Reachability in Stochastic Hybrid Systems
[Ongoing Work]

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Outline

- **Verification** of models combining:
  - **stochastic** aspects (e.g., Markov chains);
  - **hybrid** aspects (with both discrete and continuous transitions);

\[ \Rightarrow \text{stochastic hybrid systems}. \]

- Properties about the **reachability** of states (is some set of states reached with probability 1? Can we compute the probability of reaching a set?).

Goal

Identify a **decidability frontier** for reachability in stochastic hybrid systems.

Method

Follow an approach that has been successful for **infinite Markov chains**.
Reachability in infinite Markov chains

Let $\mathcal{M}$ be a countable Markov chain.

Let $B \subseteq S$ be a subset of states, $s \in S$ be an initial state.

**Goal**

Compute (or approximate) $\text{Prob}^\mathcal{M}_s(\Diamond B)$.

We set

$$\tilde{B} = \{s \in S \mid \text{Prob}^\mathcal{M}_s(\Diamond B) = 0\}.$$
How to approximate the probability of reaching $B$?

Approximation procedure (for a given $\epsilon > 0$)$^1$

We define

\[
\begin{align*}
    p_n^\text{Yes} &= \text{Prob}_s^M(\diamondsuit \leq n B) \\
    p_n^\text{No} &= \text{Prob}_s^M(\diamondsuit \leq n \tilde{B}).
\end{align*}
\]

For all $n$, $p_n^\text{Yes} \leq \text{Prob}_s^M(\diamondsuit B) \leq 1 - p_n^\text{No}$.

We stop when

\[
(1 - p_n^\text{No}) - p_n^\text{Yes} < \epsilon.
\]

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Example

\[
\begin{align*}
\text{Target: } \{a\} & \quad \Rightarrow \quad \overline{\{a\}} = \{d\}.
\end{align*}
\]

\[
\begin{align*}
\Rightarrow & \quad \frac{1}{4} \leq \text{Prob}_c^M(\Diamond \{a\}) \leq 1 - \frac{5}{8} = \frac{3}{8}. & \Rightarrow & \quad \text{Always terminates?}
\end{align*}
\]
Counterexample: diverging random walk

The procedure does not terminate for this infinite Markov chain:

\[
\begin{align*}
M &
\begin{array}{c}
\begin{tikzpicture}[node distance=2cm,auto]
\node (s0) [state, fill=green!30] {$s_0$};
\node (s1) [state, right of=s0] {$s_1$};
\node (s2) [state, right of=s1] {$s_2$};
\node (s3) [state, right of=s2] {$\cdots$};

\path[->] (s0) edge [loop left] node {$\frac{1}{3}$} (s0)
(s0) edge [bend left] node {$1$} (s1)
(s1) edge [bend left] node {$\frac{2}{3}$} (s0)
(s1) edge [loop above] node {$\frac{2}{3}$} (s1)
(s1) edge [bend left] node {$\frac{1}{3}$} (s2)
(s2) edge [bend left] node {$\frac{2}{3}$} (s1)
(s2) edge [loop right] node {$\frac{1}{3}$} (s2)
(s2) edge [bend left] node {$\frac{1}{3}$} (s3);
\end{tikzpicture}
\end{array}
\end{align*}
\]

Initial state: $s_1$, target state: $B = \{s_0\} \implies \tilde{B} = \emptyset$. For all $n$,

- \( p_n^{\text{Yes}} = \text{Prob}_s (\Diamond \leq_n B) \leq \text{Prob}_s (\Diamond B) = \frac{1}{2} \).

- \( p_n^{\text{No}} = \text{Prob}_s (\Diamond \leq_n \tilde{B}) = 0 \).

\( \implies \) For all $n$, \( (1 - p_n^{\text{No}}) - p_n^{\text{Yes}} \geq \frac{1}{2} \cdots \)
Decisiveness

Let $\mathcal{M} = (S, P)$ be a countable Markov chain, $B \subseteq S$.

**Decisiveness**

$\mathcal{M}$ is **decisive** w.r.t. $B \subseteq S$ if for all $s \in S$, $\text{Prob}_s^{\mathcal{M}}(\Diamond B \lor \Diamond \bar{B}) = 1$.

**Theorem**

If $\mathcal{M}$ is decisive w.r.t. $B$, then the approximation procedure is correct and **terminates**.

- The diverging random walk is not decisive w.r.t. $B = \{s_0\}$.
- Decisiveness also allows for a procedure to verify **almost-sure reachability**.

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Hybrid systems

- \((L, E)\) is a finite graph.
- A number \(n\) of continuous variables \(\leadsto\) states of the system are in \(L \times \mathbb{R}^n \leadsto\) uncountable!
- For each \(\ell \in L, \gamma_\ell : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n\) is a continuous dynamics.
- For each edge \(e \in E, G(e) \subseteq \mathbb{R}^n\) is a guard.
- For each edge \(e \in E, R(e) : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}\) is a reset map.
Transitions of hybrid systems

States: $L \times \mathbb{R}^n$ (discrete location $\times$ value of the continuous variables).

A transition combines a **continuous evolution** and a **discrete transition**. Example: initial state is $s = (\ell_1, (2, 0))$;

- we stay in $\ell_1$ for some **time** $\tau \geq 0$;
- we take an **edge** whose guard is satisfied;
- we take a value among the possible **resets**, e.g. $s' = (\ell_2, (\frac{1}{2}, \frac{1}{2}))$. 

\[
\begin{align*}
\ell_3 & \quad \text{y} \leq -1 \\
\ell_1 & \quad \text{x, y := 0} \\
\ell_2 & \quad y \geq 1 \\
\end{align*}
\][x, y \in [-1, 1]]

We replace the nondeterminism of hybrid systems with probability distributions on the:

- waiting time from a given state;
- edge choice;
- choice of a reset value.

\[\rightsquigarrow \text{Stochastic hybrid systems (SHSs)}\]
Undecidability

Undecidability of reachability for SHSs

Given an SHS $\mathcal{H}$, an initial distribution $\mu$ on the states of $\mathcal{H}$ and a target set $B \subseteq L \times \mathbb{R}^n$, the reachability problems

- $\text{Prob}_{\mu}^{\mathcal{H}}(\diamondsuit B) = 1$?
- $\text{Prob}_{\mu}^{\mathcal{H}}(\diamondsuit B) = 0$?
- is a value $\varepsilon$-close to $\text{Prob}_{\mu}^{\mathcal{H}}(\diamondsuit B)$?

are undecidable.

$\leadsto$ inspired from an undecidability proof for hybrid systems.\(^3\)

Goal

Find a setting in which reachability is decidable.

\(^3\)Henzinger et al., “What’s Decidable about Hybrid Automata?”, 1998.
Reachability problems in **stochastic** systems

To deal with an uncountable number of states ⇞ “finite abstraction”.

**Abstraction of a stochastic hybrid system**

- **Abstraction** whenever $p > 0 \Leftrightarrow q > 0$.
- **Sound** abstraction whenever

$$\Pr^{\mathcal{T}_2}(\Diamond B) = 1 \implies \Pr^{\mathcal{T}_1}(\Diamond \alpha^{-1}(B)) = 1.$$
Decidable classes for reachability

Hybrid systems: existence of a finite time-abstract bisimulation

- Timed automata\(^4\) ($\dot{x} = 1, x := 0$; region graph);
- Initialized rectangular hybrid systems;\(^5\)
- O-minimal hybrid systems\(^6\) (rich dynamics, all variables have to be reset at every discrete transition).

SHSs: existence of a finite and \textbf{sound} abstraction

- Single-clock stochastic timed automata;\(^7\)
- Reactive stochastic timed automata.\(^7\)

$\rightsquigarrow$ Proof of soundness: \textit{finite abstraction} $+$ \textit{decisiveness}.

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\(^7\)Bertrand et al., “When are stochastic transition systems tameable?”, 2018.
Plan to make reachability decidable: strong resets

We restrict our focus to SHSs with **strong resets**.\(^8\)

Strong reset $=$ reset that does not depend on the value of the variables.

Example: $x$ follows a uniform dist. in $[x - 1, x + 1]$ is **not** a strong reset.

$x$ follows a uniform distribution in $[-1, 1]$ is a strong reset.

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Consequences of strong resets

**Proposition**

If an SHS has (at least) one **strong reset** per cycle of the discrete graph, it

- has a **finite abstraction**;
- is **decisive** w.r.t. any set of states.

\[
\text{strong resets} \iff \text{finite abstraction} \quad + \quad \text{decisiveness} \quad \overset{\text{}}{\implies} \begin{array}{l}
\text{sound and finite} \\
\text{abstraction}
\end{array}
\]

\[\leadsto \text{Reachability is decidable when the abstraction is computable!}\]
Conclusion: decidable classes of hybrid systems

Hybrid systems: existence of a finite time-abstract bisimulation

- Timed automata;\(^9\)
- Initialized rectangular hybrid systems;\(^{10}\)
- O-minimal hybrid systems.\(^{11}\)

SHSs: existence of a sound and finite abstraction

- Single-clock stochastic timed automata;\(^{12}\)
- Reactive stochastic timed automata;\(^{12}\)
- Strongly-reset stochastic hybrid systems.

\[\rightarrow\] Reachability is **decidable** under effectiveness assumptions.

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\(^{10}\) Henzinger et al., “What’s Decidable about Hybrid Automata?”, 1998.


\(^{12}\) Bertrand et al., “When are stochastic transition systems tameable?”, 2018.