Quasi Kepler's third law for quantum many-body systems

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Abstract Approximate analytical solutions are computed for quantum self-graviting particles with different masses. The results give strong indications in favour of the existence of a quasi Kepler's third law for such systems. The relevance of the proposal is checked with accurate numerical data for the ground state of self-graviting identical bosons and with numerical estimations for systems with identical particles plus a different one. Connections between the quantum and classical systems are presented.

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I. INTRODUCTION

Kepler's third law played an important role in the history of physics. It recently reappeared in the spotlight when several accurate numerical computations showed strong indications that a generalized Kepler's third law exists for the periodic three-body system [1–4]. For two-body systems with a total energy E and a period T, the Kepler's third law states that $T\,|E|^{3/2} = \frac{\pi}{\sqrt{2}}\,G\,f(m_1, m_2)$, where G is the gravitation constant and $f(m_1, m_2)$ a given function of the two masses. Let us call this invariant the reduced time τ. For three-body system, the definition of this quantity is given by

$$\tau = T^*|E|^{3/2}, \tag{1}$$

where T^* takes into account the topology of the orbit around the three two-body collision points. $T^* = T/L_f$, where L_f is the "free group element" of the orbit [5]. Because τ is found by numerical computation to be approximately equal to a universal constant for three identical bodies, a generalization of Kepler's third law for N-body periodic orbits has been proposed in [6], using arguments based on dimensional analysis.

Strong connections exist between classical and quantum theories, the most famous one being certainly the Ehrenfest theorem, showing that expectation values obey Newton's second law. So, one can ask if Kepler's third law can also be relevant for quantum N-body systems. This problem has been addressed in [7] for systems with identical particles. An invariant reduced time has also been found, but different from the one proposed for classical N-body systems. Within the quantum calculations, the periodic orbit is replaced by a stationary quantum state, and a quantum definition of the period must be used. A first definition of this period is proposed in [7] on the basis of a semiclassical approximation, but a more relevant definition (but numerically identical to the previous one) is given in [8]. In the following, the quantum reduced time is computed (approximately but analytically) for a general many-body system and found to be identical to the proposal made in [9], using again arguments based on dimensional analysis.

The quantum reduced time is defined in Sect. II. The case of identical particles is already treated in [7], but it is presented in Sect. III for completeness, and because its relevance is checked with a particular example. Results for a general system are computed in Sect. IV, where the relevance of the analysis is checked on a particular system composed of a set identical particles plus a different one. Some concluding remarks and outlook are given in Sect. V.

II. QUANTUM REDUCED TIME

The Hamiltonian for a system of self-graviting particles is given by

$$H = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2\,m_i} - \sum_{i<j=2}^{N} \frac{G\,m_i\,m_j}{|\boldsymbol{r}_{ij}|}. \tag{2}$$

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The quantum reduced time τ_q is computed with a stationary eigenstate of this Hamiltonian by the formula

$$\tau_q = T_q\,|E|^{3/2}, \tag{3}$$

where $E = \langle H \rangle$ is the corresponding eigenvalue and T_q the equivalent period. It is given by [8]

$$T_q = \frac{\pi\, I}{\langle T \rangle}, \tag{4}$$

where I is the action for the state and $\langle T \rangle$ the mean value of T, the kinetic part of H. The virial theorem implies that

$$E = -\langle T \rangle = \frac{\langle V \rangle}{2}, \tag{5}$$

where V is the potential part of H. Actually, (5) is still valid if $(-G\, m_i\, m_j)$ is replaced by k_{ij}. The k_{ij} can be positive or negative, provided a bound state exists. This equality can be checked by using the accurate numerical results from [10] about the ground state of "self graviting bosons" and "two-component Coulombic systems". To be fair, two results from the last examples are decked out with a relative error around 10%, instead of less than 1% for all other results. We suspect simple misprints or unnoticed lacks of convergence. Using (4) and (5), (3) reduces to

$$\tau_q = \pi\, I\, |E|^{1/2}. \tag{6}$$

The procedures to compute I and E are explained in the following sections.

III. QUANTUM SYSTEMS WITH IDENTICAL PARTICLES

The envelope theory (ET) [11–13], also known as the auxiliary field method [14, 15], allows the solution of N-body quantum systems, eigenvalues and eigenvectors. In favourable situations, analytical upper or lower bounds can be obtained. The method is simple to implement and can provide fairly reliable results [16–18]. For the Hamiltonian (2) with identical particles ($m_i = m,\ \forall\, i$), it gives the following result [16]

$$E_{\text{id}} = -\frac{N^2(N-1)^3}{16}\frac{G^2\, m^5}{Q_\phi(N)^2\, \hbar^2}, \tag{7}$$

where $Q_\phi(N)$ is a global quantum number given by

$$Q_\phi(N) = \sum_{i=1}^{N-1} (\phi\, n_i + l_i) + (N-1)\frac{1+\phi}{2}, \tag{8}$$

n_i and l_i being the usual radial and orbital quantum numbers. In the genuine method, $\phi = 2$ and E_{id} are upper bounds. By using the ET in combination with a generalisation of the dominantly orbital state method, it is shown in [17] that the choice $\phi = 1$ can dramatically improve the approximate energies, but the variational character is then lost. Results from [8] show that $Q_\phi(N)\,\hbar$ is a good estimation of I for the eigenstates of H. The quantum reduced time is then

$$\tau_q = \frac{\pi}{4}\, G\, m^{5/2} N(N-1)^{3/2}. \tag{9}$$

Let us remark that this result is independent from the value of $Q_\phi(N)$, as this quantity cancels out in the calculation. For this reason, (9) can possibly give an assumed "exact" result, since the main source of inaccuracy in the ET calculation is the structure of the global quantum number [16, 17]. Moreover, as the quantum character of I and E is carried by the quantity $Q_\phi(N)\,\hbar$, this cancellation explains why the result (9) could be relevant for a classical system. Unfortunately, this value is $N(N-1)/2$ times the classical value given in [6]. This discrepancy can be due to the introduction in the classical computation of the free group element L_f for which it is difficult to find a quantum equivalent.

Thanks to the accurate energies computed in [10] for the ground state of self graviting bosons with $\hbar = G = m = 1$, it is possible to check the relevance of the notion of quantum reduced time on a particular example. Values for τ_q computed with (9) and with (6), under the assumption that $I = Q_1(N)\,\hbar$ (the value of $\phi = 1$ giving the best agreement for the energies), are compared in Fig. 1. One can see that the agreement is very good. The two calculations coincide for $N = 2$, as already mentioned in [8]. This shows that the choice of $Q_1(N)\,\hbar$ for I seems quite reasonable.

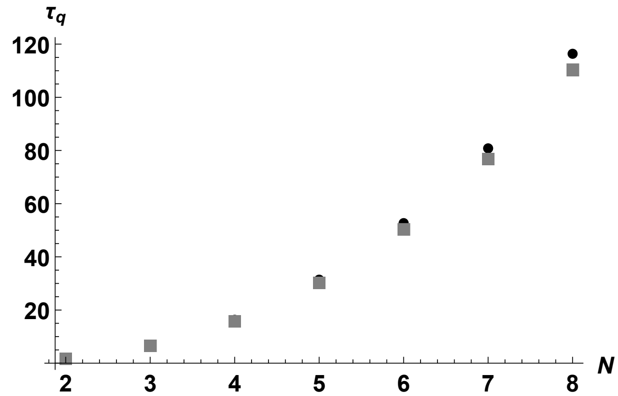

Figure 1: Reduced time τ_q for the ground state of self-graviting bosons ($\hbar = G = m = 1$) as a function of N. Black circle: values of (9); Gray square: results from (6) with $I = Q_1(N)\,\hbar$ and E taken from data in [10].

IV. GENERAL QUANTUM SYSTEMS

The ET has been generalised to study systems with different particles [19], but the procedure is then more complicated to implement. In order to well understand the approximations involved, let us detail a little bit the calculations. The first step of the procedure is to build an auxiliary Hamiltonian [19]

$$\tilde{H} = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2m_i} + \sum_{i<j=2}^{N} \left[\rho_{ij}\, \boldsymbol{r}_{ij}^2 - \frac{3}{2^{2/3}} \left(G\, m_i\, m_j\right)^{2/3} \rho_{ij}^{1/3} \right]. \tag{10}$$

The upper bounds of the eigenvalues of H are then determined by minimising the eigenvalues $\tilde{E}\left(\{\rho_{ij}\}\right)$ of $\tilde{H}$ with respect to the auxiliary parameters $\{\rho_{ij}\}$. Unfortunately, the values $\tilde{E}$ cannot be analytically computed in general when $N > 5$ [14]. As our purpose is to obtain general analytical results, we must resort to a supplementary approximation. Slightly worse upper bounds will be obtained if we impose the constraint

$$\rho_{ij} = \rho\, m_i\, m_j. \tag{11}$$

The eigenvalues $\tilde{E}$ can then be exactly computed using the procedure detailed in [14]

$$\tilde{E}(\rho) = \sqrt{2\rho \sum_{i=1}^{N} m_i}\, Q_\phi(N)\, \hbar - \frac{3}{2^{2/3}} G^{2/3} \rho^{1/3} \sum_{i<j=2}^{N} m_i\, m_j. \tag{12}$$

The minimisation with respect to ρ gives upper bounds of the upper bounds

$$\tilde{E} = -\frac{G^2}{2\, Q_\phi(N)^2\, \hbar^2} \frac{\left(\sum_{i<j=2}^{N} m_i\, m_j\right)^3}{\sum_{i=1}^{N} m_i}. \tag{13}$$

This result coincides exactly with the formula (10) in [9], guessed solely on the basis of (14) below. If we assume again that $Q_\phi(N)\,\hbar$ is a good estimation of I, (6) gives

$$\tau_q = \frac{\pi}{\sqrt{2}} G \left[\frac{\left(\sum_{i<j=2}^{N} m_i\, m_j\right)^3}{\sum_{i=1}^{N} m_i} \right]^{1/2}, \tag{14}$$

which is the relation (2) in [9], determined on the basis of dimensional arguments. So, an universal invariant reduced time is also obtained for general systems. It is easy to see that (9) is recovered when $m_i = m,\ \forall\, i$.

In order to check the error made with the approximation (11), let us consider a system composed of $(N-1)$ particles with a mass m_a and the last one with a mass m_b. In this case, $\rho_{ij} = \rho_{aa}$ for $i \le j < N$ and $\rho_{iN} = \rho_{ab}$, and $\tilde{H}$ can be solved [19]

$$\tilde{E}(\rho_{aa},\rho_{ab}) = \sqrt{\frac{2\left((N-1)\,\rho_{aa}+\rho_{ab}\right)}{m_a}}\,Q_\phi(N-1)\,\hbar + \sqrt{\frac{2\left((N-1)\,m_a+m_b\right)\rho_{ab}}{m_a m_b}}\,Q_\phi(2)\,\hbar$$
$$- \frac{3\,(N-1)(N-2)}{2^{5/3}}(G\,m_a^2)^{2/3}\rho_{aa}^{1/3} - \frac{3\,(N-1)}{2^{2/3}}(G\,m_a\,m_b)^{2/3}\rho_{ab}^{1/3} \quad (15)$$

The genuine upper bound of the ET can be numerically obtained by computing $\tilde{E} = \min_{\rho_{aa},\rho_{ab}} \tilde{E}(\rho_{aa},\rho_{ab})$. The reduced time $\tilde{\tau}_q$ is calculated with this value $\tilde{E}$ and $I = Q_\phi(N)\,\hbar$, and compared with the value τ_q computed with (14) by using the relative error Δ defined by

$$\Delta = \frac{\tilde{\tau}_q - \tau_q}{\tilde{\tau}_q}. \quad (16)$$

This error is presented in Fig. 2 for bosonic ground states with $\phi = 2$, but calculations show that the results are practically independent from ϕ. The cancellation of ϕ effects seems nearly perfect in this case. It is remarkable that the error due to the calculation of τ_q with (14) is quite small, except when the number of particles is small and the different particle is much lighter than the other ones.

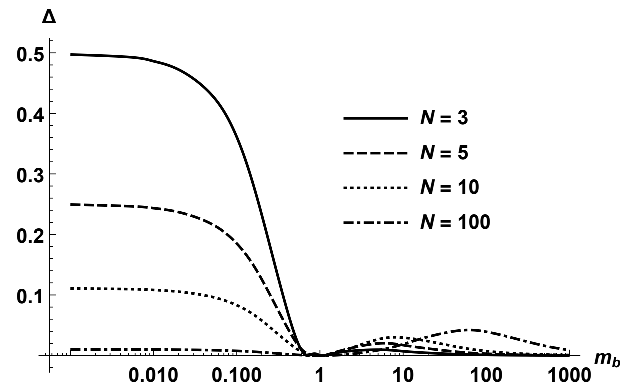

Figure 2: Relative error Δ with $\phi = 2$ (see text) for bosonic ground states as a function of m_b with $\hbar = G = m_a = 1$, for some values of N.

V. CONCLUDING REMARKS

Calculations above indicate that a quasi Kepler's third law exists for quantum self-graviting particles. The envelope theory used to solve approximately the quantum many-body problem predicts an exact invariant reduced time for systems with identical particles, and a quasi exact invariant for systems with different particles. For identical particles, a check is performed thanks to numerical data available in [10] for the ground states of bosons. For different particles, the relevance of the approximate invariant obtained is only verified for systems with a set of identical particles plus a different one. It is certainly desirable for more types of systems to be studied. Above all, accurate numerical calculations like the ones achieved in [10] should be extended to systems with different particles.

The invariant obtained in the general quantum case is different from the one computed for the classical equivalent system [6, 9]. This is probably due to the introduction in the classical calculations of information about the topology of the classical orbits, for which no equivalent is found in quantum calculations. Nevertheless, if a quasi Kepler's third law exists for quantum self-graviting particles, it is worth considering its existence for equivalent many-body classical systems. So, supplementary studies like [1–4] are certainly desirable for classical orbits with more than three particles.

Let us mention that an invariant also seems to exist for collisionless periodic orbits in the Coulomb potential for three charged particle, one positive and two negative [20]. The presence of attractive and repulsive interactions in this system makes it different from the purely attractive cases presented above. So further research involving various Coulombic systems seem desirable to check the existence of some universal relation for such systems.

If these quasi Kepler's third laws for quantum and classical many-body systems are something else than happy coincidences, it is worth searching for some fundamental principle at work. This problem certainly deserves further research.

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