Games Where You Can Play Optimally with Arena-Independent Finite Memory

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Outline

**Strategy synthesis for two-player turn-based games**

Design optimal controllers for systems interacting with an antagonistic environment.

“Optimal” w.r.t. an objective or a specification.

**Goal: interest in “simple” controllers**

Finite-memory determinacy: when do finite-memory controllers suffice?

**Inspiration**

Results by Gimbert and Zielonka¹ about memoryless determinacy.

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1 Memoryless determinacy

2 The need for memory

3 Arena-independent finite memory
1 Memoryless determinacy

2 The need for memory

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Two-player turn-based zero-sum games on graphs

- Finite two-player arenas: $S_1$ (circles, for $P_1$) and $S_2$ (squares, for $P_2$), edges $E$.
- Set $C$ of colors. Edges are colored.
- “Objectives” given by preference relations $\sqsubseteq \in C^\omega \times C^\omega$ (total preorder). Zero-sum, $\sqsubseteq^{-1}$.
- A strategy for $P_i$ is a (partial) function $\sigma : E^* \to E$. 

$C = \{\top, \bot\}$

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Question

Given a preference relation, do “simple” strategies suffice to play optimally in all arenas?

A strategy $\sigma$ of $\mathcal{P}_i$ is \textit{memoryless} if it is a function $\exists^* \mathcal{S}_i \rightarrow E$.

E.g., for reachability, \textit{memoryless} strategies suffice. Also suffice for safety, Büchi, co-Büchi, parity, mean-payoff, energy, average-energy...
Memoryless determinacy

Good understanding of memoryless determinacy:

• **sufficient** conditions to guarantee memoryless optimal strategies for both players.\(^2\),\(^3\)

• **sufficient** conditions to guarantee memoryless optimal strategies for one player.\(^4\),\(^5\),\(^6\)

• **characterization** of the preference relations admitting optimal memoryless strategies for both players.\(^7\)

Gimbert and Zielonka’s characterization\textsuperscript{8}

Let $\sqsubseteq$ be a preference relation. Two results:

1. Characterization of memoryless determinacy w.r.t. properties of $\sqsubseteq$.
2. Corollary:

One-to-two-player memoryless lifting

If

- in all one-player arenas of $\mathcal{P}_1$, $\mathcal{P}_1$ has an optimal memoryless strategy,
- in all one-player arenas of $\mathcal{P}_2$, $\mathcal{P}_2$ has an optimal memoryless strategy,

then both players have an optimal memoryless strategy in all two-player arenas.

Extremely useful in practice. Very easy to recover memoryless determinacy of, e.g., mean-payoff and parity games.

\textsuperscript{8}Gimbert and Zielonka, “Games Where You Can Play Optimally Without Any Memory”, 2005.
1 Memoryless determinacy

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The need for memory

Memoryless strategies do not always suffice.

\[ (1, -1) \rightarrow (s_1, -1,-1) \rightarrow (s_2, -1,1) \rightarrow (B, -1,1) \]

- Büchi(A) \land Büchi(B): requires finite memory.

\[ (1, -1) \rightarrow (m_1, -1,-1) \rightarrow (m_2, -1,1) \rightarrow (B, -1,1) \]

- Mean payoff \( \geq 0 \) in both dimensions: requires infinite memory.\(^9\)

\[ \Rightarrow \text{Combinations of objectives usually require memory.} \]

An attempt at lifting [GZ05] to FM determinacy

- Lack of a good understanding of finite-memory determinacy.

- **Related work**: sufficient properties to preserve FM determinacy in Boolean combinations of objectives.  

- Our approach:

  **Hope**: extend Gimbert and Zielonka’s results

  One-to-two-player lifting for memoryless finite-memory determinacy?

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Counterexample

Let \( C \subseteq \mathbb{Z} \). \( P_1 \) wants to achieve a play \( \pi = c_1 c_2 \ldots \in C^\omega \) s.t.

\[
\lim \sup_n \sum_{i=0}^n c_i = +\infty \quad \text{or} \quad \exists \infty n, \sum_{i=0}^n c_i = 0.
\]

Optimal FM strategies in one-player arenas... ... but not in two-player arenas: \( P_1 \) wins but needs infinite memory.

Intuition:
In one-player arenas, \( P_1 \) can bound the memory he needs in advance.
In two-player arenas, \( P_2 \) can generate arbitrarily long sequences.
1 Memoryless determinacy

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Arena-independent memory

- For Büchi($A$) $\land$ Büchi($B$), this structure suffices to play optimally on all arenas for $P_1$.

![Diagram](image)

- The counterexample fails because in one-player arenas, the size of the memory is dependent on the size of the arena.
- Observation: for many objectives, one fixed memory structure suffices for all arenas.

"For all $A$, does there exist $M$...?"

$\rightarrow$ "Does there exist $M$, for all $A$...?"

Method: reproducing the approach of Gimbert and Zielonka given a memory structure $M$. 
Characterization of arena-independent determinacy

Let $\sqsubseteq$ be preference relation, $\mathcal{M}$ be a memory structure.

1. Characterization of “playing with $\mathcal{M}$ is sufficient” in terms of properties of $\sqsubseteq$.

2. Corollary:

One-to-two-player lifting

If
- in all one-player arenas of $\mathcal{P}_1$, $\mathcal{P}_1$ has an optimal strategy with memory $\mathcal{M}_1$,
- in all one-player arenas of $\mathcal{P}_2$, $\mathcal{P}_2$ has an optimal strategy with memory $\mathcal{M}_2$,

then both players have an optimal strategy in all two-player arenas with memory $\mathcal{M}_1 \otimes \mathcal{M}_2$.

In short: the study of one-player arenas is sufficient to determine whether playing with arena-independent finite memory suffices.
Applicability and limits

- **Applies to** objectives with optimal *arena-independent* strategies:
  - generalized reachability,\(^{11}\)
  - generalized parity,\(^{12}\)
  - window parity,\(^{13}\)
  - lower- and upper-bounded (multi-dimensional) energy games.\(^{14,15}\)

- **Does not apply to**, e.g., multi-dimension lower-bounded energy objectives:\(^{16}\) the size of the finite memory depends on the arena.

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\(^{11}\) Fijalkow and Horn, “The surprizing complexity of reachability games”, 2010.


Conclusion

**Key observation:** for many objectives, arena-independent memory suffices.

**Contributions**

- Characterization of arena-independent finite-memory determinacy.
- One-to-two-player lifting.
- Generalization of Gimbert and Zielonka’s work.

**Future work**

Understand (arena-dependent) finite-memory determinacy through the study of one-player arenas.

Thanks!