}\hat{T}^c_{\nu\rho|}{}^\sigma C^{d\tau}_\xi\Big].
\end{aligned}
\tag{19}
$$

It is impossible to get an expression with three ghosts, one C^* and no field, by acting with δ on c_3. We can thus assume without loss of generality that c_3 vanishes, which implies that $(a_2, a_2)_{a.b.}$ should be γ-exact modulo total derivatives.

The use of the variable $\tilde{T}_{\alpha\beta} := \varepsilon^{\mu\nu}{}_\alpha\hat{T}_{\mu\nu|\beta}$ in place of $\hat{T}_{\mu\nu|\rho}(= -\frac{1}{2}\varepsilon^\alpha{}_{\mu\nu}\tilde{T}_{\alpha\rho})$ simplifies the calculations. We find, after expanding the products of ε-densities,

$$
\begin{aligned}
(a_2, a_2)_{a.b.} = \gamma\mu + d\nu + f^a{}_{bc}f_{ead}C^{*e\sigma\tau}[&C^{b\mu\alpha}\tilde{T}^c_{\mu\sigma}\tilde{T}^d_{\alpha\tau} \\
&+ C^{b\mu}_\sigma\tilde{T}^c_{\mu\alpha}\tilde{T}^{d\alpha}_\tau - \tfrac{2}{3}C^{b\mu\alpha}\tilde{T}^c_{\mu\alpha}\tilde{T}^d_{\sigma\tau} \\
&+ C^{d\mu}_\sigma\tilde{T}^b_{\mu\alpha}\tilde{T}^{c\alpha}_\tau - \tfrac{1}{3}C^d_{\sigma\tau}\tilde{T}^{b\alpha\mu}\tilde{T}^c_{\alpha\mu}].
\end{aligned}
\tag{20}
$$

We then use the only possible Schouten identity

$$
\begin{aligned}
0 &\equiv C^{*e\tau}_{[\sigma}C^{b\mu}_\alpha\tilde{T}^{c\sigma}_\mu\tilde{T}^{d\alpha}_{\tau]} \\
&= \tfrac{1}{24}[-C^{*e\sigma\tau}C^{b\mu\alpha}\tilde{T}^c_{\sigma\tau}\tilde{T}^d_{\mu\alpha} + 2C^{*e\sigma\tau}C^{b\mu\alpha}\tilde{T}^c_{\sigma\mu}\tilde{T}^d_{\alpha\tau} \\
&\quad + 2C^{*e\sigma\tau}C^b_{\sigma\mu}\tilde{T}^c_{\tau\alpha}\tilde{T}^{d\alpha\mu} - C^{*e\sigma\tau}C^b_{\sigma\tau}\tilde{T}^c_{\mu\nu}\tilde{T}^{d\mu\nu} \\
&\quad - C^{*e\sigma\tau}C^{b\mu\nu}\tilde{T}^c_{\mu\nu}\tilde{T}^d_{\sigma\tau} + 2C^{*e\sigma\tau}C^{b\mu}_\sigma\tilde{T}^c_{\mu\alpha}\tilde{T}^{d\alpha}_\tau]
\end{aligned}
\tag{21}
$$

in order to substitute in Eq. (20) the expression of $C^{*e\sigma\tau}C^{b\mu\alpha}\tilde{T}^c_{\mu\sigma}\tilde{T}^d_{\alpha\tau}$ in terms of the other summands appearing in Eq. (21). Consequently, the following expression for $(a_2, a_2)_{a.b.}$ contains only linearly independent terms:

$$
\begin{aligned}
(a_2, a_2)_{a.b.} = \gamma\mu + d\nu + C^{*e\sigma\tau}[&\tfrac{1}{2} f^a{}_{bc} f_{dea} C^{b\mu\alpha} \tilde{T}^c_{\sigma\tau} \tilde{T}^d_{\mu\alpha} \\
&+ \tfrac{1}{6} f^a{}_{bc} f_{dea} C^{b\mu\alpha} \tilde{T}^d_{\sigma\tau} \tilde{T}^c_{\mu\alpha} \\
&+ 2 f^a{}_{c(b} f_{d)ea} C^{b\mu}_{\sigma} \tilde{T}^c_{\tau\alpha} \tilde{T}^{d\alpha}{}_{\mu} \\
&+ \tfrac{1}{2} f^a{}_{b[c} f_{d]ea} C^b_{\sigma\tau} \tilde{T}^c_{\mu\alpha} \tilde{T}^{d\mu\alpha} \\
&+ \tfrac{1}{3} f^a{}_{bc} f_{dea} C^d_{\sigma\tau} \tilde{T}^c_{\mu\alpha} \tilde{T}^{b\mu\alpha}],
\end{aligned}
$$

where we used that the structure constants of $\mathcal{A}$ obey $f_{abc} \equiv \delta_{ad} f^d{}_{bc} = f_{[abc]}$.

Therefore, the above expression is a γ-cobounday modulo d if and only if $f^a{}_{bc} f_{dea} = 0$, meaning that the internal algebra $\mathcal{A}$ is nilpotent of order three. In turn, this implies[1] that $f^a{}_{bc} = 0$ and the deformation is trivial.

C. Deformation in 5 dimensions

Let us perform the same analysis for the candidate in five dimensions.

1. First-order deformation

First, a_1 must be computed from a_2 (given by (14)), using the equation $\delta a_2 + \gamma a_1 + d b_1 = 0$:

$$
\begin{aligned}
\delta a_2 &= -3 g^a{}_{(bc)} \varepsilon^{\mu\nu\rho\sigma\tau} \partial_\lambda h^{*\alpha\lambda}_a{}_\mu \partial_{[\nu} C^b_{\rho]\beta} \partial_{\alpha[\sigma} C^{c\beta}_{\tau]} d^5x \\
&= -d b_1 + 3 g^a{}_{(bc)} \varepsilon^{\mu\nu\rho\sigma\tau} h^{*\alpha\lambda}_a{}_\mu [\partial_{\lambda[\nu} C^b_{\rho]\beta} \partial_{\alpha[\sigma} C^{c\beta}_{\tau]} \\
&\quad + \partial_{[\nu} C^b_{\rho]\beta} \partial_{\lambda\alpha[\sigma} C^{c\beta}_{\tau]}] d^5x.
\end{aligned}
$$

We recall that it is a consequence of Theorem 1 that $g^a{}_{bc}$ is symmetric in its lower indices, thereby defining a commutative algebra. It is not an assumption, it comes from consistency. Therefore the first term between square bracket vanishes because of the symmetries of the structure constants $g^a{}_{bc}$ of the internal commutative algebra $\mathcal{B}$. We finally obtain, modulo trivial terms,

$$
\begin{aligned}
a_1 = \tfrac{3}{2} g^a{}_{(bc)} \varepsilon^{\mu\nu\rho\sigma\tau} h^{*\alpha\lambda}_a{}_\mu \partial_{[\nu} C^b_{\rho]}{}^\beta [&\partial_{\beta[\sigma} h^c_{\tau]\lambda\alpha} \\
&- 2\partial_{\lambda[\sigma} h^c_{\tau]\alpha\beta}] d^5x.
\end{aligned}
$$

The element a_1 gives the first-order deformation of the gauge transformations. By using the definition of the generalized de Wit-Freedman connections [12], we get the following simple expression for a_1:

$$
a_1 = g^a{}_{(bc)} \varepsilon^{\mu\nu\rho\sigma\tau} h^{*\alpha\beta}_a{}_\mu \partial_{[\nu} C^b_{\rho]}{}^\lambda \Gamma^c_{\lambda[\sigma;\tau]\alpha\beta} d^5x, \qquad (22)
$$

where $\Gamma^c_{\lambda\sigma;\tau\alpha\beta}$ is the second spin-3 connection

$$
\begin{aligned}
\Gamma^c_{\lambda\sigma;\tau\alpha\beta} = 3\partial_{(\tau}\partial_\alpha h^c_{\beta)\lambda\sigma} + \partial_\lambda \partial_\sigma h^c_{\tau\alpha\beta} - \tfrac{3}{2}(&\partial_\lambda \partial_{(\tau} h^c_{\alpha\beta)\sigma} \\
&+ \partial_\sigma \partial_{(\tau} h^c_{\alpha\beta)\lambda})
\end{aligned}
$$

transforming under a gauge transformation $\delta_\lambda h^a_{\mu\nu\rho} = 3\partial_{(\mu} \lambda^a_{\nu\rho)}$ according to

$$
\delta_\lambda \Gamma^c_{\rho\sigma;\tau\alpha\beta} = 3\partial_\tau \partial_\alpha \partial_\beta \lambda^c_{\rho\sigma}.
$$

The expression (22) for a_1 implies that the deformed gauge transformations are

$$
\overset{(1)}{\delta}_\lambda h^a_{\mu\alpha\beta} = 3\partial_\mu \lambda^a_{\alpha\beta} + g^a{}_{(bc)} \varepsilon_\mu{}^{\nu\rho\sigma\tau} \Gamma^b_{\gamma\nu;\rho\alpha\beta} \partial_\sigma \lambda^c_\tau, \qquad (23)
$$

where the right-hand side must be completely symmetrized over the indices $(\mu\alpha\beta)$.

The cubic deformation of the free Lagrangian, a_0, is obtained from a_1 by solving the top equation $\delta a_1 + \gamma a_0 + d b_0 = 0$.

Again, we consider the most general cubic expression involving four derivatives and apply γ to it, then we compute δa_1. We take the Euler-Lagrange derivative with respect to $C_{\alpha\beta}$ of the sum of the two expressions, and multiply by $C_{\alpha\beta}$ to get a sum of terms of the form $\varepsilon C \partial^4 h \partial h$ or $\varepsilon C \partial^3 h \partial^2 h$. These are not related by Schouten identities and are therefore independent; all coefficients of the obtained equation thus have to vanish. When solving this system of equations, we find that $g_{abc} \equiv \delta_{ad} g^d{}_{bc}$ must be completely symmetric. In other words, the corresponding internal commutative algebra $\mathcal{B}$ possesses an invariant norm. As for the algebra $\mathcal{A}$ of the $n = 3$ case, the positivity of energy requirement imposes a positive-definite internal metric with respect to which the norm is defined.

Finally, we obtain the following solution for a_0:

$$
\begin{aligned}
a_0 = \tfrac{3}{2} g_{(abc)} \varepsilon^{\mu\nu\rho\sigma\tau} \{ &-\tfrac{1}{8} \partial_\mu \Box h^a_\nu \partial_\rho h^b_\sigma h^c_\tau \\
&+ \tfrac{1}{2} \partial^3_{\mu\alpha\beta} h^a_\nu \partial_\rho h^{b\alpha\beta}_\sigma h^c_\tau + \tfrac{1}{4} \partial_\mu \Box h^{a\alpha\beta}_\nu \partial_\rho h^b_{\sigma\alpha\beta} h^c_\tau \\
&+ \tfrac{3}{8} \partial_\mu \Box h^a_\nu \partial_\rho h^{b\alpha\beta}_\sigma h^c_{\tau\alpha\beta} - \tfrac{1}{2} \partial_\mu \Box h^{a\alpha\beta}_\nu \partial_\rho h^b_{\sigma\alpha\gamma} h^c_{\tau\beta}{}^\gamma \\
&- \tfrac{1}{2} \partial^{3\alpha\beta}_\mu h^a_\nu \partial_\rho h^b_{\sigma\alpha\gamma} h^c_{\tau\beta}{}^\gamma - \tfrac{1}{2} \partial^{3\alpha\beta}_\mu h^a_{\nu\alpha\gamma} \partial_\rho h^b_\sigma h^c_{\tau\beta}{}^\gamma \\
&- \tfrac{1}{4} \partial^{3\alpha\beta}_\mu h^a_{\nu\alpha\beta} \partial_\rho h^{b\gamma\delta}_\sigma h^c_{\tau\gamma\delta} - \tfrac{1}{2} \partial^{3\alpha\beta}_\mu h^a_{\nu\gamma\delta} \partial_\rho h^b_{\sigma\alpha\beta} h^{c\gamma\delta}_\tau \\
&+ \partial^{3\alpha\beta}_\mu h^a_{\nu\beta\gamma} \partial_\rho h^{b\gamma\delta}_\sigma h^c_{\tau\alpha\delta} + \tfrac{1}{2} \partial^2_{\mu\alpha} h^{a\alpha\beta}_\nu \partial^{2\gamma}_\rho h^b_\sigma h^c_{\tau\beta\gamma} \\
&- \partial^2_{\mu\alpha} h^a_{\nu\beta\gamma} \partial^2_{\rho\delta} h^{b\alpha\beta}_\sigma h^{c\gamma\delta}_\tau \} d^5x.
\end{aligned}
$$

2. Second-order deformation

The next step is the equation at order 2: $(W_1, W_1)_{a.b.} = -2sW_2$. In particular, its antighost 2 component reads $(a_2, a_2)_{a.b.} = \gamma c_2 + \partial_\mu j^\mu_2$. The left-hand side is directly

[1] The internal metric δ_{ab} being Euclidean, the condition $f^a{}_{bc} f_{aef} \equiv \delta_{ad} f^a{}_{bc} f^d{}_{ef} = 0$ can be seen as expressing the vanishing of the norm of a vector in Euclidean space (fix $e = b$ and $f = c$), leading to $f^a{}_{bc} = 0$

computed from Eq. (14):

$$\begin{aligned}(a_2, a_2)_{a.b.} &= -g^a_{bc} g_{dea} \varepsilon^{\bar\mu\,\bar\nu\,\bar\rho\,\bar\sigma\,\bar\tau} \varepsilon_\mu{}^{\nu\rho\sigma\tau} \delta^{(\mu}_{\bar\tau} \delta^{\alpha)}_{\delta} \\ &\quad \times (4\partial_{\bar\mu} C^{*d\gamma}_{\bar\nu} \partial_{\gamma\bar\rho} C^{e\delta}_{\bar\sigma} + 2\partial_{\gamma\bar\mu} C^{*d\gamma}_{\bar\nu} \partial_{\bar\rho} C^{e\delta}_{\bar\sigma}\} \\ &\quad \times \partial_\nu C^{b\beta}_\rho \partial_{\alpha\sigma} C^c_{\tau\beta} \\ &= -12 g^a_{b[c} g_{d]ea} C^{*b\alpha\beta} \hat U^c{}_\alpha{}^{\gamma|\mu\nu} \hat U^d_{\beta\gamma}{}^{|\rho\sigma} \hat U^e_{\mu\nu|\rho\sigma} \\ &\quad + \gamma c_2 + \partial_\mu j^\mu_2.\end{aligned}$$

The first term appearing in the right-hand side of the above equation is a nontrivial element of $H(\gamma|d)$. Its vanishing implies that the structure constants $g_{(abc)}$ of the commutative invariant-normed algebra $\mathcal{B}$ must obey the associativity relation $g^a_{b[c} g_{d]ea} = 0$. As for the spin-2 deformation problem (see [9], Secs. 5.4 and 6), this means that, modulo redefinitions of the fields, there is no cross-interaction between different kinds of spin-3 gauge fields provided the internal metric in $\mathcal{B}$ is positive-definite—which is demanded by the positivity of energy. The cubic vertex a_0 can thus be written as a sum of independent self-interacting vertices, one for each field $h^a_{\mu\nu\rho}$, $a = 1, \ldots, N$. Without loss of generality, we may drop the internal index a and consider only one *single* self-interacting spin-3 gauge field $h_{\mu\nu\rho}$.

V. CONCLUSION

By relaxing the parity requirement imposed in [8], the present work completes the classification of the consistent non-Abelian perturbative deformations of Fronsdal's spin-3 gauge theory, under the assumptions of Poincaré invariance and locality.

In [8] the first-order cubic vertex of Berends, Burgers, and van Dam [2] was recovered. However, the latter vertex leads to inconsistencies when continued to second order in the coupling constant, as was shown in several places [3,5,6]. A new first-order non-Abelian deformation, leading to a cubic vertex, was also found in [8]. It is defined in space-time dimension $n \geqslant 5$ and passes the second-order test where the vertex of [2] shows an inconsistency.

In the present paper, by explicitly breaking parity invariance, we obtained two more consistent non-Abelian first-order deformations, leading to a cubic vertex in the Lagrangian. The first one is defined in $n = 3$ and involves a multiplet of gauge fields $h^a_{\mu\nu\rho}$ taking values in an internal, anticommutative, invariant-normed algebra $\mathcal{A}$. The second one lives in a space-time of dimension $n = 5$, the fields taking value in an internal, commutative, invariant-normed algebra $\mathcal{B}$. Taking the metrics which define the inner product in $\mathcal{A}$ and $\mathcal{B}$ positive-definite (which is required for the positivity of energy), the $n = 3$ candidate gives rise to inconsistencies when continued to perturbation order two, whereas the $n = 5$ one passes the test and can be assumed to involve only *one* kind of self-interacting spin-3 gauge field $h_{\mu\nu\rho}$, bearing no internal "color" index.

Remarkably, the cubic vertex of the $n = 5$ deformation is rather simple. Furthermore, the Abelian gauge transformations are deformed by the addition of a term involving the second de Wit-Freedman connection in a straightforward way, cf. Eq. (23). The relevance of this second generalized Christoffel symbol in relation to a hypothetical spin-3 covariant derivative was already stressed in [5].

It is interesting to compare the results of the present spin-3 analysis with those found in the spin-2 case first studied in [26]. There, two parity-breaking first-order consistent non-Abelian deformations of Fierz-Pauli theory were obtained, also living in dimensions $n = 3$ and $n = 5$. The massless spin-2 fields in the first case bear a color index, the internal algebra $\tilde{\mathcal{A}}$ being commutative and further endowed with an invariant scalar product. In the second, $n = 5$ case, the fields take value in an anticommutative, invariant-normed internal algebra $\tilde{\mathcal{B}}$. It was further shown in [26] that the $n = 3$ first-order consistent deformation could be continued to *all* orders in powers of the coupling constant, the resulting full interacting theory being explicitly written down[2]. However, it was not determined in [26] whether the $n = 5$ candidate could be continued to all orders in the coupling constant. Very interestingly, this problem was later solved in [29], where a consistency condition was obtained at second order in the deformation parameter, viz the algebra $\tilde{\mathcal{B}}$ must be nilpotent of order three. Demanding positivity of energy and using the results of [26], the latter nilpotency condition implies that there is actually no $n = 5$ deformation at all: the structure constant of the internal algebra $\mathcal{B}$ must vanish [29]. Stated differently, the $n = 5$ first-order deformation candidate of [26] was shown to be inconsistent [29] when continued at second order in powers of the coupling constant, in analogy with the spin-3 first-order deformation written in [2].

In the present spin-3 case, the situation is somehow the opposite. Namely, it is the $n = 3$ deformation which shows inconsistencies when going to second order, whereas the $n = 5$ deformation passes the first test. Also, in the $n = 3$ case the fields take values in an anticommutative, invariant-normed internal algebra $\mathcal{A}$ whereas the fields in the $n = 5$ case take value in a commutative, invariant-normed algebra $\mathcal{B}$. However, the associativity condition deduced from a second-order consistency condition is obtained for the latter case, which implies that the algebra $\mathcal{B}$ is a direct sum of one-dimensional ideals. We summarize the previous discussion in Table I.

[2]Since the deformation is consistent, starting from $n = 3$ Fierz-Pauli, the complete $n = 3$ interacting theory of [26] describes no propagating physical degree of freedom. On the contrary, the topologically massive theory in [27,28] describes a massive graviton with *one* propagating degree of freedom (and not *two*, as was erroneously typed in [26]; N. B. wants to thank S. Deser for having pointed this out to him).

TABLE I. Internal algebras for the parity-breaking first-order deformations of spin-2 and spin-3 free gauge theories.

	$s=2$	$s=3$
$n=3$	$\tilde{\mathcal{A}}$ commutative, invariant-normed	$\mathcal{A}$ anticommutative, invariant-normed and nilpotent of order 3
$n=5$	$\tilde{\mathcal{B}}$ anticommutative, invariant-normed and nilpotent of order 3	$\mathcal{B}$ commutative, invariant-normed and associative

It would be of course very interesting to investigate further the $n=5$ deformation exhibited here, since if the deformation can be consistently continued to all orders in powers of the coupling constant, this would give the first consistent interacting Lagrangian for a higher-spin gauge field. We hope to come back to this issue in the near future.

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APPENDIX A: SCHOUTEN IDENTITIES

The Schouten identities are identities due to the fact that in n dimensions the antisymmetrization over any $n+1$ indices vanishes. These identities obviously depend on the dimension and relate functions of the fields. As it is important to have a real basis when solving equations, this appendix is devoted to finding a basis for various sets of functions, depending on their structure and the number of dimensions. We consider Poincaré-invariant and parity-breaking functions, and we are interested mainly in the γ-nontrivial quantities. The fields may have an internal index but we only write it when necessary.

1. Functions of the structure $\varepsilon C^*\hat{T}\hat{T}$ in $n=4$

In order to achieve the four-dimensional study of the algebra deformation in D-degree 2, a list of the Schouten identities is needed for the functions of the structure $\varepsilon C^*\hat{T}\hat{T}$. The space of these functions is spanned by

$$T_1^{bc} = \varepsilon^{\mu\nu\rho\sigma} C^{*a\alpha}_{\mu} \hat{T}^b_{\nu\rho|}{}^{\beta} \hat{T}^c_{\sigma\alpha|\beta},$$
$$T_2^{bc} = \varepsilon^{\mu\nu\rho\sigma} C^{*a\alpha}_{\mu} \hat{T}^b_{\nu\rho|}{}^{\beta} \hat{T}^c_{\sigma\beta|\alpha},$$
$$T_3^{[bc]} = \varepsilon^{\mu\nu\rho\sigma} C^{*a\alpha\beta} \hat{T}^b_{\mu\nu|\alpha} \hat{T}^c_{\rho\sigma|\beta}.$$

There are two Schouten identities. Indeed, one should first notice that all Schouten identities are linear combinations of identities with the structure $\delta^{[\alpha\beta\gamma\delta\varepsilon]}_{[\mu\nu\rho\sigma\tau]}\varepsilon^{\mu\nu\rho\sigma}C^*\hat{T}\hat{T}=0$, where the indices $\alpha\beta\gamma\delta\varepsilon\tau$ are contracted with the indices of the ghosts and where $\delta^{[\alpha\beta\gamma\delta\varepsilon]}_{[\mu\nu\rho\sigma\tau]} = \delta^{[\alpha}_{[\mu}\delta^{\beta}_{\nu}\delta^{\gamma}_{\rho}\delta^{\delta}_{\sigma}\delta^{\varepsilon]}_{\tau]}$. Furthermore, there are only two independent identities of this type:

$$\delta^{[\alpha\beta\gamma\delta\varepsilon]}_{[\mu\nu\rho\sigma\tau]}\varepsilon^{\mu\nu\rho\sigma} C^{*\tau}_{\alpha} \hat{T}^b_{\beta\gamma|\lambda} \hat{T}^c_{\delta\varepsilon|}{}^{\lambda} = 0,$$
$$\delta^{[\alpha\beta\gamma\delta\varepsilon]}_{[\mu\nu\rho\sigma\tau]}\varepsilon^{\mu\nu\rho\sigma} C^{*\lambda}_{\alpha} \hat{T}^b_{\beta\gamma|}{}^{\tau} \hat{T}^c_{\delta\varepsilon|\lambda} = 0.$$

Expanding the product of δ's, one finds that the first identity implies that T_1^{bc} is symmetric: $T_1^{bc} = T_1^{(bc)}$, while the second one relates T_2^{bc} and $T_3^{[bc]}$: $T_2^{bc} = T_3^{[bc]}$.

So, in four dimensions, a basis of the functions with the structure $\varepsilon C^*\hat{T}\hat{T}$ is given by $T_1^{(bc)}$ and $T_3^{[bc]}$.

2. Functions of the structure $\varepsilon h^*\hat{T}\hat{U}$ in $n=4$

These functions appear in the study of the algebra deformation in D-degree 3, $n=4$. They are completely generated by the following terms:

$$T_1 = \varepsilon^{\mu\nu\rho\sigma} h^{*a\alpha\beta}_{\mu} \hat{T}^b_{\nu\gamma|\beta} \hat{U}^c_{\rho\sigma|\alpha}{}^{\gamma},$$
$$T_2 = \varepsilon^{\mu\nu\rho\sigma} h^{*a\alpha\beta}_{\mu} \hat{T}^b_{\nu\beta|\gamma} \hat{U}^c_{\rho\sigma|\alpha}{}^{\gamma},$$
$$T_3 = \varepsilon^{\mu\nu\rho\sigma} h^{*a\alpha} \hat{T}^b_{\mu\nu|}{}^{\beta} \hat{U}^c_{\rho\sigma|\alpha\beta},$$
$$T_4 = \varepsilon^{\mu\nu\rho\sigma} h^{*a}_{\mu} \hat{T}^{b\alpha\beta|}{}_{\nu} \hat{U}^c_{\rho\sigma|\alpha\beta},$$
$$T_5 = \varepsilon^{\mu\nu\rho\sigma} h^{*a\alpha\beta}_{\mu} \hat{T}^b_{\nu\rho|\gamma} \hat{U}^c_{\sigma\alpha|\beta}{}^{\gamma}.$$

There are three Schouten identities:

$$\delta^{[\alpha\beta\gamma\delta\varepsilon]}_{[\mu\nu\rho\sigma\tau]}\varepsilon^{\mu\nu\rho\sigma} h^*_{\alpha\lambda}{}^{\tau} \hat{T}_{\beta\gamma|\eta} \hat{U}_{\delta\varepsilon|}{}^{\lambda\eta} = 0,$$
$$\delta^{[\alpha\beta\gamma\delta\varepsilon]}_{[\mu\nu\rho\sigma\tau]}\varepsilon^{\mu\nu\rho\sigma} h^*_{\alpha} \hat{T}_{\beta\gamma|\lambda} \hat{U}_{\delta\varepsilon|}{}^{\tau\lambda} = 0,$$
$$\delta^{[\alpha\beta\gamma\delta\varepsilon]}_{[\mu\nu\rho\sigma\tau]}\varepsilon^{\mu\nu\rho\sigma} h^*_{\alpha\eta}{}^{\lambda} \hat{T}_{\beta\gamma|\lambda} \hat{U}_{\delta\varepsilon|}{}^{\tau\eta} = 0.$$

An explicit expansion of these identities yields the useful relations

$$T_3 + 2T_2 + 2T_5 = 0, \qquad T_3 - T_4 = 0, \qquad T_1 = 0.$$

3. Functions of the structure $\varepsilon C^*\hat{U}\hat{U}$ in $n=4$

The Schouten identities for the functions of the structure $\varepsilon C^*\hat{U}\hat{U}$ in $n=4$ are needed for the analysis of the algebra deformation in D-degree four. The functions at hand are generated by $T_1^{[bc]} = \varepsilon^{\mu\nu\rho\sigma} C^{*a}_{\alpha\beta} \hat{U}^b_{\mu\nu|}{}^{\alpha\gamma} \hat{U}^c_{\rho\sigma|}{}^{\beta}{}_{\gamma}$ and $T_2^{bc} = \varepsilon^{\mu\nu\rho\sigma} C^{*a}_{\mu\beta} \hat{U}^b_{\nu\rho|}{}^{\alpha\gamma} \hat{U}^c{}_{\sigma}{}^{\beta}{}_{|\alpha\gamma}$. However, these vanish because of the Schouten identities

$$\delta^{[\alpha\beta\gamma\delta\varepsilon]}_{[\mu\nu\rho\sigma\tau]}\varepsilon^{\mu\nu\rho\sigma} C^{*\lambda}_{\alpha} \hat{U}^b_{\beta\gamma|}{}^{\tau\eta} \hat{U}^c_{\gamma\delta|\lambda\eta} = 0,$$
$$\delta^{[\alpha\beta\gamma\delta\varepsilon]}_{[\mu\nu\rho\sigma\tau]}\varepsilon^{\mu\nu\rho\sigma} C^{*\tau}_{\alpha} \hat{U}^b_{\beta\gamma|}{}^{\lambda\eta} \hat{U}^c_{\gamma\delta|\lambda\eta} = 0.$$

Indeed, they imply that $T_1^{[bc]} + T_2^{bc} = 0$ and $T_2^{bc} = T_2^{(bc)}$, which can be satisfied only if $T_1^{[bc]} = T_2^{(bc)} = 0$.

4. Functions of the structure $\varepsilon C\partial^3 hh$ and $\varepsilon C\partial^2 h\partial h$ in $n = 3$

These functions appear when solving $\delta a_1 + \gamma a_0 = db_0$ in Sec. IV B. In generic dimension ($n > 4$), there are, respectively, 45 and 130 independent functions in the sets $\varepsilon C\partial^3 hh$ and $\varepsilon C\partial^2 h\partial h$. In three dimensions, there are 108 Schouten identities relating them, which leave only 67 independent functions. We have computed all these identities and the relations between the 108 dependent functions and the 67 independent ones. However, given their numbers, they will not be reproduced here.

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